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A TREATISE

ON

CONIC SECTIONS.

By the same Author.

A TREATISE ON THE HIGHER PLANE CURVES:

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A TREATISE

ON

CONIC SECTIONS:

CONTAINING

AN ACCOUNT OF SOME OF THE MOST IMPORTANT MODERN ALGEBRAIC AND GEOMETRIC METHODS.

BY THE

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PREFACE.

A NEW EDITION of the present Work having been called for, I have endeavoured to make this Treatise more deserving of the favourable reception it has met with. Having myself used it for the last five years in teaching Analytic Geometry to beginners, I have gained some experience as to the points where learners are likely to feel difficulties. I have accordingly rewritten a considerable part of the work, introducing in the early chapters copious numerical illustrations, such as I have been in the habit of using with my class. I have also endeavoured to separate, more carefully than in the former editions, between the elementary parts of the work and those intended for more advanced readers. The learner will find all essential parts of the theory of Analytic Geometry included in Chapters I., II., V., VI., X., XI., XII., omitting the articles marked with asterisks. Should he require examples for exercise, in addition to those contained in these chapters, he will find a sufficiently extensive collection of examples in Chapters III., VII., XIII. The remaining chapters treat of the algebraic and geometrical methods which have been introduced into use of late years, but of which no systematic account had been given in any elementary work at the time that the first edition of this Treatise was published. I have made several additions to these chapters in this edition. In the alterations which I have made throughout the book, I have profited by the works on Analytic Geometry which have appeared since the first edition was published, among which I may mention in particular Mr. Gaskin's, and Mr. Walton's "Examples on Analytic Geometry," and Mr. Puckle's "Treatise on Conic Sections."

TRINITY COLLEGE, DUBLIN,
July, 1855.

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ANALYTIC GEOMETRY.

CHAPTER I.

THE POINT.

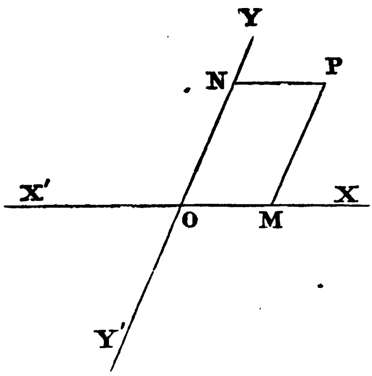
ART. 1. GEOMETRICAL theorems may be divided into two classes: theorems concerning the *magnitude* of lines, and concerning their *position*; for example, that "the square of the hypotenuse is equal to the sum of the squares of the sides," is a theorem concerning magnitude; that "the three perpendiculars of a triangle meet in a point," is a theorem concerning position.

Theorems of the former class can easily be expressed algebraically. To take the example already given, if the lengths of the sides of a right-angled triangle be a, b, c, the proposition alluded to is written $c^2 = a^2 + b^2$. The learner is probably already familiar with this application of algebra to geometry, as the propositions of the Second Book of Euclid all relate merely to the magnitude of lines, and the demonstration of them is much simplified by the use of algebraical symbols. But it is by no means so easy to see how to express algebraically theorems involving the *position* of lines. Accordingly, although algebra was, soon after its introduction into Europe, applied to the solution of the first class of questions, its use was not extended to this latter class until the year 1637, when Des Cartes, by the publication of his "*Géométrie*," laid the foundation of the science on which we are about to enter.

2. The following method of determining the position of any

point on a plane is that introduced by Des Cartes, and generally used by succeeding geometers.

We are supposed to be given the position of two fixed right lines, XX′, YY′, intersecting in the point O. Now, if through any point P we draw PM, PN, parallel to YY′ and XX′, it is plain that, if we knew the position of the point P, we should know the lengths of the parallels PM, PN, or, *vice versâ*, that if we knew the lengths of PM, PN, we should know the position of the point O.



Suppose, for example, that we were given PN = a, PM = b, we need only measure OM = a and ON = b, and draw the parallels PM, PN, which will intersect in the point required.

It is usual to denote PM parallel to OY by the letter y, and PN parallel to OX by the letter x, and the point P is said to be determined by the two equations $x = a$, $y = b$.

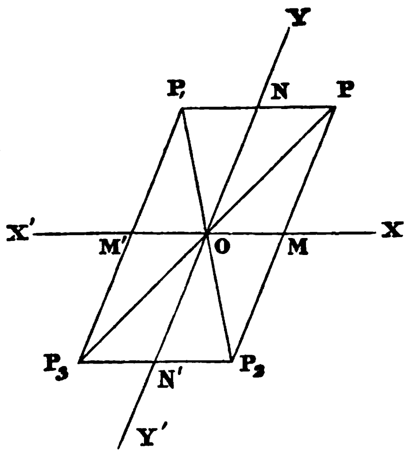
3. The parallels PM, PN, are called the *co-ordinates* of the point P; that parallel to YY′ is often called the *ordinate* of the point P; and that parallel to XX′ the *abscissa*.

The fixed lines XX′ and YY′ are termed the *axes of co-ordinates*, and the point O, in which they intersect, is called the *origin*. The axes are said to be rectangular or oblique, according as the angle at which they intersect is a right angle or oblique.

It will readily be seen that the co-ordinates of the point M on the preceding figure are $x = a$, $y = 0$; that those of the point N are $x = 0$, $y = b$; and that those of the origin itself are $x = 0$, $y = 0$.

4. In order that the equations $x = a$, $y = b$, should only be satisfied by *one* point, it is necessary to pay attention, not only to the *magnitudes*, but also to the *signs* of the co-ordinates.

If we paid no attention to the signs of the co-ordinates, we might measure OM = a and ON = b, on either side of the origin, and any of the four points, P, P_1, P_2, P_3, would satisfy the equations $x = a$, $y = b$. It is possible, however, to distinguish algebraically between the lines OM, OM′ (which are equal in magnitude, but opposite in direction) by giving them different signs. We lay down a rule, that if lines measured in one direction be considered as positive, lines measured in the opposite direction must be considered as negative. It is, of course, arbitrary in which direction we measure positive lines, but it is customary to consider OM (measured to the *right* hand) and ON (measured *upwards*) as positive, and OM′, ON′ (measured in the opposite directions) as negative lines.



Introducing these conventions, the four points, P, P_1, P_2, P_3, are easily distinguished. Their co-ordinates are, respectively,

$$\left.\begin{matrix} x = +a \\ y = +b \end{matrix}\right\}, \quad \left.\begin{matrix} x = -a \\ y = +b \end{matrix}\right\}, \quad \left.\begin{matrix} x = +a \\ y = -b \end{matrix}\right\}, \quad \left.\begin{matrix} x = -a \\ y = -b \end{matrix}\right\}.$$

These distinctions of sign can present no difficulty to the learner, who is supposed to be already familiar with the principles of trigonometry.

It appears from what has been said, that the points

$$x = +a,\ y = +b, \text{ and } x = -a,\ y = -b,$$

lie on a right line passing through the origin; that they are equidistant from the origin, and on opposite sides of it.

N. B.—The points whose co-ordinates are $x = a$, $y = b$, or $x = x'$, $y = y'$, are generally briefly designated as the point ab, the point $x'y'$.

5. *Given the co-ordinates of two points $x'y'$, $x''y''$, to express the distance between them, the axes of co-ordinates being supposed rectangular.*

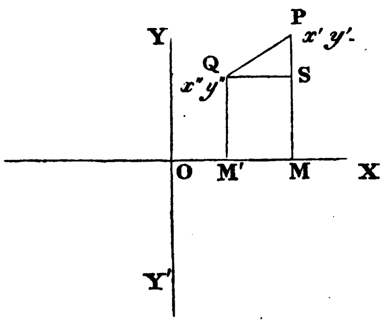
By Euclid, I. 47,

$$PQ^2 = PS^2 + SQ^2, \text{ but } PS = PM - QM' = y' - y'',$$
$$\text{and } QS = OM - OM' = x' - x'';$$

hence $$\delta^2 = PQ^2 = (x' - x'')^2 + (y' - y'')^2.$$

To express the distance of any point from the origin, we must make $x'' = 0$, $y'' = 0$, in the above, and we find

$$\delta^2 = x'^2 + y'^2.$$



6. In the following pages we shall but seldom have occasion to make use of oblique co-ordinates, since formulæ are, in general, much simplified by the use of rectangular axes; as, however, oblique co-ordinates may sometimes be employed with advantage, we shall give the principal formulæ in their most general form.

Suppose, in the last figure, the angle YOX oblique and $= \omega$,

then $$PSQ = 180° - \omega,$$

and $$PQ^2 = PS^2 + QS^2 - 2PS.QS.\cos PSQ,$$

or, $$PQ^2 = (y' - y'')^2 + (x' - x'')^2 + 2(y' - y'')(x' - x'')\cos\omega.$$

Similarly, the square of the distance of a point, $x'y'$, from the origin $= x'^2 + y'^2 + 2x'y' \cos\omega$.

In applying these formulæ, attention must be paid to the signs of the co-ordinates. If the point Q, for example, were in the angle XOY′, the sign of y'' would be changed, and the line PS would be the *sum* and not the *difference* of y' and y''.

Ex. 1. To find the lengths of the sides of a triangle the co-ordinates of whose vertices are $x' = 2$, $y' = 3$; $x'' = 4$, $y'' = -5$; $x''' = -3$, $y''' = -6$, the axes being rectangular.
Ans. $\sqrt{68}$, $\sqrt{50}$, $\sqrt{106}$.

Ex. 2. Find the lengths of the sides of a triangle the co-ordinates of whose vertices are the same as in the last example, the axes being inclined at an angle of 60°.
Ans. $\sqrt{52}$, $\sqrt{57}$, $\sqrt{151}$.

7. *Given the co-ordinates of two points, $x'y'$, $x''y''$, to find the co-ordinates of the point cutting the line joining them, in a given ratio $m:n$.*

Let x, y be the co-ordinates of the point R which we seek to determine, then

$m : n :: \text{PR} : \text{RQ} :: \text{MS} : \text{SN}$,

or

$m : n :: x' - x : x - x''$,

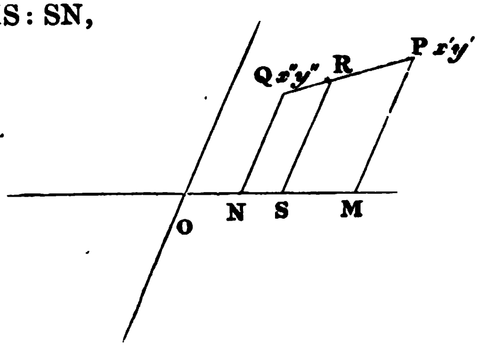
or $mx - mx'' = nx' - nx$,

hence

$$x = \frac{mx'' + nx'}{m + n}.$$

In like manner

$$y = \frac{my'' + ny'}{m + n}.$$



If the line were to be cut *externally* in the given ratio, we should have

$$m : n :: x - x' : x - x'',$$

and therefore

$$x = \frac{mx'' - nx'}{m - n}, \quad y = \frac{my'' - ny'}{m - n}.$$

We can sufficiently distinguish the cases of internal and external section, if we agree that to cut a line in the ratio $m : + n$ shall denote to cut it *internally* in a certain ratio; and that to cut in the ratio $m : - n$ shall denote to cut it *externally* in the same ratio: for the formulæ for external section are obtained from those for internal section by changing the sign of either m or n.

Ex. 1. To find the co-ordinates of the middle point of the line joining the points $x'y'$, $x''y''$.

Ans. $x = \frac{x' + x''}{2}$, $y = \frac{y' + y''}{2}$.

Ex. 2. To find the co-ordinates of the middle points of the sides of the triangle the co-ordinates of whose vertices are (2, 3), (4, – 5), (– 3, – 6).

Ans. $(3, -1)$, $\left(\frac{1}{2}, -\frac{11}{2}\right)$, $\left(-\frac{1}{2}, -\frac{3}{2}\right)$.

Ex. 3. The line joining the points (2, 3), (4, – 5) is trisected: to find the co-ordinates of the point of trisection nearest the former point.

Ans. $x = \frac{8}{3}$, $y = \frac{1}{3}$.

Ex. 4. The co-ordinates of the vertices of a triangle being $x'y'$, $x''y''$, $x'''y'''$, to find the co-ordinates of the point of trisection (remote from the vertex) of the line joining any vertex to the middle point of the opposite side.

Ans. $x = \frac{x' + x'' + x'''}{3}$, $y = \frac{y' + y'' + y'''}{3}$.

Ex. 5. To find the co-ordinates of the intersection of the bisectors of sides of the triangle, the co-ordinates of whose vertices are given in Ex. 2.

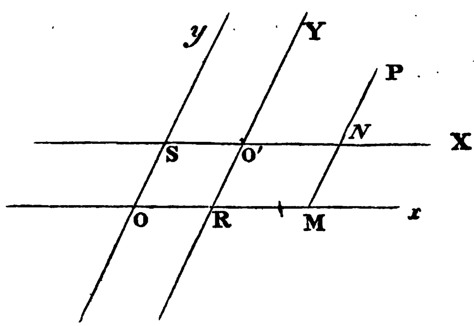
Ans. $x = 1,\ y = -\frac{8}{3}$.

Ex. 6. Any side of a triangle is cut in the ratio $m : n$, and the line joining this to the opposite vertex is cut in the ratio $m + n : l$; to find the co-ordinates of the point of section.

Ans. $x = \frac{lx' + mx'' + nx'''}{l + m + n},\ y = \frac{ly' + my'' + ny'''}{l + m + n}$.

8. *Transformation of Co-ordinates.*—When we know the co-ordinates of a point referred to one pair of axes, it is frequently necessary to find its co-ordinates referred to another pair of axes. This operation is called *the transformation of co-ordinates.*

We shall consider three cases separately: first, we shall suppose the origin changed, but the new axes parallel to the old; secondly, we shall suppose the direction of the axes changed, but the origin to remain unaltered; and thirdly, we shall examine the case when both origin and direction of the axes are altered.



First. Let the new axes be parallel to the old.

Let Ox, Oy, be the old axes, O′X, O′Y, the new axes. Let the co-ordinates of the new origin referred to the old be x', y', or O′S = x', O′R = y'. Let the old co-ordinates be x, y, the new X, Y, then we have

$$OM = OR + RM, \text{ and } PM = PN + NM,$$

that is,

$$x = x' + X, \text{ and } y = y' + Y.$$

These formulæ are, evidently, equally true, whether the axes be oblique or rectangular.

9. Next, let the direction of the axes be changed, while the origin is unaltered.

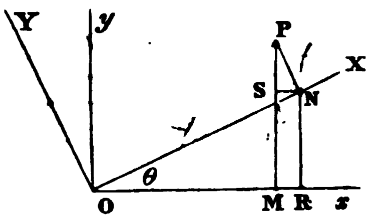
(1.) We shall commence with the case where both systems are rectangular, and we shall denote by θ the angle xOX = yOY.

Then $PM = PS + NR$; $OM = OR - SN$.

But since the angle

$$SPN = xOX = \theta,$$

$PS = PN\cos\theta$, $NR = ON\sin\theta$;
$OR = ON\cos\theta$, $SN = PN\sin\theta$.



We have, therefore,

$$y = Y\cos\theta + X\sin\theta,$$
$$x = X\cos\theta - Y\sin\theta.$$

(2.) In general let the angles between the axes be any whatever. In the figure then PS, PN are drawn parallel to Oy, OY, and NS to Ox. Then, as before,

$$PM = PS + NR.$$

We have no longer $PS = PN\cos SPN$, since PSN is not supposed a right angle; but

$$PS : PN :: \sin PNS\ (= \sin YOx) : \sin PSN\ (= \sin yOx);$$

$$\therefore PS = \frac{PN\sin YOx}{\sin yOx};$$

and

$$NR : ON :: \sin xOX : \sin NRO\ (= \sin yOx),$$

$$\therefore NR = \frac{ON\sin xOX}{\sin yOx}.$$

Hence

$$y\sin xOy = Y\sin xOY + X\sin xOX.$$

From symmetry we can write down

$$x\sin yOx = X\sin yOX + Y\sin yOY.$$

In using these formulæ, however, attention must be paid to the signs of the angles concerned in them.

The sign + is to be used when the angles xOy, xOY, xOX, are all measured on the same side of Ox; and yOx, yOX, yOY, on the same side of Oy.

In the case represented in the figure, the angle yOY lies on the opposite side of Oy from the angles yOx and yOX, and the formula would become

$$x\sin yOx = X\sin yOX - Y\sin yOY.$$

It will often be convenient to write these formulæ as follows: let the angle between the old axes $yOx = \omega$: let the angle that the new axis of X makes with the old, $XOx = a$; let $YOx = \beta$: then the formulæ become

$$y \sin\omega = X \sin a + Y \sin\beta$$
$$x \sin\omega = X \sin(\omega - a) + Y \sin(\omega - \beta).$$

10. Lastly, by combining the transformations of the two preceding articles, we can find the co-ordinates of a point referred to two new axes in any position whatever. We first find the coordinates (by Art. 8) referred to a pair of axes through the new origin parallel to the old axes, and then (by Art. 9) we can find the co-ordinates referred to the required axes.

The general expressions are obviously obtained by adding x' and y' to the values for x and y given in the last article.

Ex. 1. The co-ordinates of a point satisfy the relation $x^2 + y^2 - 4x - 6y = 18$; what will this become if the origin be transformed to the point (2, 3)?

Ans. $X^2 + Y^2 = 31$.

Ex. 2. The co-ordinates of a point to one set of rectangular axes satisfy the relation $y^2 - x^2 = 6$: what will this become if transformed to axes bisecting the angles between the given axes?

Ans. $XY = 3$.

Ex. 3. Transform the equation $2x^2 - 5xy + 2y^2 = 4$ from axes inclined to each other at an angle of 60°, to the right lines which bisect the angles between the given axes.

Ans. $X^2 - 27Y^2 + 12 = 0$.

Ex. 4. Transform the same equation to rectangular axes, retaining the old axis of x.

Ans. $3X^2 + 10Y^2 - 7XY\sqrt{3} = 6$.

Ex. 5. It is evident that when we change from one set of rectangular axes to another, $x^2 + y^2$ must $= X^2 + Y^2$, since both express the square of the distance of a point from the origin. Verify this by squaring and adding the expressions for X and Y in Art. 9.

Ex. 6. Verify in like manner in general that

$$x^2 + y^2 + 2xy \cos xOy = X^2 + Y^2 + 2XY \cos XOY.$$

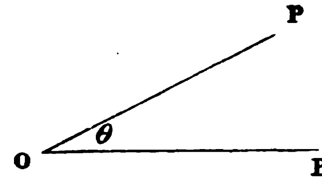
11. *The degree of any equation between the co-ordinates is not altered by transformation of co-ordinates.*

Transformation cannot *increase* the degree of the equation: for if the highest terms in the given equation be x^m, y^m, &c., those in the transformed equation will be

$$\{x' \sin\omega + x \sin(\omega - a) + y \sin(\omega - \beta)\}^m, \ (y' \sin\omega + x \sin a + y \sin\beta)^m,$$

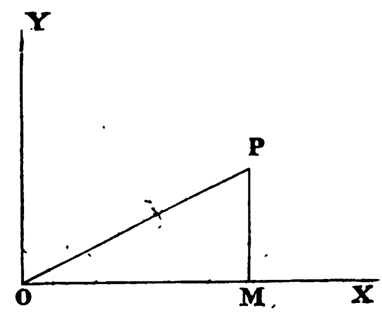
&c., which evidently cannot contain powers of x or y above the m^{th} degree. Neither can transformation *diminish* the degree of an equation, since by transforming the transformed equation back again to the old axes, we must fall back on the original equation, and if the first transformation had diminished the degree of the equation, the second should increase it, contrary to what has been just proved.

12. *Polar Co-ordinates.*—Beside the method of expressing the position of a point which we have hitherto made use of, there is also another which is often employed.



If we were given a fixed point O, and a fixed line through it, OB, it is evident that we should know the position of any point, P, if we knew the length OP, and also the angle POB. The line OP is called the *radius vector*; the fixed point is called the *pole*; and this method is called the method of *polar co-ordinates*.

It is very easy, being given the x and y co-ordinates of a point, to find its polar ones, or *vice versâ*.



First, let the fixed line coincide with the axis of x, then we have

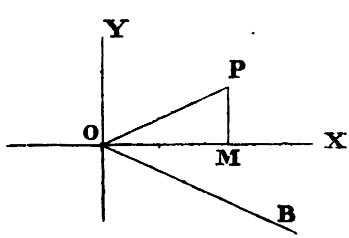
OP : PM :: sin PMO : sin POM;

denoting OP by ρ, POM by θ, and YOX by ω; then

$$\text{PM or } y = \frac{\rho \sin\theta}{\sin\omega}; \text{ and similarly, OM} = x = \frac{\rho\sin(\omega-\theta)}{\sin\omega}.$$

For the more ordinary case of rectangular co-ordinates, $\omega = 90°$, and we have simply

$$x = \rho\cos\theta \text{ and } y = \rho\sin\theta.$$



Secondly. Let the fixed line OB not coincide with the axis of x, but make with it an angle $= a$, then

$$\text{POB} = \theta \text{ and POM} = \theta - a,$$

and we have only to substitute $\theta - a$ for θ in the preceding formulæ.

For rectangular co-ordinates we have

$$x = \rho\cos(\theta - \alpha) \text{ and } y = \rho\sin(\theta - \alpha).$$

Ex. 1. Change to polar co-ordinates, the following equations in rectangular co-ordinates.

$x^2 + y^2 = 5mx$. *Ans.* $\rho = 5m\cos\theta$.

$x^2 - y^2 = a^2$. *Ans.* $\rho^2\cos 2\theta = a^2$.

Ex. 2. Change to rectangular co-ordinates the following equations in polar co-ordinates.

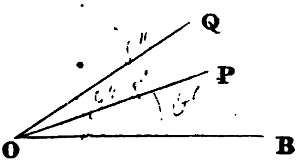
$\rho^2\sin 2\theta = 2a^2$. *Ans.* $xy = a^2$.

$\rho^2 = a^2\cos 2\theta$. *Ans.* $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

$\rho^{\frac{1}{2}}\cos\frac{1}{2}\theta = a^{\frac{1}{2}}$. *Ans.* $x^2 + y^2 = (2a - x)^2$.

$\rho^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{1}{2}\theta$. *Ans.* $(2x^2 + 2y^2 - ax)^2 = a^2(x^2 + y^2)$.

13. *To express the distance between two points, in terms of their polar co-ordinates.*



Let P and Q be the two points,

$$OP = \rho', \quad POB = \theta';$$
$$OQ = \rho'', \quad QOB = \theta'';$$

then

$$PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ \cdot \cos POQ,$$

or

$$\delta^2 = \rho'^2 + \rho''^2 - 2\rho'\rho''\cos(\theta'' - \theta').$$

CHAPTER II.

THE RIGHT LINE.

14. We saw, in the last chapter, that we could determine the position of a point, being given two equations regarding its co-ordinates, of the form $x = a$, $y = b$. It is evident that we could equally determine the point, had we been given any two equations of the first degree between its co-ordinates, such as

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

for we have here two equations between two unknown quantities, which we can solve by eliminating y and x alternately between them, and obtain two results of the form

$$x = a, \quad y = b.$$

Ex. What point is denoted by the equations $3x + 5y = 13$, $4x - y = 2$?
Ans. $x = 1$, $y = 2$.

15. Two equations of higher order between the co-ordinates would represent, not *one*, but a determinate number of points. For, eliminating y between the equations, we obtain an equation containing x only; let its roots be a_1, a_2, a_3, &c. Now, if we substitute any of these values (a_1) for x in the original equations, we get two equations in y, which must have a common root (since the result of elimination between the equations is rendered $= 0$ by the supposition $x = a_1$). Let this common root be $y = \beta_1$. Then the point whose co-ordinates are $x = a_1$, $y = \beta_1$, will at once satisfy both the given equations; and so, in like manner, will the point whose co-ordinates are $x = a_2$, $y = \beta_2$, &c.

If the given equations were of the m^{th} and n^{th} degrees respectively, the equation in x would (by the theory of elimination, see *Lacroix's Algebra*, § 196, p. 278; *Young's Algebra*, § 124, p. 229) be of the mn^{th} degree, and consequently there would be mn roots a_1, a_2, &c., and, therefore, mn points represented by the two equations.

Ex. 1. What points are represented by the two equations $x^2 + y^2 = 5$, $xy = 2$? Eliminating y between the equations, we get $x^4 - 5x^2 + 4 = 0$. The roots of this equation are $x^2 = 1$ and $x^2 = 4$, and, therefore, the four values of x are

$$x = +1, \quad x = -1, \quad x = +2, \quad x = -2.$$

Substituting any of these in the second equation, we obtain the corresponding values of y,

$$y = +2, \quad y = -2, \quad y = +1, \quad y = -1.$$

The two given equations, therefore, represent the four points

$$(+1, +2), \; (-1, -2), \; (+2, +1), \; (-2, -1).$$

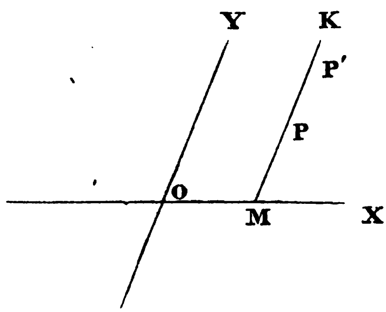
Ex. 2. What points are denoted by the equations
$x - y = 1$, $x^2 + y^2 = 25$? *Ans.* (4, 3), (– 3, – 4).

Ex. 3. What points are denoted by the equations
$x^2 - 5x + y + 3 = 0$, $x^2 + y^2 - 5x - 3y + 6 = 0$? *Ans.* (1, 1), (2, 3), (3, 3), (4, 1).

16. Having seen that any *two* equations between the co-ordinates represent geometrically one or more *points*, we proceed to inquire the geometrical signification of *a single* equation between the co-ordinates. We shall find the case to be similar to the solution of a class of geometrical problems, with which the learner is familiar. We can determine a triangle, being given the base and any other two conditions, but had we been given only one other condition, the vertex, though no longer determined in position, would still be limited to a certain *locus*. So we shall find,

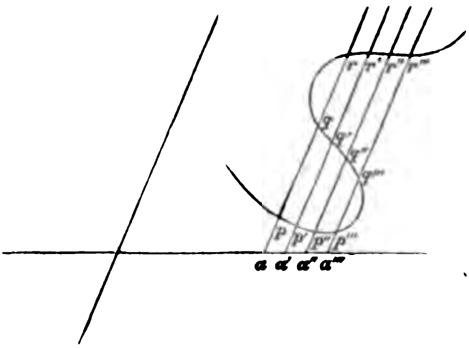
that although one equation between the two co-ordinates is not sufficient to *determine* a point, it is, however, sufficient to *limit it to a certain locus.* In fact, the equation asserts, that a certain relation subsists between the co-ordinates of every point represented by it. Now, although this relation will not in general subsist between the co-ordinates of any point *taken at random*, yet there will be more points than one for which this relation *will* be true; the assemblage of these points will form a locus of points whose co-ordinates satisfy the equation, and this locus is considered the geometrical signification of the given equation.

That a single equation between the co-ordinates signifies a locus, we shall first illustrate by the simplest example. Let us recall the construction by which (p. 2) we determined the position of a point from the two equations $x = a$, $y = b$. We took OM $= a$; we drew MK parallel to OY; and then, measuring MP$=b$, we found P, the point required. Had we been given a different value of y, $x = a$, $y = b'$, we should proceed as before, and we should find a point P′ still situated on the line MK, but at a different distance from M. Lastly, if the value of y were left wholly indeterminate, and we were merely given the single equation $x = a$, we should know that the point P was situated *somewhere* on the line MK, but its position in that line would not be determined. Hence the line MK is the locus of all the points represented by the equation $x = a$, since, whatever point we take on the line MK, the x of that point will always $= a$.



17. In general, if we were given an equation of any degree between the co-ordinates, let us assume for x any value we please ($x = a$), and the equation will enable us to determine a finite number of values of y answering to this particular value of x, and, consequently, the equation will be satisfied for each of the points (p, q, r, &c.), whose x is the assumed value, and whose y is that found from the equation. Again, assume for x any other value

($x = a'$), and we find, in like manner, another series of points, p', q', r', whose co-ordinates satisfy the equation. So again, if we assume $x = a''$ or $x = a'''$, &c. Now, if x take successively all possible values, the assemblage of points found as above will form a *locus*, every point of which satisfies the conditions of the equation, and which is, therefore, its geometrical signification. We see then that every equation we can write down between the co-ordinates x and y must represent geometrically a locus of some kind. It is on this consideration that the whole science of Analytic Geometry is founded.



18. It is the business of Analytic Geometry to investigate the nature of the different loci represented by different equations. Then, having once ascertained the locus represented by a given equation (for example, $Ax + By + C = 0$), if we find this relation subsisting between the co-ordinates of any point, we shall be sure that this point lies on the locus so determined, and, *vice versâ*, if we take any point on the locus, we shall know that this relation will exist between its co-ordinates.

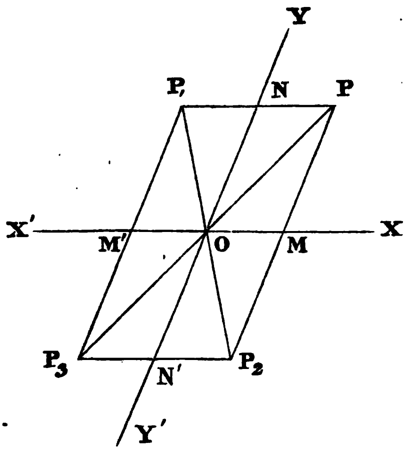
These loci are classified according to the degrees of the equations representing them, being said to be of the m^{th}, n^{th}, or p^{th}, &c., degree, according as the equations representing them are of the m^{th}, n^{th}, or p^{th} degree between x and y.

We commence with the equation of the first degree, and we shall find that this always represents a *right line*, and, conversely, that the equation of a right line is always of the first degree.

19. We have already (Art. 16) examined the simplest case of an equation of the first degree, namely, the equation $x = a$, and we found that an equation of this form represents a right line PM parallel to the axis OY, and meeting the axis OX at a distance from the origin OM = to a. Similarly, the equation $y = b$ repre-

sents a line PN parallel to the axis OX, and meeting the axis OY at a distance from the origin ON = b.

Let us now proceed to examine the case next in order of simplicity, that of a right line passing through the origin, and let us consider what relation subsists between the co-ordinates of points situated on such a line.



If we take any point P on such a line, we see that *both* the co-ordinates PM, OM, will vary in length, but that the *ratio* PM : OM will be constant, being = to the ratio

sin POM : sin MPO.

Hence we see, that the equation

$$y = \frac{\sin POM}{\sin MPO} x,$$

will be satisfied for every point of the line OP, and, therefore, this equation is said to be the equation of the line OP.

Conversely, if we were asked what locus was represented by the equation

$$y = mx,$$

write the equation in the form $\frac{y}{x} = m$, and the question is, "to find the locus of a point P, such that, if we draw PM, PN parallel to two fixed lines, the ratio PM : PN may be constant." Now this locus evidently is a right line OP, passing through O, the point of intersection of the two fixed lines, and dividing the angle between them in such a manner that

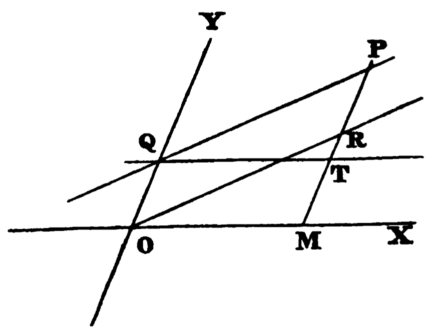
$$\sin POM = m \sin PON.$$

If the axes be rectangular, sin PON = cos POM, therefore, m = tan POM, and the equation $y = mx$ represents a right line passing through the origin, and making an angle with the axis of x, whose tangent is m.

20. An equation of the form $y = + mx$ will denote a line OP,

situated in the angles YOX, Y'OX'. On the contrary, an equation of the form $y = - mx$ will denote a line OP', situated in the angles Y'OX, YOX'.

For it appears, from the equation $y = + mx$, that whenever x is positive y will be positive, and whenever x is negative y will be negative. Points, therefore, represented by this equation, must have their co-ordinates either both positive or both negative, and such points we saw (Art. 4) lie only in the angles YOX, Y'OX'. On the contrary, in order to satisfy the equation $y = - mx$, if x be positive y must be negative, and if x be negative y must be positive. Points, therefore, satisfying this equation, will have their co-ordinates of *different* signs, and must, therefore (Art. 4), lie in the angles Y'OX, YOX'.



21. Let us now examine how to represent a right line PQ, situated in any manner with regard to the axes.

Draw OR through the origin parallel to PQ, and let the ordinate PM meet OR in R. Now it is plain (as in Art. 19), that the ratio RM : OM will be always constant (RM always equal, suppose, to m . OM); but the ordinate PM differs from RM by the constant length PR = OQ, which we shall call b. Hence we may write down the equation

$$PM = RM + PR, \text{ or } PM = m \cdot OM + PR,$$

that is,

$$y = mx + b.$$

The equation, therefore, $y = mx + b$, being satisfied by every point of the line PQ, is said to be the equation of that line.

It appears from the last Article, that m will be positive or negative according as OR, parallel to the right line PQ, lies in the angle YOX, or Y'OX. And, again, b will be positive or negative according as the point Q, in which the line meets OY, lies *above* or *below* the origin.

Conversely, the equation $y = mx + b$ will always denote a right line; for the equation can be put into the form

$$\frac{y - b}{x} = m.$$

Now, since if we draw the line QT parallel to OM, TM will be $= b$, and PT therefore $= y - b$, the question becomes: "To find the locus of a point, such that, if we draw PT parallel to OY to meet the fixed line QT, PT may be to QT in a constant ratio;" and this locus evidently is the right line PQ passing through Q.

The most general equation of the first degree, $Ax + By + C = 0$, can obviously be reduced to the form $y = mx + b$, since it is equivalent to

$$y = -\frac{A}{B}x - \frac{C}{B};$$

this equation therefore *always* represents a right line.

22. From the last Articles we are able to ascertain the geometrical meaning of the constants in the equation of a right line. If the right line represented by the equation $y = mx + b$ make an angle $= \alpha$ with the axis of x, and $= \beta$ with the axis of y, then (Art. 19)

$$m = \frac{\sin\alpha}{\sin\beta};$$

and if the axes be rectangular, $m = \tan\alpha$.

We saw (Art. 21) that b is the intercept which the line cuts off on the axis of y.

If the equation be given in the general form $Ax + By + C = 0$, we can reduce it, as in the last Article, to the form $y = mx + b$, and we find that

$$-\frac{A}{B} = \frac{\sin\alpha}{\sin\beta},$$

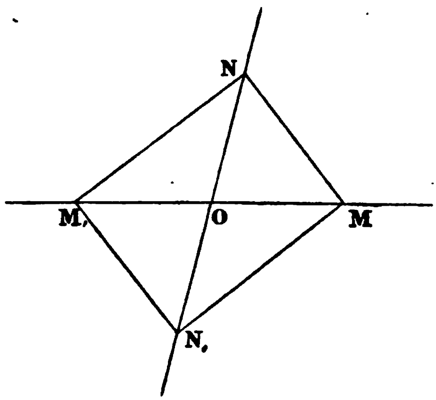
or if the axes be rectangular $= \tan\alpha$; and that $-\frac{C}{B}$ is the length of the intercept made by the line on the axis of y.

Cor.—The lines $y = mx + b$, $y = m'x + b'$ will be parallel to each other if $m = m'$, since then they will both make the same angle with the axis. Similarly the lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$, will be parallel if

$$\frac{A}{B} = \frac{A'}{B'}.$$

Beside the forms $Ax + By + C = 0$ and $y = mx + b$, there are two other forms in which the equation of a right line is frequently used; these we next proceed to lay before the reader.

23. *To find the lengths of the intercepts which the line* MN, *whose equation is* $Ax + By + C = 0$, *cuts off on the axes.*



We found in the last Article the length of one of these intercepts, by comparing the present equation with the equation $y = mx + b$. We prefer, however, in the present Article, to investigate the same question directly, by the help of an important principle already alluded to (Art. 18). The co-ordinates of *every* point of the line MN must of course satisfy the given equation, therefore so must the co-ordinates of the point M, where this line meets the axis of x. Now for every point on the axis of x, $y = 0$ (Art. 3), therefore, for the point M, the equation gives $Ax + C = 0$, but the x of the point M is the intercept OM, whose length is required; therefore,

$$OM = -\frac{C}{A}.$$

Similarly,

$$ON = -\frac{C}{B}.$$

Hence it is easy to find the equation of a line which shall cut off intercepts on the axes, $OM = a$ and $ON = b$.

The general equation of a right line is

$$Ax + By + C = 0, \text{ or } \frac{A}{C}x + \frac{B}{C}y + 1 = 0;$$

but

$$\frac{A}{C} = -\frac{1}{OM} = -\frac{1}{a}, \text{ and } \frac{B}{C} = -\frac{1}{ON} = -\frac{1}{b};$$

therefore, the equation of the right line required is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

This is *the equation of the right line in terms of the intercepts it cuts off on the axes.* It evidently holds whether the axes be oblique or rectangular.

It is plain that the position of the line will vary with the signs of the quantities a and b. For example, the given equation $\frac{x}{a}+\frac{y}{b}=1$, which cuts off positive intercepts on both axes, represents the line MN on the preceding figure;

$\frac{x}{a}-\frac{y}{b}=1$, cutting off a positive intercept on the axis of x, and a negative intercept on the axis of y, represents MN′.

Similarly, $\frac{y}{b}-\frac{x}{a}=1$ represents NM′;

and $\frac{x}{a}+\frac{y}{b}=-1$ represents M′N′.

The student will find no difficulty in examining for himself how changes in the signs of A, B, or C affect the position of the line represented by the general equation

$$Ax + By + C = 0.$$

Ex. 1. Examine the position of the following lines, and find the intercepts they make on the axes. $2x - 3y = 7$; $3x + 4y + 9 = 0$; $3x + 2y = 6$; $4y - 5x = 20$.

Ex. 2. The sides of a triangle being taken for axes, form the equation of the line joining the points which cut off the m^{th} part of each, and show, by Art. 22, that it is parallel to the base. *Ans.* $\frac{x}{ma}+\frac{y}{mb}=1$.

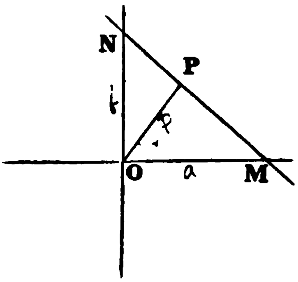
24. If we suppose $A = 0$ in the general equation, the intercept $-\frac{C}{A}$ made by the line on the axis of x becomes infinite. Hence the line $By + C = 0$ cuts the axis of x at an infinite distance, or, in other words, is parallel to it. This agrees with Art. 16.

The distance from the origin at which this parallel meets the axis of y (Art. 22), is $-\frac{C}{B}$. If, therefore, $C = 0$, this distance will vanish, and the equation $y = 0$ represents the axis of x itself.

Similarly, $Ax + C = 0$ denotes a line parallel to the axis of y, and $x = 0$ the axis of y itself.

25. *To express the equation of a right line in terms of the length of the perpendicular on it from the origin, and of the angles which this perpendicular makes with the axes.*

Let the length of the perpendicular OP = p, the angle POM which it makes with the axis of $x = a$, PON = β, OM = a, ON = b.



We saw (Art. 23) that the equation of the right line MN was

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Multiply this equation by p, and we have

$$\frac{p}{a}x + \frac{p}{b}y = p.$$

But $\frac{p}{a} = \cos a$; $\frac{p}{b} = \cos\beta$; therefore the equation of the line is

$$x \cos a + y \cos\beta = p.$$

In rectangular co-ordinates, which we shall most generally use, we have $\beta = 90° - a$. Hence, $x \cos a + y \sin a = p$ is the equation, referred to rectangular co-ordinates, of a line, the perpendicular on which from the origin makes an angle = a with the axis of x, and is in length = p.

If we had been given the equation of a right line in the general form $Ax + By + C = 0$, it is easy to reduce it to the form $x \cos a + y \sin a = p$; for, divide the first by $\sqrt{(A^2 + B^2)}$, and we have

$$\frac{A}{\sqrt{(A^2+B^2)}}x + \frac{B}{\sqrt{(A^2+B^2)}}y + \frac{C}{\sqrt{(A^2+B^2)}} = 0.$$

But we may take

$$\frac{A}{\sqrt{(A^2+B^2)}} = \cos a, \text{ and } \frac{B}{\sqrt{(A^2+B^2)}} = \sin a,$$

since the sum of squares of these two quantities = 1.

Hence we learn, that $\frac{A}{\sqrt{(A^2+B^2)}}$ and $\frac{B}{\sqrt{(A^2+B^2)}}$ are respectively the cosine and sine of the angle which the perpendicular from the origin on the line $(Ax + By + C = 0)$ makes with the axis of x, and that $\frac{C}{\sqrt{(A^2+B^2)}}$ is the length of this perpendicular.

The square root in these values is, of course, susceptible of a double sign; since the equation may be reduced to either of the forms

$$x\cos\alpha + y\cos\beta - p = 0, \quad x\cos(\alpha+180^\circ) + y\cos(\beta+180^\circ) + p = 0.$$

* 26. *To reduce the equation* $Ax + By + C = 0$ (*referred to oblique co-ordinates*), *to the form* $x\cos\alpha + y\cos\beta = p$.

Let us suppose that the given equation when multiplied by a certain factor R is reduced to the required form, then $RA = \cos\alpha$, $RB = \cos\beta$. But it can easily be proved that, if α and β be any two angles whose sum is ω, we shall have

$$\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\omega = \sin^2\omega.$$

Hence $$R^2(A^2 + B^2 - 2AB\cos\omega) = \sin^2\omega,$$

and the equation reduced to the required form is

$$\frac{A\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}}x + \frac{B\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}}y + \frac{C\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}} = 0.$$

And we learn that

$$\frac{A\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}}, \quad \frac{B\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}},$$

are respectively the cosines of the angles that the perpendicular from the origin on the line $Ax + By + C = 0$, makes with the axes of x and y; and that $\dfrac{C\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}}$ is the length of this perpendicular. This length may be more easily calculated by dividing the double area of the triangle NOM, ($ON\,.\,OM\sin\omega$) by the length of MN, expressions for which are easily found.

27. *To find the length of the perpendicular from any point* $x'y'$, *on the line whose equation is* $x\cos\alpha + y\cos\beta - p = 0$.

We shall show that it is found by substituting the co-ordinates x', y', for x and y in the given equation, and is equal to

$$\pm(x'\cos\alpha + y'\cos\beta - p).$$

For, from the given point Q draw QR parallel to the given line, and QS perpendicular. Then

$$OK = x', \text{ and } OT \text{ will be } = x'\cos\alpha.$$

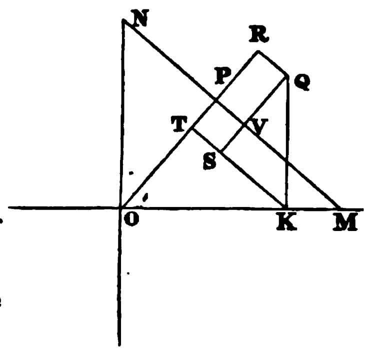
Again, since

$$SQK = \beta, \text{ and } QK = y',$$

$$RT = QS = y' \cos\beta;$$

hence

$$x' \cos\alpha + y' \cos\beta = OR.$$



Subtract OP the perpendicular from the origin, and

$x' \cos\alpha + y' \cos\beta - p = PR =$ the perpendicular QV. Q. E. D.

But if in the figure the point Q had been taken on the side of the line next the origin, we should have obtained for the perpendicular the expression $p - x' \cos\alpha - y' \cos\beta$; and we see that the perpendicular changes sign as we pass from one side of the line to the other. It is arbitrary on which side of the line we shall regard the perpendicular as positive. If we choose that form to represent the perpendicular in which the absolute term is positive, then it is easy to see that the perpendiculars which fall on the side of the line next the origin are to be regarded as positive, and those on the other side as negative; and *vice versâ* if we choose the other form.

If the equation of the line had been given in the form $Ax + By + C = 0$, we have only to reduce it to the form

$$x \cos\alpha + y \cos\beta - p = 0,$$

and the length of the perpendicular from any point $x'y'$,

$$= \frac{Ax' + By' + C}{\sqrt{(A^2 + B^2)}}, \text{ or } \frac{(Ax' + By' + C) \sin\omega}{\sqrt{(A^2 + B^2 - 2AB\cos\omega)}},$$

according as the axes are rectangular or oblique. By comparing the expression for the perpendicular from $x'y'$ with that for the perpendicular from the origin, we see that $x'y'$ lies on the same side of the line as the origin when $Ax' + By' + C$ has the same sign as C, and *vice versâ*.

The condition that any point $x'y'$ should be on the right line $Ax + By + C = 0$, is, of course, that the co-ordinates $x'y'$ should satisfy the given equation, or

$$Ax' + By' + C = 0.$$

And the present Article shows that this condition is merely the algebraical statement of the fact, that the perpendicular from the point $x'y'$ on the given line is $= 0$.

Ex. 1. Find the length of the perpendicular from the origin on the line

$$3x + 4y + 20 = 0,$$

axes being rectangular. *Ans.* 4.

Ex. 2. Find the length of the perpendicular from the point (2, 3) on $2x + y - 4 = 0$.

Ans. $\frac{3}{\sqrt{5}}$: and the given point is on the side remote from the origin.

Ex. 3. Find the length of the perpendicular from $(3, -4)$ on $4x + 2y - 7$, the angle between the axes being 60°. *Ans.* $\frac{3}{4}$: and the point is on the side next the origin.

Ex. 4. Find the length of the perpendicular from the origin on

$$a(x-a) + b(y-b) = 0.$$ *Ans.* $\sqrt{(a^2+b^2)}$.

28. *To find the equation of a right line passing through a given point $x'y'$.*

The general equation of a right line, we have seen, can be put under the form $y = mx + b$, where m and b are as yet unknown, and are to be determined by any conditions we are given respecting the line. Now suppose a point on the line given, the equation $y = mx + b$, which is true for *every* point on the line, must be true for the point $x'y'$. Hence we get the condition $y' = mx' + b$. As we are given no other condition, we are not able to determine *both* the unknown quantities m and b, but by means of this condition we can determine *one* of them, $b = y' - mx'$. Substituting this value in the general equation, we get

$$y = mx + y' - mx',$$

or

$$y - y' = m(x - x'),$$

for the equation of a right line passing through the point $x'y'$. m remains indeterminate, as it ought, since an infinite number of lines can be drawn through the point $x'y'$.

29. *To find the equation of a right line passing through two given points, $x'y'$, $x''y''$.*

The condition that the right line must pass through a second point will now enable us to determine the constant m which was left indeterminate in the last Article.

By the last Article the equation of a right line through $x'y'$ is

$$y - y' = m(x - x'),$$

or

$$\frac{y - y'}{x - x'} = m.$$

But since the line must also pass through the point $x''y''$, this equation must be satisfied when the co-ordinates x'', y'', are substituted for x and y; hence

$$\frac{y'' - y'}{x'' - x'} = m.$$

Substituting this value of m, the equation of the line becomes

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'}.$$

In this form the equation can be easily remembered, but, clearing it of fractions, we obtain it in a form which is sometimes more convenient,

$$(y' - y'')x - (x' - x'')y + x'y'' - y'x'' = 0.$$

Cor.—The equation of the line joining the point $x'y'$ *to the origin* is $y'x = x'y$.

It will sometimes happen that we can write down, without calculation, the equation of the line joining two points. If we happen to know beforehand that the co-ordinates of both points are connected by the relations

$$Ax' + By' + C = 0 \text{ and } Ax'' + By'' + C = 0,$$

then it is evident that the equation of the line joining them is $Ax + By + C = 0$, for it is the equation of a right line, and is satisfied by the co-ordinates of both points.

Ex. 1. Form the equations of the sides of a triangle, the co-ordinates of whose vertices are (2, 1), (3, – 2), (– 4, – 1). *Ans.* $3x + y = 7$, $x + 7y + 11 = 0$, $3y - x = 1$.

Ex. 2. Find the lengths of the perpendiculars from each vertex of this triangle on the opposite side. *Ans.* $2\sqrt{2}$, $\sqrt{10}$, $2\sqrt{10}$, and the origin is within the triangle.

Ex. 3. Form the equations of the sides of the triangle formed by (2, 3), (4, – 5), (– 3, – 6). *Ans.* $4x + y = 11$, $x - 7y = 39$, $9x - 5y = 3$.

Ex. 4. Form the equation of the line joining the points

$$x'y' \text{ and } \frac{mx' + nx''}{m + n}, \quad \frac{my' + ny''}{m + n}.$$

Ans. $(y' - y'')x - (x' - x'')y + x'y'' - y'x'' = 0.$

Ex. 5. Form the equation of the line joining

$$x'y' \text{ and } \frac{x'' + x'''}{2}, \quad \frac{y'' + y'''}{2}.$$

Ans. $(y'' + y''' - 2y')x - (x'' + x''' - 2x')y + x''y' - y''x' + x'''y' - y'''x' = 0.$

Ex. 6. Form the equations of the bisectors of the sides of the triangle described in Ex. 3. *Ans.* $5x - 6y = 21$; $17x - 3y = 25$; $7x + 9y + 17 = 0$.

Ex. 7. Form the equation of the line joining

$$\frac{lx' - mx''}{l - m}, \frac{ly' - my''}{l - m} \text{ to } \frac{lx' - nx'''}{l - n}, \frac{ly' - ny'''}{l - n}.$$

Ans. $$x\{l(m-n)y' + m(n-l)y'' + n(l-m)y'''\} - y\{l(m-n)x' + m(n-l)x'' + n(l-m)x'''\}$$
$$= lm(y'x'' - x'y'') + mn(y''x''' - x''y''') + nl(y'''x' - y'x''').$$

30. *To find the condition that three points shall lie on one right line.*

We found (in Art. 29) the equation of the line joining two of them, and we have only to see if the co-ordinates of the third will satisfy this equation.

The condition, therefore, is

$$(y_1 - y_2)x_3 - (x_1 - x_2)y_3 + (x_1y_2 - x_2y_1) = 0,$$

which can be put into the more symmetrical form,

$$y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2) = 0.*$$

31. *To find the area of the triangle formed by three points.*

If we multiply the length of the line joining two of the points, by the perpendicular on that line from the third point, we shall have double the area. Now the length of the perpendicular from x_3y_3 on the line joining x_1y_1, x_2y_2, the axes being rectangular, is (Art. 27)

$$\frac{(y_1 - y_2)x_3 - (x_1 - x_2)y_3 + x_1y_2 - x_2y_1}{\sqrt{\{(y_1 - y_2)^2 + (x_1 - x_2)^2\}}},$$

and the denominator of this fraction is the length of the line joining x_1y_1, x_2y_2, hence

$$y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2)$$

represents double the area formed by the three points.

If the axes be oblique, it will be found on repeating the investigation with the formulæ for oblique axes, that the only change that will occur is that the expression just given is to be multiplied by $\sin\omega$.

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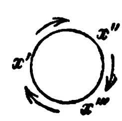
* In using this and other similar formulæ, which we shall afterwards have occasion to employ, the learner must be careful to take the co-ordinates in a fixed order (see engraving). For instance, in the second member of the formula just given y_2 takes the place of y_1, x_3 of x_2, and x_1 of x_3. Then, in the third member, we advance from y_2 to y_3, from x_3 to x_1, and from x_1 to x_2, always proceeding in the order just indicated.



COR. 1.—Double the area of the triangle formed by the lines joining the points x_1y_1, x_2y_2 to the origin, is $y_1x_2 - y_2x_1$, as appears by making $x_3 = 0$, $y_3 = 0$, in the preceding formula.

COR. 2.—The condition that three points should be on one right line, when interpreted geometrically, asserts that the area of the triangle formed by the three points becomes $= 0$.

32. *To express the area of a polygon in terms of the co-ordinates of its angular points.*

Take any point xy within the polygon, and connect it with all the vertices x_1y_1, x_2y_2, ... x_ny_n; then evidently the area of the polygon is the sum of the areas of all the triangles into which the figure is thus divided. But by the last Article double these areas are respectively

$$x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1,$$
$$x(y_2 - y_3) - y(x_2 - x_3) + x_2y_3 - x_3y_2,$$
$$x(y_3 - y_4) - y(x_3 - x_4) + x_3y_4 - x_4y_3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x(y_{n-1} - y_n) - y(x_{n-1} - x_n) + x_{n-1}y_n - x_ny_{n-1},$$
$$x(y_n - y_1) - y(x_n - x_1) + x_ny_1 - x_1y_n.$$

When we add these together, the parts which multiply x and y vanish, as they evidently ought to do, since the value of the total area must be independent of the manner in which we divide it into triangles; and we have for double the area

$$(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + \cdots (x_ny_1 - x_1y_n).$$

This may be otherwise written,

$$x_1(y_2 - y_n) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + \cdots x_n(y_1 - y_{n-1}),$$

or else

$$y_1(x_n - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_4) + \cdots y_n(x_{n-1} - x_1).$$

Ex. 1. Find the area of the triangle (2, 1), (3, – 2), (– 4, – 1). *Ans.* 10.

Ex. 2. Find the area of the triangle (2, 3), (4, – 5), (– 3, – 6). *Ans.* 29.

Ex. 3. Find the area of the quadrilateral (1, 1), (2, 3), (3, 3), (4, 1). *Ans.* 4.

33. *To find the co-ordinates of the point of intersection of two right lines whose equations are given.*

Each equation expresses a relation which must be satisfied by the co-ordinates of the point required; we find its co-ordinates, therefore, by solving for the two unknown quantities x and y, from the two given equations. Let the equations be given in the most general form,

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

then x will be found $= \dfrac{BC' - B'C}{AB' - BA'}$, and $y = \dfrac{AC' - A'C}{BA' - AB'}$.

We said (Art. 14) that the position of a point was determined, being given two equations between its co-ordinates. The reader will now perceive that each equation represents a locus on which the point must lie, and that the point is the intersection of the two loci represented by the equations. Even the simplest equations to represent a point, viz., $x = a$, $y = b$, are the equations of two parallels to the axes of co-ordinates, the intersection of which is the required point.

The reader will also now understand why two equations of the first degree only represent one point, and why two equations of higher degree represent more points than one (Art. 15). In the first case each equation represents a right line, and two right lines can only intersect in one point. In the more general case, the loci represented by the equations are curves of higher dimensions, which will intersect each other in more points than one.

34. *To find the condition that three right lines shall meet in a point.*

Let their equations be

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0, \quad A''x + B''y + C'' = 0.$$

If they intersect, the co-ordinates of the intersection of two of them must satisfy the third equation; and using the values found in the last article, we get, for the required condition,

$$A''(BC' - B'C) + B''(CA' - C'A) + C''(AB' - BA') = 0,$$

which may be also written in either of the forms

$$A(B'C'' - B''C') + B(C'A'' - C''A') + C(A'B'' - A''B') = 0,$$
$$A(B'C'' - B''C') + A'(B''C - BC'') + A''(BC' - B'C) = 0.$$

Ex. 1. To find the co-ordinates of the vertices of the triangle the equations of whose sides are $x + y = 2$; $x - 3y = 4$; $3x + 5y + 7 = 0$.

Ans. $\left(\frac{5}{2}, -\frac{1}{2}\right), \left(-\frac{1}{14}, -\frac{19}{14}\right), \left(\frac{17}{2}, -\frac{13}{2}\right)$.

Ex. 2. To find the co-ordinates of the intersections of

$$3x + y - 2 = 0;\quad x + 2y = 5;\quad 2x - 3y + 7 = 0.$$

Ans. $\left(-\frac{1}{5}, \frac{13}{5}\right), (-1, 3), \left(-\frac{1}{11}, \frac{25}{11}\right).$

Ex. 3. Find the co-ordinates of the intersections of

$$2x + 3y = 13;\quad 5x - y = 7;\quad x - 4y + 10 = 0.$$

Ans. They meet in the point (2, 3).

Ex. 4. Find the co-ordinates of the vertices, and the equations of the diagonals, of the quadrilateral the equations of whose sides are

$$2y - 3x = 10,\ 2y + x = 6,\ 16x - 10y = 33,\ 12x + 14y + 29 = 0.$$

Ans. $\left(-1, \frac{7}{2}\right), \left(3, \frac{3}{2}\right), \left(\frac{1}{2}, -\frac{5}{2}\right), \left(-3, \frac{1}{2}\right);\ 6y - x = 6,\ 8x + 2y + 1 = 0.$

Ex. 5. Find the intersections of opposite sides of the same quadrilateral and the equation of the line joining them. *Ans.* $\left(83, \frac{259}{2}\right), \left(-\frac{71}{5}, \frac{101}{10}\right);\ 162y - 199x = 4462.$

Ex. 6. Find the diagonals of the parallelogram formed by $x = a,\ x = a',\ y = b,\ y = b'$.

Ans. $(b - b')x - (a - a')y = a'b - ab';\ (b - b')x + (a - a')y = ab - a'b'.$

Ex. 7. The axes of co-ordinates being the base of a triangle and the bisector of the base, form the equations of the two bisectors of sides, and find the co-ordinates of their intersection. Let the co-ordinates of the vertex be $0, y'$, those of the base angles $x', 0$; and $-x', 0$.

Ans. $3x'y - y'x - x'y' = 0;\ 3x'y + y'x - x'y' = 0;\ \left(0, \frac{y'}{3}\right).$

Ex. 8. The equations of the sides of a quadrilateral are

$$\frac{x}{a} + \frac{y}{b} = 1,\ \frac{x}{a} - \frac{y}{b'} = 1,\ \frac{x}{a'} + \frac{y}{b'} + 1 = 0,\ \frac{y}{b} - \frac{x}{a'} = 1,$$

find the co-ordinates of the intersections of opposite sides and of the middle point of the line joining them.

Ans. $\left\{\frac{aa'(b + b')}{ba' - ab'}, \frac{bb'(a + a')}{ab' - a'b}\right\}, \left\{\frac{aa'(b + b')}{a'b' - ab}, \frac{bb'(a + a')}{a'b' - ab)}\right\},$

$$\frac{1}{2(a'b' - ab)(ab' - a'b)}\{aa'(a - a')(b + b')^2,\ bb'(b' - b)(a + a')^2\}.$$

Ex. 9. Find the equation of the line joining the middle points of the diagonals of the same quadrilateral. *Ans.* $\frac{2x}{a - a'} + \frac{2y}{b - b'} = 1.$

Ex. 10. Verify that the co-ordinates of the middle point found in Ex. 8 satisfy this equation.

* 35. *To find the area of the triangle formed by the three lines* $Ax + By + C = 0,\ A'x + B'y + C' = 0,\ A''x + B''y + C'' = 0.$

We find the co-ordinates of the vertices by Art. 33, and substituting in the formula of Art. 31, we obtain for the double area the expression

$$\frac{BC' - B'C}{AB' - BA'}\left\{\frac{A'C'' - C'A''}{B'A'' - A'B''} - \frac{A''C - C''A}{B''A - A''B}\right\}$$
$$+ \frac{B'C'' - B''C'}{A'B'' - B'A''}\left\{\frac{A''C - C''A}{B''A - A''B} - \frac{AC' - CA'}{BA' - AB'}\right\}$$
$$+ \frac{B''C - BC''}{A''B - B''A}\left\{\frac{AC' - CA'}{BA' - AB'} - \frac{A'C'' - C'A''}{B'A'' - A'B''}\right\}.$$

But if we reduce to a common denominator, and observe that the numerator of the fraction between the first brackets is

$$\{A''(BC' - B'C) + A(B'C'' - B''C') + A'(B''C - C''B)\}$$

multiplied by A″; and that the numerators of the fractions between the second and third brackets are the same quantity multiplied respectively by A and A′, we get for the double area the expression

$$\frac{\{A(B'C'' - B''C') + A'(B''C - BC'') + A''(BC' - B'C)\}^2}{(AB' - BA')(A'B'' - B'A'')(A''B - B''A)}.$$

If the three lines meet in a point, this expression for the area vanishes (Art. 34); if any two of them are parallel, it becomes infinite (Art. 22).

36. *Given the equations of two right lines, to find the equation of a third through their point of intersection.*

The method of solving this question, which will first occur to the reader, is to obtain the co-ordinates of the point of intersection by Art. 33, and then to substitute these values for $x'y'$ in the equation of Art. 28, viz., $y - y' = m(x - x')$. The question, however, admits of an easier solution by the help of the following important principle: *If* $S = 0$, $S' = 0$, *be the equations of any two loci, then the locus represented by the equation* $S + kS' = 0$ (*where* k *is any constant*) *passes through every point common to the two given loci.* For it is plain that any co-ordinates which satisfy the equation $S = 0$, and also satisfy the equation $S' = 0$, must likewise satisfy the equation $S + kS' = 0$.

Thus, then, the equation

$$(Ax + By + C) + k(A'x + B'y + C') = 0,$$

which is obviously the equation of a right line, denotes one passing through the intersection of the right lines

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

for if the co-ordinates of the point common to them both be substituted in the equation $(Ax + By + C) + k(A'x + B'y + C') = 0$, they will satisfy it, since they make each member of the equation separately $= 0$.

Ex. 1. To find the equation of the line joining to the origin the intersection of

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0.$$

Multiply the first by C', the second by C, and subtract, and the equation of the required line is $(AC' - A'C)x + (BC' - CB')y = 0$; for it passes through the origin (Art. 19), and by the present article it passes through the intersection of the given lines.

Ex. 2. To find the equation of the line drawn through the intersection of the same lines, parallel to the axis of x. *Ans.* $(BA' - AB')y + CA' - AC' = 0$.

Ex. 3. To find the equation of the line joining the intersection of the same lines to the point $x'y'$. Writing down by this article the general equation of a line through the intersection of the given lines, we determine k from the consideration that it must be satisfied by the co-ordinates $x'y'$, and find for the required equation

$$(Ax + By + C)(A'x' + B'y' + C') = (Ax' + By' + C)(A'x + B'y + C').$$

Ex. 4. Find the equation of the line joining the point (2, 3) to the intersection of $2x + 3y + 1 = 0$, $3x - 4y = 5$.

Ans. $11(2x + 3y + 1) + 14(3x - 4y - 5) = 0$; or $64x - 23y = 59$.

37. The principle established in the last article gives us a test for three lines intersecting in the same point, often more convenient in practice than that given in Art. 34. *Three right lines will pass through the same point if their equations being multiplied each by any constant quantity, and added together, the sum is identically* $= 0$: that is to say, if the following relation be true, no matter what x and y are—

$$l(Ax + By + C) + m(A'x + B'y + C') + n(A''x + B''y + C'') = 0.$$

For then those values of the co-ordinates which make the first two members severally $= 0$ must also make the third $= 0$.

Ex. 1. The three bisectors of the sides of a triangle meet in a point. Their equations are (Art. 29, Ex. 5)—

$$(y' + y'' - 2y''')x - (x' + x'' - 2x''')y + (x'y''' - y'x''') + (x''y''' - y''x''') = 0.$$
$$(y'' + y''' - 2y')x - (x'' + x''' - 2x')y + (x''y' - y''x') + (x'''y' - y'''x') = 0.$$
$$(y''' + y' - 2y'')x - (x''' + x' - 2x'')y + (x'''y'' - y'''x'') + (x'y'' - y'x'') = 0.$$

And since the three equations when added together vanish identically, the lines represented by them meet in a point. Its co-ordinates are found (Art. 33)

$$\left(\frac{x' + x'' + x'''}{3}, \frac{y' + y'' + y'''}{3}\right).$$

Ex. 2. Prove the same thing, taking for axes two sides of the triangle whose lengths are a and b. *Ans.* $\frac{2x}{a}+\frac{y}{b}-1=0,\ \frac{x}{a}+\frac{2y}{b}-1=0,\ \frac{x}{a}-\frac{y}{b}=0.$

* 38. *To find the co-ordinates of the intersection of the line joining the points $x'y'$, $x''y''$, with the right line $Ax+By+C=0$.*

We might solve this question by forming the equation of the line joining the two points, and then determining, by Art. 33, its intersection with the given line. There is, however, another method (which we shall frequently employ) of determining the point in which the line joining two given points is met by a given locus. We know (Art. 7) that the co-ordinates of any point on the line joining the given points must be of the form

$$x=\frac{mx''+nx'}{m+n},\quad y=\frac{my''+ny'}{m+n};$$

and we may take as our unknown quantity $\frac{m}{n}$, the ratio, namely, in which the line joining the points is cut by the given locus, and we may determine this unknown quantity from the condition, that the co-ordinates just written shall satisfy the equation of the locus. Thus, in the present example we have

$$A\frac{mx''+nx'}{m+n}+B\frac{my''+ny'}{m+n}+C=0;$$

hence

$$\frac{m}{n}=-\frac{Ax'+By'+C}{Ax''+By''+C};$$

and consequently the co-ordinates of the required point are

$$x=\frac{(Ax'+By'+C)\,x''-(Ax''+By''+C)\,x'}{(Ax'+By'+C)-(Ax''+By''+C)};$$

with a similar expression for y. This value for the ratio $m:n$ might also have been deduced geometrically from the consideration that the ratio in which the line joining $x'y'$, $x''y''$ is cut, is equal to the ratio of the perpendiculars from these points upon the given line; but (Art. 27) these perpendiculars are

$$\frac{Ax'+By'+C}{\sqrt{(A^2+B^2)}}\ \text{and}\ \frac{Ax''+By''+C}{\sqrt{(A^2+B^2)}}.$$

The negative sign in the preceding value arises from the fact that in the case of *internal* section to which the positive sign of $m:n$ corresponds (Art. 7), the perpendiculars fall on opposite sides of the given line, and must, therefore, be understood as having different signs (Art. 27).

If a right line cut the sides of a triangle BC, CA, AB, *in the points* LMN, *then*

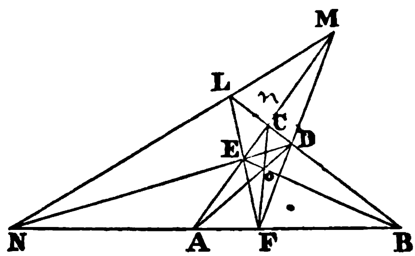
$$\frac{BL \cdot CM \cdot AN}{CL \cdot AM \cdot BN} = -1.$$

Let the co-ordinates of the vertices be $x'y'$, $x''y''$, $x'''y'''$, then

$$\frac{BL}{CL} = -\frac{Ax'' + By'' + C}{Ax''' + By''' + C};$$

$$\frac{CM}{AM} = -\frac{Ax''' + By''' + C}{Ax' + By' + C};$$

$$\frac{AN}{BN} = -\frac{Ax' + By' + C}{Ax'' + By'' + C};$$



and the truth of the theorem is manifest.

* 39. *To find the ratio in which the line joining two points* x_1y_1, x_2y_2, *is cut by the line joining two other points* x_3y_3, x_4y_4.

The equation of this latter line is (Art. 29)

$$(y_3 - y_4)\,x - (x_3 - x_4)\,y + x_3y_4 - x_4y_3 = 0.$$

Therefore by the last article

$$\frac{m}{n} = -\frac{(y_3 - y_4)\,x_1 - (x_3 - x_4)\,y_1 + x_3y_4 - x_4y_3}{(y_3 - y_4)\,x_2 - (x_3 - x_4)\,y_2 + x_3y_4 - x_4y_3}.$$

It is plain (by Art. 31) that this is the ratio of the two triangles whose vertices are

$$x_1y_1,\ x_3y_3,\ x_4y_4, \text{ and } x_2y_2,\ x_3y_3,\ x_4y_4,$$

as also is geometrically evident.

If the lines connecting any assumed point with the vertices of a triangle meet the opposite sides BC, CA, AB, *respectively, in* D, E, F, then

$$\frac{BD \cdot CE \cdot AF}{CD \cdot AE \cdot BF} = +1.$$

Let the assumed point be x_4y_4, and the vertices x_1y_1, x_2y_2, x_3y_3,

then
$$\frac{BD}{CD}=\frac{x_1(y_2-y_4)+x_2(y_4-y_1)+x_4(y_1-y_2)}{x_1(y_4-y_3)+x_4(y_3-y_1)+x_3(y_1-y_4)},$$

$$\frac{CE}{AE}=\frac{x_2(y_3-y_4)+x_3(y_4-y_2)+x_4(y_2-y_3)}{x_1(y_2-y_4)+x_2(y_4-y_1)+x_4(y_1-y_2)},$$

$$\frac{AF}{BF}=\frac{x_1(y_4-y_3)+x_4(y_3-y_1)+x_3(y_1-y_4)}{x_2(y_3-y_4)+x_3(y_4-y_2)+x_4(y_2-y_3)},$$

and the truth of the theorem is evident.

40. *To find the angle between two lines, whose equations with regard to rectangular co-ordinates are given.*

The angle between the lines is manifestly equal to the angle between the perpendiculars on the lines from the origin; if therefore these perpendiculars make with the axis of x the angles α, α', we have (Art. 25)

$$\cos\alpha=\frac{A}{\sqrt{(A^2+B^2)}};\ \sin\alpha=\frac{B}{\sqrt{(A^2+B^2)}};$$

$$\cos\alpha'=\frac{A'}{\sqrt{(A'^2+B'^2)}};\ \sin\alpha'=\frac{B'}{\sqrt{(A'^2+B'^2)}};$$

Hence
$$\sin(\alpha-\alpha')=\frac{BA'-AB'}{\sqrt{(A^2+B^2)}\sqrt{(A'^2+B'^2)}};$$

$$\cos(\alpha-\alpha')=\frac{AA'+BB'}{\sqrt{(A^2+B^2)}\sqrt{(A'^2+B'^2)}};$$

and therefore
$$\tan(\alpha-\alpha')=\frac{BA'-AB'}{AA'+BB'}.$$

Cor. 1.—The two lines are parallel to each other when

$$BA'-AB'=0 \text{ (Art. 22)},$$

since then the angle between them vanishes.

Cor. 2.—The two lines are perpendicular to each other when

$$AA'+BB'=0,$$

since then the tangent of the angle between them becomes infinite.

If the equations of the lines had been given in the form

$$y=mx+b,\ y=m'x+b';$$

since the angle between the lines is the difference of the angles they make with the axis of x, and since (Art. 22) the tangents of these angles are m and m', it follows that the tangent of the required angle is $\dfrac{m - m'}{1 + mm'}$; that the lines are parallel if $m = m'$; and perpendicular to each other if $mm' + 1 = 0$.

41. *To find the angle between two lines, the co-ordinates being oblique.*

We proceed as in the last article, using the expressions of Art. 26.

$$\cos a = \frac{A \sin \omega}{\sqrt{(A^2 + B^2 - 2AB \cos \omega)}},$$

$$\cos a' = \frac{A' \sin \omega}{\sqrt{(A'^2 + B'^2 - 2A'B' \cos \omega)}};$$

consequently,

$$\sin a = \frac{B - A \cos \omega}{\sqrt{(A^2 + B^2 - 2AB \cos \omega)}},$$

$$\sin a' = \frac{B' - A' \cos \omega}{\sqrt{(A'^2 + B'^2 - 2A'B' \cos \omega)}}.$$

Hence

$$\sin (a - a') = \frac{(BA' - AB') \sin \omega}{\sqrt{(A^2 + B^2 - 2AB \cos \omega)} \sqrt{(A'^2 + B'^2 - 2A'B' \cos \omega)}},$$

$$\cos (a - a') = \frac{BB' + AA' - (AB' + A'B) \cos \omega}{\sqrt{(A^2 + B^2 - 2AB \cos \omega)} \sqrt{(A'^2 + B'^2 - 2A'B' \cos \omega)}},$$

$$\tan (a - a') = \frac{(BA' - AB') \sin \omega}{AA' + BB' - (AB' + BA') \cos \omega}.$$

Cor. 1.—The lines are parallel if $BA' = AB'$.

Cor. 2.—The lines are perpendicular to each other if

$$AA' + BB' = (AB' + BA') \cos \omega.$$

42. *To find the equation of a line passing through a given point and making a given angle, ϕ, with a given line $y = mx + b$ (the axes of co-ordinates being rectangular).*

Let the equation of the required line be

$$y - y' = m'(x - x'),$$

and the formula of Art. 40,

$$\tan \phi = \frac{m - m'}{1 + mm'},$$

enables us to determine

$$m' = \frac{m - \tan \phi}{1 + m \tan \phi}.$$

To find the equation of a right line passing through a given point, and perpendicular to a given line, $y = mx + b$.

The condition that two lines should be perpendicular, being $mm' = -1$, we have at once for the equation of the required perpendicular

$$y - y' = -\frac{1}{m}(x - x').$$

It is easy, from the above, to see that the equation of the perpendicular from the point $x'y'$ on the line $Ax + By + C = 0$ is

$$A(y - y') = B(x - x'),$$

that is to say, we interchange the coefficients of x and y, and alter the sign of one of them.

Ex. 1. To find the equations of the perpendiculars from each vertex on the opposite side, of the triangle $(2, 1)$, $(3, -2)$, $(-4, -1)$.

The equations of the sides are (Art. 29, Ex. 1)

$$x + 7y + 11 = 0, \quad 3y - x = 1, \quad 3x + y = 7;$$

and the equations of the perpendiculars

$$7x - y = 13, \quad 3x + y = 7, \quad 3y - x = 1.$$

The triangle is consequently right-angled.

Ex. 2. To find the equations of the perpendiculars at the middle points of the sides of the same triangle. The co-ordinates of the middle points being

$$\left(-\frac{1}{2}, -\frac{3}{2}\right), (-1, 0), \left(\frac{5}{2}, -\frac{1}{2}\right).$$

The perpendiculars are

$$7x - y + 2 = 0, \quad 3x + y + 3 = 0, \quad 3y - x + 4 = 0, \quad \text{intersecting in } \left(-\frac{1}{2}, -\frac{3}{2}\right).$$

Ex. 3. Find the equations of the perpendiculars from the vertices of the triangle $(2, 3)$, $(4, -5)$, $(-3, -6)$ (see Art. 29, Ex. 3).

Ans. $7x + y = 17$, $5x + 9y + 25 = 0$, $x - 4y = 21$: intersecting in $\left(\frac{89}{29}, -\frac{130}{29}\right)$.

Ex. 4. Find the equations of the perpendiculars at the middle points of the sides of the same triangle.

Ans. $7x + y + 2 = 0$, $5x + 9y + 16 = 0$, $x - 4y = 7$; intersecting in $\left(-\frac{1}{29}, -\frac{51}{29}\right)$.

Ex. 5. To find in general the equations of the perpendiculars from the vertices on the opposite sides of a triangle the co-ordinates of whose vertices are given.

Ans.
$$(x'' - x''')x + (y'' - y''')y + (x'x''' + y'y''') - (x'x'' + y'y'') = 0,$$
$$(x''' - x')x + (y''' - y')y + (x''x' + y''y') - (x''x''' + y''y''') = 0,$$
$$(x' - x'')x + (y' - y'')y + (x'''x'' + y'''y'') - (x'''x' + y'''y') = 0.$$

By Art. 30 these lines intersect in a point, since the equations added together vanish identically.

Ex. 6. Find the equations of the perpendiculars at the middle points of the sides of a triangle, and show that they meet in a point.

Ans. $(x''-x''')x+(y''-y''')y+\frac{1}{2}(x''^2-x'''^2)+\frac{1}{2}(y''^2-y'''^2)=0,$

$(x'''-x')x+(y'''-y')y+\frac{1}{2}(x'''^2-x'^2)+\frac{1}{2}(y'''^2-y'^2)=0,$

$(x'-x'')x+(y'-y'')y+\frac{1}{2}(x'^2-x''^2)+\frac{1}{2}(y'^2-y''^2)=0.$

Ex. 7. Taking for axes the base of a triangle and the perpendicular on it from the vertex, find the equations of the other two perpendiculars, and the co-ordinates of their intersection. The co-ordinates of the vertex are now $(0, y')$, and of the base angles $(x', 0)$, $(-x'', 0)$.

Ans. $x''(x-x')+y'y=0,\ \ x'(x+x'')-y'y=0,\ \ \left(0, \frac{x'x''}{y'}\right).$

Ex. 8. Using the same axes, find the equations of the perpendiculars at the middle points of sides, and the co-ordinates of their intersection.

Ans. $2(x''x+y'y)=y'^2-x''^2,\ 2(x'x-y'y)=x'^2-y'^2,\ 2x=x'-x'',\left(\frac{x'-x''}{2}, \frac{y'^2-x'x''}{2y'}\right).$

43. *To find the equation of the bisector of the angle between two lines,* $x\cos\alpha+y\sin\alpha-p=0$, $x\cos\beta+y\sin\beta-p'=0$.

We find the equation of this line most simply by expressing algebraically the property that the perpendiculars let fall from any point xy of the bisector on the two lines are equal. This immediately gives us the equation

$$x\cos\alpha+y\sin\alpha-p=x\cos\beta+y\sin\beta-p',$$

since each side of this equation denotes the length of one of those perpendiculars (Art. 27).

The reader will remember (Art. 27) that the sign of the perpendicular changes as we pass from one side of a line to the other; consequently the equation

$$x\cos\alpha+y\sin\alpha-p=-(x\cos\beta+y\sin\beta-p')$$

denotes the bisector of the supplemental angle between the two lines.

If the equations had been given in the form $Ax+By+C=0$, $A'x+B'y+C'=0$, the equation of the pair of bisectors would be

$$\frac{Ax+By+C}{\sqrt{(A^2+B^2)}}=\pm\frac{A'x+B'y+C'}{\sqrt{(A'^2+B'^2)}}.$$

If we choose that sign which will make the two constant terms of the same sign, it follows from Art. 27 that we shall have the bisector of that angle in which the origin lies; and if we give the constant terms opposite signs, we shall have the equation of the bisector of the supplemental angle.

Ex. 1. Reduce the equation of the bisectors of the angles between two lines, to the form $x \cos \alpha + y \sin \alpha = p$.

Ans. $$x \cos\{\tfrac{1}{2}(\alpha + \alpha') + 90°\} + y \sin\{\tfrac{1}{2}(\alpha + \alpha') + 90°\} = \frac{p - p'}{2 \sin \frac{1}{2}(\alpha - \alpha')};$$

$$x \cos \tfrac{1}{2}(\alpha + \alpha') + y \sin \tfrac{1}{2}(\alpha + \alpha') + \frac{p + p'}{2 \cos \frac{1}{2}(\alpha - \alpha')} = 0.$$

Ex. 2. Prove that the three bisectors of the angles of a triangle meet in a point. The origin being anywhere within the triangle, their equations are

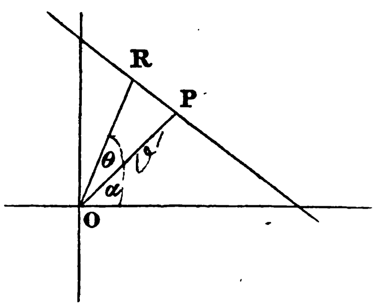
$$(x \cos \alpha + y \sin \alpha - p) - (x \cos \beta + y \sin \beta - p') = 0,$$
$$(x \cos \beta + y \sin \beta - p') - (x \cos \gamma + y \sin \gamma - p'') = 0,$$
$$(x \cos \gamma + y \sin \gamma - p'') - (x \cos \alpha + y \sin \alpha - p) = 0.$$

Ex. 3. Find the equations of the bisectors of the angles between

$$3x + 4y - 9 = 0, \quad 12x + 5y - 3 = 0.$$

Ans. $7x - 9y + 34 = 0, \quad 9x + 7y = 12.$

44. *To find the polar equation of a right line* (see Art. 12).



Suppose we take, as our fixed axis, the perpendicular on the given line, then let OR be any *radius vector* drawn from the pole to the given line

$$OR = \rho, \; ROP = \theta,$$

but, plainly,

$$OR \cos \theta = OP,$$

hence, the equation is

$$\rho \cos \theta = p.$$

If the fixed axis make an angle α with the perpendicular, the equation is

$$\rho \cos(\theta - \alpha) = p.$$

This equation may also be obtained by transforming the equation with regard to rectangular co-ordinates,

$$x \cos \alpha + y \sin \alpha = p.$$

Rectangular co-ordinates are transformed to polar by writing for x, $\rho \cos \theta$, and for y, $\rho \sin \theta$ (see Art. 12); hence the equation becomes

$$\rho (\cos \theta \cos \alpha + \sin \theta \sin \alpha) = p;$$

or, as we got before,

$$\rho \cos (\theta - \alpha) = p.$$

An equation of the form

$$\rho (\mathrm{A} \cos \theta + \mathrm{B} \sin \theta) = \mathrm{C}$$

can be (as in Art. 25) reduced to the form $\rho \cos (\theta - \alpha) = p$, by dividing by $\sqrt{(\mathrm{A}^2 + \mathrm{B}^2)}$; we shall then have

$$\cos \alpha = \frac{\mathrm{A}}{\sqrt{(\mathrm{A}^2 + \mathrm{B}^2)}}, \quad \sin \alpha = \frac{\mathrm{B}}{\sqrt{(\mathrm{A}^2 + \mathrm{B}^2)}}, \quad p = \frac{\mathrm{C}}{\sqrt{(\mathrm{A}^2 + \mathrm{B}^2)}}.$$

Ex. 1. Reduce to rectangular co-ordinates the equation

$$\rho = 2a \sec \left(\theta + \frac{\pi}{6}\right).$$

Ex. 2. Find the polar co-ordinates of the intersection of the following lines, and also the angle between them: $\rho \cos \left(\theta - \frac{\pi}{2}\right) = 2a$, $\rho \cos \left(\theta - \frac{\pi}{6}\right) = a$.

Ans. $\rho = 2a$, $\theta = \frac{\pi}{2}$, angle $= \frac{\pi}{8}$.

Ex. 3. Find the polar equation of the line passing through the points whose polar co-ordinates are ρ', θ'; ρ'', θ''.

Ans. $\rho'\rho'' \sin(\theta' - \theta'') + \rho''\rho \sin(\theta'' - \theta) + \rho\rho' \sin(\theta - \theta') = 0.$

CHAPTER III.

EXAMPLES ON THE RIGHT LINE.

45. Having in the last chapter laid down principles by which we are able to express algebraically the position of any point or right line, we proceed to give some further examples of the application of this method to the solution of geometrical problems. The learner should diligently exercise himself in working out such questions, until he has acquired quickness and readiness in the use of this method. The examples given in this chapter being introduced, not so much for their own sake, as to show how such questions may be solved algebraically, will often be such as admit of simpler *geometrical* solutions. It must not be supposed, however, that because in these instances the geometrical method has the advantage, it is in all cases to be preferred. Each method has its peculiar recommendations. If the geometrical solutions of some

questions are clearer and more simple, the algebraical method proceeds with more uniformity, and reaches its end with greater certainty. It should be the student's aim to make himself master of both instruments of investigation, so as to be able to apply either, according as the nature of the subject demands. We shall give examples of some of the classes of problems which are of most frequent occurrence: the student who has mastered these will find no difficulty in applying the same method to any others that may present themselves.

46. *Problems where it is required to prove that three lines meet in a point.*

It seems unnecessary to add any illustrations to those given in the last chapter, on this subject. The process we pursue is as follows: We form the equations of the three lines: it may then happen that we observe at once that the three equations vanish identically when added together (multiplied, it may be, by suitable constants): if this be the case, we know, by Art. 37, that the lines represented by the equations meet in a point. Otherwise, we find the co-ordinates of the intersection of two of them, and examine whether they satisfy the equation of the third; or else we apply to the equations the test of Art. 34.

In the solution of this and every other class of geometrical problems, our equations may generally be much simplified by a judicious choice of axes of co-ordinates: since, by choosing for axes two of the most remarkable lines on the figure, several of our expressions will often be much shortened. On the other hand, it will sometimes happen that by choosing axes unconnected with the figure, the equations will gain in symmetry more than an equivalent for what they lose in simplicity. The reader may compare the two solutions of the same question, given Ex. 1 and 2, Art. 37, where, though the first solution is the longest, it has the advantage that the equation of one bisector being formed, those of the others can be written down without further calculation.

Since expressions containing angles become more complicated by the use of oblique co-ordinates, it will be generally advisable to use rectangular axes in any question in which the consideration of angles is involved.

Problems where it is required to prove that three points lie in one right line.

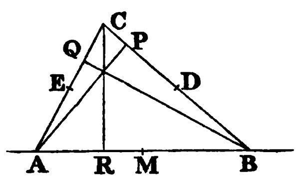
It may happen that we observe that, the co-ordinates of two of them being $x'y'$, $x''y''$, those of the third are of the form

$$\frac{mx''+nx'}{m+n},\ \frac{my''+ny'}{m+n},$$

in which case it is obvious, Art. 7, that the three points are in one right line. Otherwise we form the equation of the line joining two of them, and examine whether it is satisfied by the co-ordinates of the third.

47. *Loci.*—Analytic geometry adapts itself with peculiar readiness to the investigation of loci. We have only to find what relation the conditions of the question assign between the co-ordinates of the point whose locus we seek, and then the statement of this relation in algebraical language gives us at once the *equation* of the required locus.

Ex. 1. Given base and difference of squares of sides of a triangle, to find the locus of vertex.



Take the base for axis of x, and a perpendicular through one extremity A for axis of y. Call the length of base c, and let the co-ordinates of vertex be x, y. Then

$$BC^2 = CR^2 + RB^2 = y^2 + (c-x)^2.$$

$$AC^2 = x^2 + y^2,$$

therefore $$BC^2 - AC^2 = c^2 - 2cx;$$

and, putting this equal to a constant,

$$c^2 - 2cx = m^2$$

is the equation of the locus of vertex; but this is (Art. 15) the equation of a line perpendicular to the base at a distance from the origin $= \frac{c^2-m^2}{2c}$. Subtracting this from c, the other segment will be $\frac{c^2+m^2}{2c}$, and we easily verify that the locus will cut the base so that the difference of the squares of segments = the difference of squares of sides (Euc. I. 47, Cor. 4).

Ex. 2. Given base and sum of sides of a triangle, if the perpendicular be produced beyond the vertex until its whole length is equal to one of the sides, to find the locus of the extremity of the perpendicular.

Take the same axes, and let us inquire what relation exists between the co-ordinates of the point whose locus we are seeking. The x of this point plainly is AR, and the y is, by hypothesis, = AC; and if m be the given sum of sides,

$$BC = m - y.$$

Now (Euclid, II. 13),

$$BC^2 = AB^2 + AC^2 - 2AB \, . \, AR; \text{ or, denoting AB by } c,$$

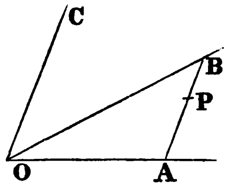
$$(m - y)^2 = c^2 + y^2 - 2cx.$$

Reducing this equation, we get

$$2my - 2cx = m^2 - c^2,$$

the equation of a right line.

Ex. 3. Given two fixed lines, OA and OB, if any line be drawn to intersect them parallel to a third fixed line, OC, to find the locus of the point where AB is cut in a given ratio.



We may here employ oblique axes, since angles are not concerned (Art. 46). Let us take the fixed line OA for axis of x, and the fixed line OC for axis of y, then the equation of OB must be of the form $y = mx$, and it is required to find the locus of the point P cutting AB, so that AP may, for instance, $= n$AB.

Since the point B lies on the line whose equation is $y = mx$, we have

$$AB = mOA,$$

therefore $$AP = mnOA,$$

but AP is the y of the point P, and OA its x, therefore the locus of P is expressed by the equation $$y = mnx,$$

and is, therefore, a right line through the point O.

Ex. 4. Given bases and sum of areas of any number of triangles having a common vertex, to find its locus.

Let the equations of the bases be

$$x \cos\alpha + y \sin\alpha - p = 0, \quad x \cos\beta + y \sin\beta - p_1 = 0,$$

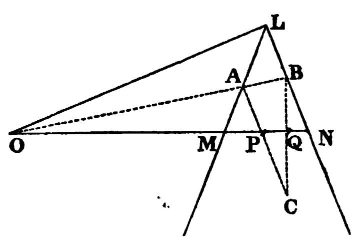
$$x \cos\gamma + y \sin\gamma - p_2 = 0, \text{ \&c.}$$

and their lengths, a, b, c, &c.; and let the given sum $= m^2$; then, since (Art. 27) $x\cos\alpha + y\sin\alpha - p$ denotes the perpendicular from the point xy on the first line, $a(x\cos\alpha + y\sin\alpha - p)$ will be double the area of the first triangle, &c., and the equation of the locus will be

$$a(x\cos\alpha + y\sin\alpha - p) + b(x\cos\beta + y\sin\beta - p_1) + c(x\cos\gamma + y\sin\gamma - p_2) + \text{\&c.} = 2m^2,$$

which, since it contains x and y only in the first degree, will represent a right line.

Ex. 5. Two vertices of a triangle ABC move on fixed right lines LM, LN, and the three sides pass through three fixed points O, P, Q which lie on a right line; find the locus of the third vertex.



Take for axis of x the right line OP, containing the three fixed points, and for axis of y the line OL joining the intersection of the two fixed lines to the point O through which the base passes. Let the co-ordinates of C be $x'y'$, and let

$$OL = b, \quad OM = a, \quad ON = a',$$

$$OP = c, \quad OQ = c'.$$

Then obviously the equations of LM, LN are

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ and } \frac{x}{a'} + \frac{y}{b} = 1.$$

The equation of CP through $x'y'$ and P ($y = 0$, $x = c$) is

$$(x' - c)\,y - y'x + cy' = 0.$$

The co-ordinates of A, the intersection of this line with

$$\frac{x}{a} + \frac{y}{b} = 1,$$

are $$x_1 = \frac{ab\,(x' - c) + acy'}{b\,(x' - c) + ay'}; \quad y_1 = \frac{b\,(a - c)\,y'}{b\,(x' - c) + ay'}.$$

The co-ordinates of B are found by simply accentuating the letters in the preceding:

$$x_2 = \frac{a'b\,(x' - c') + a'c'y'}{b\,(x' - c') + a'y'}; \quad y_2 = \frac{b\,(a' - c')\,y'}{b\,(x' - c') + a'y'}.$$

Now the condition that two points, x_1y_1, x_2y_2, shall lie on a right line passing through the origin, is (Art. 30) $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

Applying this condition we have

$$\frac{b\,(a - c)\,y'}{ab\,(x' - c) + acy'} = \frac{b\,(a' - c')\,y'}{a'b\,(x' - c') + a'c'y'}.$$

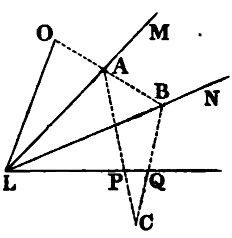
This being a relation then which must always be satisfied by the co-ordinates $x'y'$, the equation of the locus is obtained by simply removing the accents from $x'y'$; and clearing of fractions, we have

$$(a - c)\,[a'b\,(x - c') + a'c'y] = (a' - c')\,[ab\,(x - c) + acy],$$

or $$\frac{(ac' - a'c)\,x}{cc'\,(a - a') - aa'\,(c - c')} + \frac{y}{b} = 1,$$

the equation of a right line through the point L.

Ex. 6. If in the last example the points P, Q lie on a right line passing not through O but through L, find the locus of vertex.



Take for axis of x the line LP, and for axis of y the line LO. Let LP = a, LQ = a', LO = b, and let the equations of LM, LN be $y = mx$ and $y = m'x$. The equation of CP through $x'y'$ and $(a, 0)$ is

$$y'\,(x - a) = (x' - a)\,y.$$

The co-ordinates therefore of the point A where this line meets $y = mx$ are

$$x_1 = \frac{ay'}{y' - mx' + am}, \quad y_1 = \frac{amy'}{y' - mx' + am}.$$

In like manner the co-ordinates of B are $x_2 = \frac{a'y'}{y' - m'x' + a'm'}$, $y_2 = \frac{a'm'y'}{y' - m'x' + a'm'}$.

We must now express that the line joining these points passes through O. Substitute the co-ordinates $x = 0$, $y = b$, in the equation of the line joining two points x_1y_1, x_2y_2 (Art. 29), and it becomes

$$(x^1 - x_2)\,b = x_1y_2 - y_1x_2; \quad \text{or,} \quad x_1\,(y_2 - b) = x_2\,(y_1 - b).$$

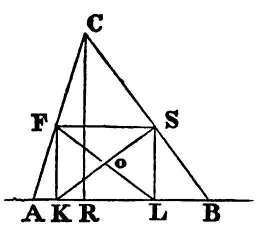
Substitute in this the values just obtained for x_1, y_1, x_2, y_2, and clear of fractions; the equation becomes divisible by y', and we have for the relation to be satisfied by the point $x'y'$,

$$a\{(a'm' - b)y + m'b(x - a')\} = a'\{(am - b)y + mb(x - a)\},$$

the equation of a right line. It passes (Art. 86) through the intersection of the lines found by equating each side of the equation separately to 0. It will be found that these are the lines joining P and Q to the points where a parallel to LQ through O meets LM, LN.

48. It is often convenient, instead of expressing the conditions of the problem directly in terms of the co-ordinates of the point whose locus we are seeking, to express them in the first instance in terms of some other lines of the figure; we must then obtain as many relations as are necessary in order to eliminate the indeterminate quantities thus introduced, so as to have remaining a relation between the co-ordinates of the point whose locus is sought. The following Examples will sufficiently illustrate this method.

Ex. 1. To find the locus of the middle points of rectangles inscribed in a given triangle.



Let us take for axes CR and AB; let $CR = p$, $BR = s$, $AR = s'$. The equations of AC and BC are

$$\frac{y}{p} - \frac{x}{s'} = 1 \text{ and } \frac{y}{p} + \frac{x}{s} = 1.$$

Now if we draw any line FS parallel to the base at a distance $FK = k$, and whose equation, therefore, is

$$y = k,$$

we can find the abscissæ of the points F and S, in which the line FS meets AC and BC, by substituting in the equations of AC and BC this value, $y = k$. Thus we get from the first equation

$$\frac{k}{p} - \frac{x}{s'} = 1 \therefore x \text{ or } KR = -s'\left(1 - \frac{k}{p}\right);$$

and from the second equation

$$\frac{k}{p} + \frac{x}{s} = 1 \therefore x \text{ or } RL = s\left(1 - \frac{k}{p}\right).$$

Having the abscissæ of F and S, we have (by Art. 7) the abscissa of the middle point of FS, viz., $x = \frac{s - s'}{2} \cdot \left(1 - \frac{k}{p}\right)$. This is evidently the abscissa of the middle point of the rectangle. But its ordinate is $y = \frac{k}{2}$. Now we want to find a relation which will subsist between this ordinate and abscissa whatever k be. We have only then to eliminate k between these equations, by substituting in the first the value of k $(= 2y)$, derived from the second, when we have

$$2x = (s - s')\left(1 - \frac{2y}{p}\right).$$

or $$\frac{2x}{s-s'}+\frac{2y}{p}=1.$$

This is the equation of the locus which we seek. It obviously represents a right line, and if we examine the intercepts which it cuts off on the axes we shall find it to be the line joining the middle point of the perpendicular CR to the middle point of the base.

Ex. 2. A parallel is drawn to the base of a triangle, and perpendiculars to the sides erected at its extremities, find the locus of their intersection.

Take the same axes as in Ex. 1. Then the line FQ, which is a perpendicular to the line AC $\left(\frac{y}{p}-\frac{x}{s'}=1\right)$, through the point F$\left\{-s'\left(1-\frac{k}{p}\right), k\right\}$ has for its equation

$$\frac{1}{p}\left\{x+s'\left(1-\frac{k}{p}\right)\right\}+\frac{1}{s'}(y-k)=0.$$

In like manner, the equation of SQ is

$$\frac{1}{p}\left\{x-s\left(1-\frac{k}{p}\right)\right\}-\frac{1}{s}(y-k)=0.$$

Now since the point whose locus we are seeking lies on both the lines FQ, SQ, each of the equations just written expresses a relation which must be satisfied by its co-ordinates. Still, since these equations involve k, they express relations which are only true for that particular point of the locus which corresponds to the case where the parallel FS is drawn at a height k above the base. If, however, between the equations, we eliminate the indeterminate k, we shall obtain a relation involving only the co-ordinates and known quantities, and which, since it must be satisfied whatever be the position of the parallel FS, will be the required equation of the locus.

In order, then, to eliminate k between the equations, put them into the form

$$\text{FQ}\qquad \frac{x}{p}+\frac{s'}{p}+\frac{y}{s'}=k\left(\frac{1}{s'}+\frac{s'}{p^2}\right),\ \text{and}$$

$$\text{SQ}\qquad \frac{y}{s}-\frac{x}{p}+\frac{s}{p}=k\left(\frac{1}{s}+\frac{s}{p^2}\right);$$

and eliminating, we have for the equation of the locus,

$$\left(\frac{1}{s}+\frac{s}{p^2}\right)\left(\frac{x}{p}+\frac{s'}{p}+\frac{y}{s'}\right)=\left(\frac{1}{s'}+\frac{s'}{p^2}\right)\left(\frac{y}{s}-\frac{x}{p}+\frac{s}{p}\right);$$

but this is evidently the equation of a right line, since x and y are only in the first degree, and it will be found that it passes through the vertex of the given triangle, for the co-ordinates of the vertex $x=0$, $y=p$, will satisfy the equation. It also passes (Art. 36) through the intersection of the lines formed by equating each side of the equation separately to 0. It will be found that these are the lines drawn at the extremities of the base perpendicular to the conterminous sides.

Ex. 3. A line is drawn parallel to the base of a triangle, and the points where it meets the sides joined to any two fixed points on the base; to find the locus of the point of intersection of the joining lines.

We shall preserve the same axes, &c., as in Ex. 2, and let the co-ordinates of the fixed points, T and V, on the base, be for T $(m, 0)$, and for V $(n, 0)$.

The equation of FT will be found to be

$$\left\{s'\left(1-\frac{k}{p}\right)+m\right\}y+kx-km=0.$$

and that of SV to be

$$\left\{s\left(1-\frac{k}{p}\right)-n\right\}y-kx+kn=0.$$

Putting the equations into the form

FT $$(s'+m)y-k\left(\frac{s'}{p}y-x+m\right)=0,$$

and SV $$(s-n)y-k\left(\frac{s}{p}y+x-n\right)=0;$$

and, eliminating k, we get for the equation of the locus

$$(s-n)\left(\frac{s'}{p}y-x+m\right)=(s'+m)\left(\frac{s}{p}y+x-n\right).$$

But this is the equation of a right line, since x and y are only in the first degree.

Ex. 4. A line is drawn parallel to the base of a triangle, and its extremities joined transversely to those of the base; to find the locus of the point of intersection of the joining lines.

This is a particular case of the foregoing, but admits of a simple solution by choosing for axes the sides of the triangle AC and CB. Let the lengths of those lines be a, b, and let the lengths of the proportional intercepts made by the parallel be μa, μb. Then the equations of the transversals will be

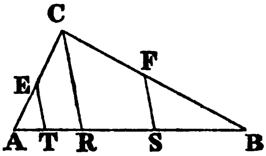
$$\frac{x}{a}+\frac{y}{\mu b}=1 \text{ and } \frac{x}{\mu a}+\frac{y}{b}=1.$$

Subtract one from the other; divide by the constant $1-\frac{1}{\mu}$, and we get for the equation of the locus

$$\frac{x}{a}-\frac{y}{b}=0,$$

which we have elsewhere found (see p. 30) to be the equation of the bisector of the base of the triangle.

Ex. 5. If on the base of a triangle we take any portion AT, and on the other side of the base another portion BS, in a fixed ratio to AT, and draw ET and FS parallel to a fixed line CR, to find the locus of O, the point of intersection of EB and FA.



Take CR for axis of y; let AT $= k$, BR $= s$, AR $= s'$, CR $= p$, let the fixed ratio be m, then BS will $= mk$; the co-ordinates of S will be $(s-mk, 0)$, and of T$\{-(s'-k), 0\}$.

The ordinates of E and F will be found by substituting these values of x in the equations of AC and BC. We get for

$$\text{E}, \quad x=-(s'-k), \quad y=\frac{pk}{s'},$$

and for

$$\text{F}, \quad x=s-mk, \quad y=\frac{mpk}{s}.$$

Now form the equations of the transverse lines, and the equation of EB is

$$(s+s'-k)y+\frac{pk}{s'}x-\frac{pks}{s'}=0,$$

and the equation of AF is

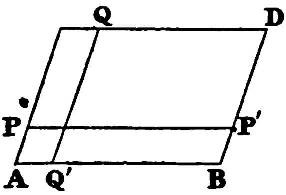
$$(s+s'-mk)y-\frac{mpk}{s}x-\frac{mpks'}{s}=0.$$

To eliminate k, subtract one equation from the other, and the result, divided by k, will be

$$(m-1)y+\left(\frac{mp}{s}+\frac{p}{s'}\right)x+\left(\frac{mps'}{s}-\frac{ps}{s'}\right)=0,$$

which is the equation of a right line.

Ex. 6. PP′ and QQ′ are any two parallels to the sides of a parallelogram; to find the locus of the intersection of the lines PQ and P′Q′.



Let us take two of the sides for our axes, and let the lengths of the sides be a and b, and let AQ′ = m, AP = n. Then the equation of PQ, joining P $(0, n)$ to Q (m, b) is

$$(b-n)x-my+mn=0,$$

and the equation of P′Q′ joining P′ (a, n) to Q′ $(m, 0)$ is

$$nx-(a-m)y-mn=0.$$

There being *two* indeterminates, m and n, we should at first suppose that it would not be possible to eliminate them from *two* equations. However, if we add the above equations, it will be found that both vanish together, and we get for our locus

$$bx-ay=0,$$

the equation of the diagonal of the parallelogram.

Ex. 7. Given a point and two fixed lines: draw any two lines through the fixed point, and join transversely the points where they meet the fixed lines, to find the locus of intersection of the transverse lines.

Take the fixed lines for axes, and let the equations of the lines through the fixed point be

$$\frac{x}{m}+\frac{y}{n}=1, \quad \text{and } \frac{x}{m'}+\frac{y}{n'}=1.$$

The condition that these lines should pass through the fixed point $x'y'$ gives us

$$\frac{x'}{m}+\frac{y'}{n}=1, \quad \text{and } \frac{x'}{m'}+\frac{y'}{n'}=1;$$

or, subtracting,

$$x'\left(\frac{1}{m}-\frac{1}{m'}\right)+y'\left(\frac{1}{n}-\frac{1}{n'}\right)=0.$$

Now the equations of the transverse lines clearly are

$$\frac{x}{m}+\frac{y}{n'}=1, \quad \text{and } \frac{x}{m'}+\frac{y}{n}=1;$$

or, subtracting,

$$x\left(\frac{1}{m}-\frac{1}{m'}\right)-y\left(\frac{1}{n}-\frac{1}{n'}\right)=0.$$

Now from this and the equation just found we can eliminate

$$\left(\frac{1}{m}-\frac{1}{m'}\right) \text{ and } \left(\frac{1}{n}-\frac{1}{n'}\right),$$

and we have

$$x'y + y'x = 0,$$

the equation of a right line through the origin.

Ex. 8. At any point of the base of a triangle is drawn a line of given length, parallel to a given one, and so as to be bisected by the base: find the locus of the intersection of the lines joining its extremities to those of the base.

Ex. 9. The base of a triangle is given, and the sides meet a fixed line AB parallel to the base in points C, D, such that the ratio of AC : BD is given; find the locus of vertex.

Ex. 10. Given the vertical angle of a triangle and the sum of sides, find the locus of the point where the base is cut in a given ratio.

Ex. 11. Given two fixed points, A, B, one on each of the axes: if A′ and B′ be taken on the axes, so that OA′ + OB′ = OA + OB, find the locus of the intersection of AB′, A′B.

49. *Problems where it is required to prove that a moveable right line passes through a fixed point.*

We have seen (Art. 36) that the line

$$Ax + By + C + k(A'x + B'y + C') = 0;$$

or, what is the same thing,

$$(A + kA')x + (B + kB')y + C + kC' = 0,$$

where k is indeterminate, always passes through a fixed point, namely, the intersection of the lines

$$Ax + By + C = 0, \text{ and } A'x + B'y + C' = 0.$$

Hence, *if the equation of a right line contain an indeterminate quantity in the first degree, the right line will always pass through a fixed point.*

Ex. 1. Given vertical angle of a triangle and the sum of the reciprocals of the sides; the base will always pass through a fixed point.

Take the sides for axes; the equation of the base is $\frac{x}{a}+\frac{y}{b}=1$, and we are given the condition

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}, \quad \text{or } \frac{1}{b}=\frac{1}{c}-\frac{1}{a},$$

therefore, equation of base is

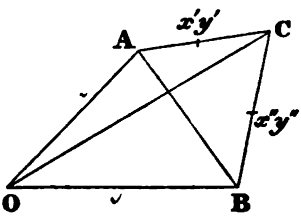
$$\frac{x}{a}+\frac{y}{c}-\frac{y}{a}=1,$$

where c is constant and a indeterminate, that is,

$$\frac{1}{a}(x-y)+\frac{y}{c}-1=0,$$

where $\frac{1}{a}$ is indeterminate. Hence the base must always pass through the intersection of the two lines $$x - y = 0, \quad \text{and } y = c.$$

Ex. 2. Given three fixed lines, OA, OB, OC, meeting in a point, if the three vertices of a triangle move one on each of these lines, and two sides of the triangle pass through fixed points, to prove that the remaining side passes through a fixed point.



Take for axes the fixed lines OA, OB, on which the base angles move, then the line OC on which the vertex moves will have an equation of the form $y = mx$, and let the fixed points be $x'y'$, $x''y''$. Now, in any position of the vertex, let its co-ordinates be $x = a$, and, consequently, $y = ma$; then the equation of AC is

$$(x' - a)y - (y' - ma)x + a(y' - mx') = 0.$$

Similarly, the equation of BC is $$(x'' - a)y - (y'' - ma)x + a(y'' - mx'') = 0.$$

Now, the length of the intercept OA is found by making $x = 0$ in equation AC, or

$$y = -\frac{a(y' - mx')}{x' - a}.$$

Similarly, OB is found by making $y = 0$ in BC, or

$$x = \frac{a(y'' - mx'')}{y'' - ma}.$$

Hence, from these intercepts, equation of AB is

$$x\frac{y'' - ma}{y'' - mx''} - y\frac{x' - a}{y' - mx'} = a.$$

But since a is indeterminate, and only in the first degree, this line always passes through a fixed point. The particular point is found by arranging the equation in the form

$$\frac{y''}{y'' - mx''}x - \frac{x'}{y' - mx'}y - a\left(\frac{mx}{y'' - mx''} - \frac{y}{y' - mx'} + 1\right) = 0.$$

Hence the line passes through the intersection of the two lines

$$\frac{y''}{y'' - mx''}x - \frac{x'}{y' - mx'}y = 0,$$

and

$$\frac{mx}{y'' - mx''} - \frac{y}{y' - mx'} + 1 = 0.$$

Ex. 3. If in the last example the line on which the vertex C moves do not pass through O, to determine whether in any case the base will pass through a fixed point.

We retain the same axes and notation as before, with the only difference that the equation of the line on which C moves will be $y = mx + n$, and the co-ordinates of the vertex in any position will be a, and $ma + n$. Then the equation of AC is

$$(x' - a)y - (y' - ma - n)x + a(y' - mx') - nx' = 0.$$

The equation of BC is

$$(x'' - a)y - (y'' - ma - n)x + a(y'' - mx'') - nx'' = 0.$$

$$\text{OA} = -\frac{a(y' - mx') - nx'}{x' - a}; \quad \text{OB} = \frac{a(y'' - mx'') - nx''}{y'' - ma - n}.$$

The equation of AB is therefore

$$x \cdot \frac{y'' - ma - n}{a(y'' - mx'') - nx''} - y \cdot \frac{x' - a}{a(y' - mx') - nx'} = 1.$$

Now when this is cleared of fractions, it will in general contain a in the second degree, and therefore, the base will in general not pass through a fixed point; *if, however, the points $x'y'$, $x''y''$, lie in a right line ($y = kx$) passing through* O, we may substitute in the denominators $y'' = kx''$, and $y' = kx'$, and the equation becomes

$$x \cdot \frac{y'' - ma - n}{x''} - y \cdot \frac{x' - a}{x'} = a(k - m) - n,$$

which only contains a *in the first degree*, and, therefore, denotes a right line passing through a fixed point.

Ex. 4. If a line be such that the sum of the perpendiculars let fall on it from a number of fixed points, each multiplied by a constant, may = 0, it will pass through a fixed point.

Let the equation of the line be

$$x \cos a + y \sin a - p = 0,$$

then the perpendicular on it from $x'y'$ is

$$x' \cos a + y' \sin a - p,$$

and the conditions of the problem give us

$$m'(x' \cos a + y' \sin a - p) + m''(x'' \cos a + y'' \sin a - p) + m'''(x''' \cos a + y''' \sin a - p) + \&c. = 0,$$

or, using the abbreviations $\Sigma(mx')$ for the sum* of the mx, that is,

$$m'x' + m''x'' + m'''x''' + \&c.,$$

and in like manner $\Sigma(my')$ for

$$m'y' + m''y'' + m'''y''' + \&c.,$$

and $\Sigma(m)$ for the sum of the m's or

$$m' + m'' + m''' + \&c.$$

We may write the preceding equation

$$\Sigma(mx') \cos a + \Sigma(my') \sin a - p\Sigma(m) = 0.$$

Substituting in the original equation the value of p, hence obtained, we get for the equation of the moveable line

$$x\Sigma(m) \cos a + y\Sigma(m) \sin a - \Sigma(mx') \cos a - \Sigma(my') \sin a = 0,$$

or

$$x\Sigma(m) - \Sigma(mx') + \{y\Sigma(m) - \Sigma(my')\} \tan a = 0.$$

Now as this equation involves the indeterminate $\tan a$ in the first degree, the line passes through the fixed point determined by the equations

$$x\Sigma(m) - \Sigma(mx') = 0, \quad \text{and } y\Sigma(m) - \Sigma(my') = 0,$$

or, writing at full length,

$$x = \frac{m'x' + m''x'' + m'''x''' + \&c.}{m' + m'' + m''' + \&c.} \qquad y = \frac{m'y' + m''y'' + m'''y''' + \&c.}{m' + m'' + m''' + \&c.}$$

This point has sometimes been called the *centre of mean position* of the given points.

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* By sum we mean the *algebraic* sum, for any of the quantities m', m'', &c., may be negative.

50. If the equation of any line involve the co-ordinates of a certain point in the first degree, thus,

$$(Ax' + By' + C)\,x + (A'x' + B'y' + C')\,y + (A''x' + B''y' + C'') = 0.$$

Then if the point $x'y'$ move along a right line, the line whose equation has just been written will always pass through a fixed point. For, suppose the point always to lie on the line

$$Lx' + My' + N = 0,$$

then if, by the help of this relation, we eliminate x' from the given equation, the indeterminate y' will remain in it of the first degree, therefore the line will pass through a fixed point.

Or, again, *if the coefficients in the equation* $Ax + By + C = 0$, *be connected by the relation* $aA + bB + cC = 0$ (*where* a, b, c *are constant and* A, B, C *may vary*) *the line represented by this equation will always pass through a fixed point.*

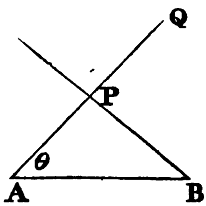
For by the help of the given relation we can eliminate C and write the equation

$$(cx - a)\,A + (cy - b)\,B = 0,$$

a right line passing through the point $\left(x = \frac{a}{c},\quad y = \frac{b}{c}\right)$.

51. *Polar co-ordinates.*—It is, in general, convenient to use this method, if the question be to find the locus of the extremities of lines drawn through a fixed point according to any given law.

Ex. 1. A and B are two fixed points; draw through B any line, and let fall on it a perpendicular from A, AP; produce AP so that the rectangle AP · AQ may be constant: to find the locus of the point Q.

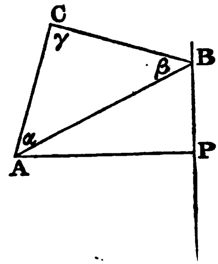


Take A for the pole, and AB for the fixed axis, then AQ is our radius vector, designated by ρ, and the angle QAB = θ, and our object is to find the relation existing between ρ and θ. Let us call the constant length AB = c, and from the right-angled triangle APB we have AP = $c\cos\theta$, but AP · AQ = const = k^2, therefore,

$$\rho c\cos\theta = k^2, \text{ or } \rho\cos\theta = \frac{k^2}{c};$$

but we have seen (Art. 44) that this is the equation of a right line perpendicular to AB, and at a distance from A = $\frac{k^2}{c}$.

Ex. 2. Given the angles of a triangle; one vertex A is fixed, another B moves along a fixed right line: to find the locus of the third.



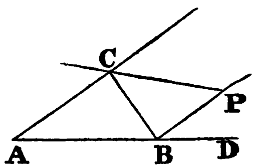
Take the fixed vertex A for pole, and AP perpendicular to the fixed line for axis, then AC = ρ, CAP = θ. Now since the angles of ABC are given, AB is in a fixed ratio to AC (= mAC) and BAP = $\theta - \alpha$; but AP = AB cos BAP; therefore, if we call AP, a, we have

$$m\rho\cos(\theta - \alpha) = a,$$

which (Art. 44) is the equation of a right line, making an angle α with the given line, and at a distance from

$$A = \frac{a}{m}.$$

Ex. 8. Given base and sum of sides of a triangle, if at either extremity of the base B a perpendicular be erected to the conterminous side BC: to find the locus of P the point where it meets the external bisector of vertical angle CP.



Let us take the point B for our pole, then BP will be our radius vector ρ; and let us take the base produced for our fixed axis, then PBD = θ, and our object is to express ρ in terms of θ. Let us designate the sides and opposite angles of the triangle a, b, c, A, B, C, then it is easy to see, that the angle BCP = $90° - \frac{1}{2}C$, and from the triangle PCB, that $a = \rho\tan\frac{1}{2}C$. Hence it is evident, that if we could express a and $\tan\frac{1}{2}C$ in terms of θ, we could express ρ in terms of θ. Now from the triangle ABC we have

$$b^2 = a^2 + c^2 - 2ac\cos B,$$

but if the given sum of sides be m, we may substitute for b, $m - a$; and cos B plainly $= \sin\theta$; hence

$$m^2 - 2am + a^2 = a^2 + c^2 - 2ac\sin\theta,$$

and

$$a = \frac{m^2 - c^2}{2(m - c\sin\theta)}.$$

Thus we have expressed a in terms of θ and constants, and it only remains to find an expression for $\tan\frac{1}{2}C$.

Now
$$\tan\tfrac{1}{2}C = \frac{b\sin C}{b(1 + \cos C)}.$$

But $\quad b\sin C = c\sin B = c\cos\theta$; and $b\cos C = a - c\cos B = a - c\sin\theta$.

Hence
$$\tan\tfrac{1}{2}C = \frac{c\cos\theta}{m - c\sin\theta}.$$

We are now able to express ρ in terms of θ, for, substitute in the equation $a = \rho\tan\frac{1}{2}C$ the values we have found for a and $\tan\frac{1}{2}C$, and we get

$$\frac{m^2 - c^2}{2(m - c\sin\theta)} = \frac{\rho c\cos\theta}{(m - c\sin\theta)}, \quad \text{or } \rho\cos\theta = \frac{m^2 - c^2}{2c}.$$

Hence the locus is a line perpendicular to the base of the triangle at a distance from

$$B = \frac{m^2 - c^2}{2c}.$$

The student may exercise himself with the corresponding locus, if CP had been the *internal* bisector, and if the *difference* of sides had been given.

Ex. 4. Given n fixed right lines and a fixed point O; if through this point any radius vector be drawn meeting the right lines in the points $r_1, r_2, r_3 \ldots r_n$, and on this a point R be taken such that $\frac{n}{OR} = \frac{1}{Or_1} + \frac{1}{Or_2} + \frac{1}{Or_3} + \ldots \frac{1}{Or_n}$, to find the locus of R.

Let the equations of the right lines be

$$\rho \cos(\theta - \alpha) = p_1; \ \rho \cos(\theta - \beta) = p_2, \text{ \&c.}$$

Hence it is easy to see that the equation of the locus is

$$\frac{n}{\rho} = \frac{\cos(\theta - \alpha)}{p_1} + \frac{\cos(\theta - \beta)}{p_2} + \text{\&c.}$$

the equation of a right line (Art. 44). This theorem is only a particular case of a general one which we shall prove afterwards.

* CHAPTER IV.

APPLICATION OF ABRIDGED NOTATION TO THE EQUATION OF THE RIGHT LINE.

52. We have seen (Art. 36) that the line

$$(x \cos\alpha + y \sin\alpha - p) - k(x \cos\beta + y \sin\beta - p') = 0$$

denotes a line passing through the intersection of the lines

$$x \cos\alpha + y \sin\alpha - p = 0, \quad x \cos\beta + y \sin\beta - p' = 0.$$

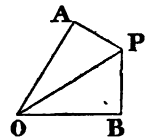
We shall often find it convenient to use abbreviations for these quantities. Let us call

$$x \cos\alpha + y \sin\alpha - p, \ \alpha: \quad x \cos\beta + y \sin\beta - p', \ \beta.$$

Then the theorem just stated may be more briefly expressed, the equation $\alpha - k\beta = 0$, denotes a line passing through the intersection of the two lines denoted by $\alpha = 0$, $\beta = 0$. We shall for brevity call these the lines α, β, and their point of intersection the point $\alpha\beta$. We shall, too, have occasion often to use abbreviations for the equations of lines in the form $Ax + By + C = 0$. We shall in these cases make use of Roman letters, reserving the letters of the Greek alphabet to intimate that the equation is in the form

$$x \cos\alpha + y \sin\alpha - p = 0.$$

53. We proceed to examine the meaning of the coefficient k in the equation $\alpha - k\beta = 0$. We saw (Art. 27) that the quantity α (that is, $x\cos\alpha + y\sin\alpha - p$) denoted the length of the perpendicular let fall from any point xy, on the line OA (which we suppose represented by α). Similarly, that β is the length of the perpendicular from the point xy, on the line OB, represented by β. Hence the equation



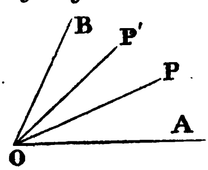
$$\alpha - k\beta = 0, \quad \left(\text{or } \frac{\alpha}{\beta} = k\right),$$

asserts, that if from any point of the locus represented by it, perpendiculars be let fall on the lines OA, OB, the ratio of these perpendiculars, that is, $\frac{PA}{PB}$, will be constant, and $= k$. Hence the locus represented by $\alpha - k\beta = 0$ is a right line through O, and

$$k = \frac{PA}{PB}, \quad \text{or} = \frac{\sin POA}{\sin POB}.$$

It follows from the conventions concerning signs (Art. 27) that $\alpha + k\beta = 0$ denotes a right line dividing *externally* the angle AOB into parts such that $\frac{\sin POA}{\sin POB} = k$. It is of course assumed in what we have said that the perpendiculars PA, PB are those which we agree to consider positive; those on the opposite sides of α, β being regarded as negative.

54. The reader is probably already acquainted with the following fundamental geometrical theorem:—"*If a pencil of four right lines meeting in a point* O *be intersected by any transverse right line in the four points* A, P, P′, B, *then the ratio* $\frac{AP \cdot P'B}{AP' \cdot PB}$ *is constant, no matter how the transverse line be drawn.*" This ratio is called the *anharmonic ratio* of the pencil. In fact, let the perpendicular from O on the transverse line $= p$: then $p \cdot AP = OA \cdot OP \cdot \sin AOP$ (both being double the area of the triangle AOP); $p \cdot P'B = OP' \cdot OB \sin P'OB$; $p \cdot AP' = OA \cdot OP' \sin AOP'$; $p \cdot PB = OP \cdot OB \cdot \sin POB$; hence



$$p^2 \cdot AP \cdot P'B = OA \cdot OP \cdot OP' \cdot OB \cdot \sin AOP \cdot \sin P'OB;$$
$$p^2 \cdot AP' \cdot PB = OA \cdot OP' \cdot OP \cdot OB \cdot \sin AOP' \cdot \sin POB;$$

$$\frac{AP \cdot P'B}{AP' \cdot PB} = \frac{\sin AOP \cdot \sin P'OB}{\sin AOP' \cdot \sin POB};$$

but the latter is a constant quantity, independent of the position of the transverse line.

55. If $\alpha - k\beta = 0$, $\alpha - k'\beta = 0$, be the equations of two lines, then $\frac{k}{k'}$ will be the *anharmonic ratio* of the pencil formed by the four lines α, β, $\alpha - k\beta$, $\alpha - k'\beta$, for

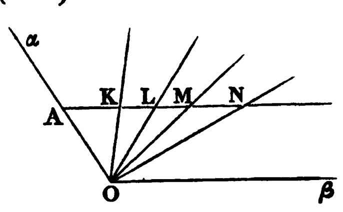
$$k = \frac{\sin AOP}{\sin POB}, \quad k' = \frac{\sin AOP'}{\sin P'OB},$$

$$\therefore \frac{k}{k'} = \frac{\sin AOP \cdot \sin P'OB}{\sin AOP' \cdot \sin POB},$$

but this is the anharmonic ratio of the pencil.

The pencil is a *harmonic* pencil when $\frac{k}{k'} = -1$, for then the angle AOB is divided internally and externally into parts whose sines are in the same ratio. Hence we have the important theorem, *two lines whose equations are $\alpha - k\beta = 0$, $\alpha + k\beta = 0$, form with α, β a harmonic pencil.*

56. In general the anharmonic ratio of four lines $\alpha - k\beta$, $\alpha - l\beta$, $\alpha - m\beta$, $\alpha - n\beta$, is $\frac{(n-l)(m-k)}{(n-m)(l-k)}$. For let the pencil be cut by any parallel to β in the four points K, L, M, N, and the ratio is $\frac{NL \cdot MK}{NM \cdot LK}$. But since β has the same value for each of these four points, the perpendiculars from these



points on α are (by virtue of the equations of the lines) proportional to k, l, m, n; and AK, AL, AM, AN, are evidently proportional to these perpendiculars; hence NL is proportional to $n - l$; MK to $m - k$; NM to $n - m$; and LK to $l - k$.

Ex. 1. To express in this notation the proof that the three bisectors of the angles of a triangle meet in a point.

The equations of three bisectors are obviously (see Arts. 43, 53) $\alpha - \beta = 0$, $\beta - \gamma = 0$, $\gamma - \alpha = 0$, which, added together, vanish identically.

Ex. 2. Any two of the external bisectors of the angles of a triangle meet on the third internal bisector.

Attending to the convention about signs, it is easy to see that the equations of two external bisectors are $\alpha + \beta = 0$, $\alpha + \gamma = 0$, and subtracting one from the other we get $\beta - \gamma = 0$, the equation of the third internal bisector.

Ex. 3. The three perpendiculars of a triangle meet in a point.

Let the angles opposite to the sides α, β, γ, be A, B, C, respectively. Then since the perpendicular divides any angle of the triangle into parts, which are the complements of the remaining two angles, therefore (by Art. 53) their equations are

$$\alpha \cos A - \beta \cos B = 0, \quad \beta \cos B - \gamma \cos C = 0, \quad \gamma \cos C - \alpha \cos A = 0,$$

which obviously meet in a point.

Ex. 4. The three bisectors of the sides of a triangle meet in a point.

The ratio of the perpendiculars on the sides from the point where the bisector meets base plainly is $\sin A : \sin B$. Hence the equations of the three bisectors are

$$\alpha \sin A - \beta \sin B = 0, \quad \beta \sin B - \gamma \sin C = 0, \quad \gamma \sin C - \alpha \sin A = 0.$$

Ex. 5. To form the equation of a perpendicular to the base of a triangle at its extremity. *Ans.* $\alpha + \beta \cos C = 0$.

Ex. 6. If there be two triangles such that the perpendiculars from the vertices of one on the sides of the other meet in a point, then, *vice versâ*, the perpendiculars from the vertices of the second on the sides of the first will meet in a point.

Let the sides be $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, and let us denote by $(\alpha\beta)$ the angle between α and β.

Then the equation of the perpendicular

$$\text{from } \alpha\beta \text{ on } \gamma' \text{ is } \alpha \cos(\beta\gamma') - \beta \cos(\alpha\gamma') = 0,$$
$$\text{from } \beta\gamma \text{ on } \alpha' \text{ is } \beta \cos(\gamma\alpha') - \gamma \cos(\beta\alpha') = 0,$$
$$\text{from } \alpha\gamma \text{ on } \beta' \text{ is } \gamma \cos(\alpha\beta') - \alpha \cos(\gamma\beta') = 0.$$

The condition that these should meet in a point is found by eliminating β between the first two, and examining whether the resulting equation coincides with the third. It is

$$\cos(\alpha\beta') \cos(\beta\gamma') \cos(\gamma\alpha') = \cos(\alpha'\beta) \cos(\beta'\gamma) \cos(\gamma'\alpha).$$

But the symmetry of this equation shows that this is also the condition that the perpendiculars from the vertices of the second triangle on the sides of the first should meet in a point.

57. The lines $\alpha - k\beta = 0$, and $k\alpha - \beta = 0$, are plainly such that one makes the same angle with the line α which the other makes with the line β, and are therefore equally inclined to the bisector $\alpha - \beta$.

Ex. If through the vertices of a triangle there be drawn any three lines meeting in a point, the three lines drawn through the same angles, equally inclined to the bisectors of the angles, will also meet in a point.

Let the sides of the triangle be α, β, γ, and let the equations of the first three lines be

$$l\alpha - m\beta = 0, \quad m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0,$$

which, by the principle of Art. 36, are the equations of three lines meeting in a point, and which obviously pass through the points $\alpha\beta$, $\beta\gamma$, and $\gamma\alpha$. Now, from this Article, the equations of the second three lines will be

$$\frac{\alpha}{l} - \frac{\beta}{m} = 0, \quad \frac{\beta}{m} - \frac{\gamma}{n} = 0, \quad \text{and } \frac{\gamma}{n} - \frac{\alpha}{l} = 0,$$

which (by Art. 36) must also meet in a point.

58. *Given the equations of three right lines, forming a triangle,* $L = 0$, $M = 0$, $N = 0$, *the equation of every right line can be thrown into the form* $lL + mM + nN = 0$.

Let $L = Ax + By + C$, $M = A'x + B'y + C'$, $N = A''x + B''y + C''$, then in order to throw the equation of any fourth line

$$ax + by + c = 0$$

into the form $lL + mM + nN = 0$, we should have three equations to determine three unknown quantities, namely,

$$lA + mA' + nA'' = a, \quad lB + mB' + nB'' = b, \quad lC + mC' + nC'' = c;$$

$$\text{whence } l = \frac{a(B'C'' - B''C') + b(C'A'' - A'C'') + c(A'B'' - B'A'')}{A(B'C'' - B''C') + B(C'A'' - A'C'') + C(A'B'' - B'A'')},$$

with corresponding values for m and n. It is plain (Art. 34) that if the three right lines L, M, N meet in a point, the theorem of this article would not be true, since the values of l, m, n would then become infinite. We have used in this article equations of the form $Ax + By + C = 0$, because it was with regard to equations in this form that the condition for three lines meeting in a point (Art. 34) was given; but had the equations been given in the form $x \cos\alpha + y \sin\beta = p$, it would of course be equally true that the equation of any fourth line can be thrown into the form

$$l\alpha + m\beta + n\gamma = 0.$$

59. *To write in the form* $l\alpha + m\beta + n\gamma = 0$ *the equation of the line joining two given points* $x'y'$, $x''y''$.

Let α' denote the quantity $x' \cos\alpha + y' \sin\alpha - p$ found by substituting the co-ordinates $x'y'$ in the equation $\alpha = 0$, &c. Then the condition that the co-ordinates $x'y'$ shall satisfy the equation $l\alpha + m\beta + n\gamma = 0$, may be written

$$l\alpha' + m\beta' + n\gamma' = 0.$$

Similarly we have $$l\alpha'' + m\beta'' + n\gamma'' = 0.$$

Solving for $\frac{l}{n}, \frac{m}{n}$, from these two equations, and substituting in the given form, we obtain for the equation of the line joining the two points

$$\alpha (\beta'\gamma'' - \gamma'\beta'') + \beta (\gamma'\alpha'' - \gamma''\alpha') + \gamma (\alpha'\beta'' - \alpha''\beta') = 0.$$

This article obviously proves, independently of the last article, that the equation of every right line can be thrown into the form $l\alpha + m\beta + n\gamma = 0$.

Ex. 1. To find the equation of the line joining two points given by the equations

$$(k\alpha - \beta = 0,\ l\alpha - \gamma = 0),\quad (k'\alpha - \beta = 0,\ l'\alpha - \gamma = 0).$$

From the first set of equations, $\beta' = k\alpha'$, $\gamma' = l\alpha'$; from the second, $\beta'' = k'\alpha''$, $\gamma'' = l'\alpha''$; substituting these values in the equation of this article, it becomes divisible by $\alpha'\alpha''$, and we have

$$\alpha (kl' - k'l) + \beta (l - l') + \gamma (k' - k) = 0.$$

Otherwise thus: The required equation must (Art. 36) be capable of being thrown into either of the forms

$$(k\alpha - \beta) + A (l\alpha - \gamma) = 0,\quad (k'\alpha - \beta) + A' (l'\alpha - \gamma) = 0.$$

These equations must be identical; the coefficient of β is the same in both; and equating the coefficients of α and γ we get

$$A = A' = -\frac{k - k'}{l - l'};$$

and substituting this value we get for the required line the same equation as before.

Ex. 2. To form the equation of the line joining the intersection of the perpendiculars of a triangle to the intersection of the bisectors of sides.

We have (Art. 57, Ex. 3, 4)

$$\beta' = \frac{\alpha' \cos A}{\cos B},\ \gamma' = \frac{\alpha' \cos A}{\cos C};\quad \beta'' = \frac{\alpha'' \sin A}{\sin B},\ \gamma'' = \frac{\alpha'' \sin A}{\sin C};$$

substituting these values, the equation becomes

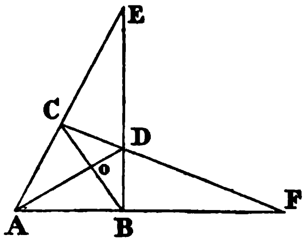
$$\alpha \sin 2A \sin (B - C) + \beta \sin 2B \sin (C - A) + \gamma \sin 2C \sin (A - B) = 0.$$

60. The following examples will further illustrate the general principle, that being given the equations of three lines, $\alpha = 0$,

$\beta = 0$, $\gamma = 0$, it is possible to express the equation of any other line in terms of these.

Ex. 1. To deduce analytically the harmonic properties of a complete quadrilateral.

Let the equation of AC be $\alpha = 0$; of AB, $\beta = 0$; of BD, $\gamma = 0$; of AD, $l\alpha - m\beta = 0$; and of BC, $m\beta - n\gamma = 0$. Then we are able to express in terms of these quantities the equations of all the other lines of the figure.



For instance, the equation of CD is

$$l\alpha - m\beta + n\gamma = 0,$$

for it is the equation of a right line passing through the intersection of $l\alpha - m\beta$ and γ, that is, the point D, and of α and $m\beta - n\gamma$, that is, the point C. Again, $l\alpha - n\gamma = 0$ is the equation of OE, for it passes through $\alpha\gamma$ or E, and it also passes through the intersection of AD and BC, since it is $= (l\alpha - m\beta) + (m\beta - n\gamma)$.

EF joins the point $\alpha\gamma$ to the point $(l\alpha - m\beta + n\gamma, \beta)$, and its equation will be found to be $l\alpha + n\gamma = 0$.

From Art. 55 it appears, that the four lines EA, EO, EB, and EF, form an harmonic pencil, for their equations have been shown to be

$$\alpha = 0,\ \gamma = 0, \text{ and } l\alpha \pm n\gamma = 0.$$

Again, the equation of FO, which joins the points $(l\alpha + n\gamma, \beta)$ and $(l\alpha - m\beta, m\beta - n\gamma)$, is

$$l\alpha - 2m\beta + n\gamma = 0.$$

Hence (Art. 55) the four lines FE, FC, FO, and FB, are an harmonic pencil, for their equations are

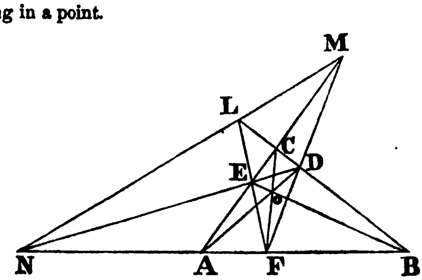
$$l\alpha - m\beta + n\gamma = 0,\ \beta = 0, \quad \text{and } l\alpha - m\beta + n\gamma \pm m\beta = 0.$$

Again, OC, OE, OD, OF, are an harmonic pencil, for their equations are

$$l\alpha - m\beta = 0, \quad m\beta - n\gamma = 0, \quad \text{and } l\alpha - m\beta \pm (m\beta - n\gamma) = 0.$$

Ex. 2. To discuss the properties of the system of lines formed by drawing through the angles of a triangle three lines meeting in a point.

Let the equation of AB be $\gamma = 0$; of AC $\beta = 0$; of BC $\alpha = 0$; then we shall assume for OC $l\alpha - m\beta$; for OA $m\beta - n\gamma$; and for OB $n\gamma - l\alpha$ (as in Art. 57); these three lines meet in a point, since these three quantities added together are $= 0$.



Now we can form the equations of all the other lines in the figure.

For example, the equation of EF is

$$m\beta + n\gamma - l\alpha = 0,$$

since it passes through the points $(\beta, n\gamma - l\alpha)$ or E, and $(\gamma, m\beta - l\alpha)$ or F.

In like manner, the equation of DF is

$$l\alpha - m\beta + n\gamma = 0,$$

and of DE

$$l\alpha + m\beta - n\gamma = 0.$$

Now we can prove, that the three points L, M, N are all in one right line, whose equation is

$$l\alpha + m\beta + n\gamma = 0,$$

for this line passes through the points $(l\alpha + m\beta - n\gamma, \gamma)$ or N, $(l\alpha - m\beta + n\gamma, \beta)$ or M, and $(m\beta + n\gamma - l\alpha, \alpha)$ or L.

The equation of CN is

$$l\alpha + m\beta = 0,$$

for this is evidently a line through (α, β) or C, and it also passes through N, since it $= (l\alpha + m\beta + n\gamma) - n\gamma$.

Hence BN is cut harmonically, for the equations of the four lines CN, CA, CF, CB are,

$$\alpha = 0, \quad \beta = 0, \quad l\alpha - m\beta = 0, \quad l\alpha + m\beta = 0.$$

We shall often afterwards meet with equations of the form discussed in this example.

Ex. 3. If two triangles be such that the intersections of the corresponding sides lie on the same right line, the lines joining the corresponding vertices meet in a point.

Let the sides of the first triangle be α, β, γ; and let the line on which the corresponding sides meet be $l\alpha + m\beta + n\gamma$: then the equation of a line through the intersection of this with α must be of the form $l'\alpha + m\beta + n\gamma = 0$, and similarly those of the other two sides of the second triangle are

$$l\alpha + m'\beta + n\gamma = 0, \qquad l\alpha + m\beta + n'\gamma = 0.$$

But subtracting successively each of the last three equations from another, we get for the equations of the lines joining corresponding vertices

$$(l - l')\,\alpha = (m - m')\beta, \quad (m - m')\beta = (n - n')\gamma, \quad (n - n')\gamma = (l - l')\alpha.$$

which obviously meet in a point.

61. We have seen that having assumed any three right lines, we can express the equation of any right line in the form

$$A\alpha + B\beta + C\gamma = 0,$$

and so solve any problem by a set of equations expressed in terms of α, β, γ, without any direct mention of x and y. This suggests a new way of looking at the principle laid down in Art. 58, &c. Instead of regarding α as a mere abbreviation for the quantity $x \cos\alpha + y \sin\alpha - p$, we may look upon it as simply denoting the length of the perpendicular from a point on the line α. We may imagine a system of *trilinear co-ordinates* in which the position of a point is defined by its distances from three fixed lines, and in which the position of any right line is defined by a homogeneous equation between these distances of the form

$$A\alpha + B\beta + C\gamma = 0.$$

The advantage of trilinear co-ordinates is, that whereas in Cartesian (or x and y) co-ordinates the utmost simplification we can introduce is by choosing *two* of the most remarkable lines in the figure for axes of co-ordinates, we can in trilinear co-ordinates obtain still more simple expressions by choosing *three* of the most remarkable lines for the lines of reference α, β, γ. The reader will compare the brevity of the expressions in Art. 56 with those corresponding in Chap. II.

62. *To reduce a non-homogeneous equation (for example,* $\alpha = 3$*) to the homogeneous form* $l\alpha + m\beta + n\gamma = 0$.

Let a, b, c be the lengths of the sides of the triangle formed by the three lines of reference; then since α denotes the length of the perpendicular from any point O on α, $a\alpha$ is double the area of the triangle OBC; in like manner $b\beta$ is double OAC; and $c\gamma$ double OAB; therefore, no matter where the point O be taken, the quantity $a\alpha + b\beta + c\gamma$ is always constant, and equal double the area of the triangle ABC. The reader may suppose that this is only true if the point O be taken *within* the triangle; but he is to remember that if the point O were on the other side of any of the lines of reference (α), we must give a negative sign to that perpendicular, and the quantity $a\alpha + b\beta + c\gamma$ would then

$$= 2(OAC + OAB - OBC),$$

that is, still = twice the area of the triangle. If, then, we call the double area M, the equation $\alpha = 3$ may be written

$$M\alpha = 3(a\alpha + b\beta + c\gamma),$$

which is the required form. If A, B, C be the angles (opposite α, β, γ respectively) of the triangle formed by the lines of reference, it is plain that $\alpha \sin A + \beta \sin B + \gamma \sin C$ is also constant, being $= \dfrac{M \sin A}{a}$.

63. *To express in trilinear co-ordinates the equation of the parallel to a given line* $A\alpha + B\beta + C\gamma = 0$.

In Cartesian co-ordinates two lines are parallel if their equations $Ax + By + C = 0$, $Ax + By + C' = 0$ differ only by a constant. It follows, then, that the equation

$$A\alpha + B\beta + C\gamma + k(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0$$

denotes a line parallel to $A\alpha + B\beta + C\gamma = 0$, since the two equations differ only by a constant.

Ex. 1. To find the equation of a parallel to the base of a triangle drawn through the vertex. *Ans.* $\alpha \sin A + \beta \sin B = 0$.

For this, obviously, is a line through $\alpha\beta$, and writing the equation in the form

$$\gamma \sin C - (\alpha \sin A + \beta \sin B + \gamma \sin C) = 0,$$

it appears that it differs only by a constant from $\gamma = 0$.

We see, also, that the parallel $\alpha \sin A + \beta \sin B$, and the bisector of the base $\alpha \sin A - \beta \sin B$ form a harmonic pencil with α, β (Art. 55).

Ex. 2. The line joining the middle points of sides of a triangle is parallel to the base.

Form (Art. 59) the equation of the line joining ($\beta \sin B - \gamma \sin C'$, α), ($\alpha \sin A - \gamma \sin C$, β), when we get $\alpha \sin A + \beta \sin B - \gamma \sin C = 0$, which, by this article, is parallel to γ.

Ex. 3. To find the equation of a perpendicular to any side at its middle point.

This is to draw a parallel to the line $\alpha \cos A - \beta \cos B = 0$ through the point ($\alpha \sin A - \beta \sin B$, γ). *Ans.* $\alpha \sin A - \beta \sin B + \gamma \sin(A - B) = 0$.

Ex. 4. The three such perpendiculars meet in a point. Their equations vanish when multiplied, respectively, by $\sin 2C$, $\sin 2B$, $\sin 2A$, and added together. The equations of the lines joining their intersection to the vertices will be found to be $\frac{\alpha}{\cos A} = \frac{\beta}{\cos B}$ &c.

Ex. 5. Verify that this point lies on the line whose equation is given Art. 59, Ex. 2.

Ex. 6. Find the length of the perpendicular from a point $\alpha'\beta'\gamma'$ on $A\alpha + B\beta + C\gamma = 0$.

Ans. $$\frac{A\alpha' + B\beta' + C\gamma'}{\sqrt{(A^2 + B^2 + C^2 - 2AB \cos C - 2BC \cos A - 2CA \cos B)}}.$$

64. *To examine what line is denoted by the equation*

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0.$$

This equation is included in the general form of an equation of a right line, but we have seen that the co-ordinates of any finite point render the quantity $\alpha \sin A + \beta \sin B + \gamma \sin C$ = a certain constant, and never = 0. Let us return, however, to the general equation of the right line, $Ax + By + C = 0$. We saw that the intercepts which this line cuts off on the axes are $-\frac{C}{A}$, $-\frac{C}{B}$; consequently, the smaller A and B become, the greater will be the intercepts on the axes, and, therefore, the more remote the line represented by $Ax + By + C = 0$. Let A and B be both = 0, then the intercepts become infinite, and the line is altogether situated at an infinite distance from the origin. Hence we arrive at the conclusion, that the paradoxical equation $C = 0$, a constant = 0, (and therefore, likewise, $\alpha \sin A + \beta \sin B + \gamma \sin C = 0$), represents a right line situated altogether at an infinite distance from the origin.

65. We saw (Art. 63) that a line parallel to the line $\alpha = 0$ has an equation of the form $\alpha + C = 0$. Now the last Article shows that this is only an additional illustration of the principle of Art. 36. For, a parallel to α may be considered as intersecting it at an infinite distance, but (Art. 36) an equation of the form $\alpha + C = 0$ represents a line through the intersection of the lines $\alpha = 0$, $C = 0$, or (Art. 64) through the intersection of the line α with the line at infinity.

66. We have to add, in conclusion, that Cartesian co-ordinates are only a particular case of trilinear. There appears, at first sight, to be an essential difference between them, since trilinear equations are always homogeneous, while we are accustomed to speak of Cartesian equations as containing an absolute term, terms of the first degree, terms of the second degree, &c. A little reflection, however, will show that this difference is only apparent, and that Cartesian equations must be equally homogeneous in reality, though not in form. The equation $x = 3$, for example, must mean that the line x is equal to three feet or three inches, or, in short, to three times some linear unit; the equation $xy = 9$ must mean that the rectangle xy is equal to nine *square* feet or *square* inches, or *to nine squares* of some linear unit; and so on.

If we wish to have our equation homogeneous in form as well as in reality, we may denote our linear unit by z, and write the equation of the right line

$$Ax + By + Cz = 0.$$

Comparing this with the equation

$$A\alpha + B\beta + C\gamma = 0;$$

and remembering (Art. 64) that when a line is at an infinite distance its equation takes the form $z = 0$, we learn that *equations in Cartesian co-ordinates are only the particular form assumed by trilinear equations when two of the lines of reference are what are called the co-ordinate axes, while the third is at an infinite distance.*

CHAPTER V.

EQUATIONS ABOVE THE FIRST DEGREE REPRESENTING RIGHT LINES.

67. Before proceeding to speak of the *curves* represented by equations above the first degree, we shall examine some cases where these equations represent *right lines.*

If we take any number of equations, L = 0, M = 0, N = 0, &c., and multiply them together, the compound equation LMN, &c. = 0 will represent the aggregate of all the lines represented by its factors; for it will be satisfied by the values of the co-ordinates which make any of its factors = 0. Conversely, *if an equation of any degree can be resolved into others of lower degrees, it will represent the aggregate of all the loci represented by its different factors.* If, then, an equation of the n^{th} degree can be resolved into n factors of the first degree, it will represent n right lines.

68. *A homogeneous equation, of the n^{th} degree between the variables, denotes n right lines passing through the origin.*

Let the equation be

$$x^n - px^{n-1}y + qx^{n-2}y^2 - \&c. \cdots + ty^n = 0.$$

Divide by y^n, and we get

$$\left(\frac{x}{y}\right)^n - p\left(\frac{x}{y}\right)^{n-1} + q\left(\frac{x}{y}\right)^{n-2} - \&c. = 0.$$

Let a, b, c, &c., be the n roots of this equation, then it is resolvable into the factors

$$\left(\frac{x}{y} - a\right)\left(\frac{x}{y} - b\right)\left(\frac{x}{y} - c\right) \&c. = 0,$$

and the original equation is therefore resolvable into the factors

$$(x - ay)(x - by)(x - cy) \&c. = 0.$$

It accordingly represents the n right lines $x - ay = 0$, &c., all of which pass through the origin. Thus, then, in particular, the homogeneous equation

$$x^2 - pxy + qy^2 = 0$$

represents the two right lines $x - ay = 0$, $x - by = 0$, where a and b are the two roots of the quadratic

$$\left(\frac{x}{y}\right)^2 - p\left(\frac{x}{y}\right) + q = 0.$$

It is proved, in like manner, that the equation

$$(x-a)^n - p(x-a)^{n-1}(y-b) + q(x-a)^{n-2}(y-b)^2 \cdots + t(y-b)^n = 0$$

denotes n right lines passing through the point (a, b).

Ex. 1. What locus is represented by the equation $xy = 0$?

Ans. The two axes, since the equation is satisfied by either of the suppositions $x = 0$, $y = 0$.

Ex. 2. What locus is represented by $x^2 - y^2 = 0$?

Ans. The bisectors of the angles between the axes, $x \pm y = 0$ (see Art. 43).

Ex. 3. What locus is represented by $x^2 - 5xy + 6y^2 = 0$?

Ans. $x - 2y = 0$, $x - 3y = 0$.

Ex. 4. What locus is represented by $x^2 - 2xy \sec\theta + y^2 = 0$?

Ans. $x = y \tan(45° \pm \frac{1}{2}\theta)$.

Ex. 5. What lines are represented by $x^2 - 2xy \tan\theta - y^2 = 0$?

Ex. 6. What lines are represented by $x^3 - 6x^2y + 11xy^2 - 6y^3 = 0$?

69. Let us examine more minutely the three cases of the solution of the equation $x^2 - pxy + qy^2 = 0$, according as its roots are real and unequal, real and equal, or both imaginary.

The first case presents no difficulty: a and b are the tangents of the angles which the lines make with the axis of y (the axes being supposed rectangular), p is therefore the sum of those tangents, and q their product.

In the second case, when $a = b$, it was once usual among geometers to say that the equation represented but one right line $(x - ay = 0)$. We shall find, however, many advantages in making the language of geometry correspond exactly to that of algebra, and as we do not say that the equation above has *only one* root, but that it has *two equal* roots, so we shall not say that it represents *only one* line, but that it represents *two coincident* right lines.

Thirdly, let the roots be both imaginary. In this case no real co-ordinates can be found to satisfy the equation, except the co-ordinates of the origin $x = 0$, $y = 0$; hence it was usual to say that in this case the equation did not represent right lines, but was the equation of the origin. Now this language appears to us very objectionable, for we saw (Arts. 14, 15) that *two* equations are required to determine any point, hence we are unwilling to acknowledge *any single* equation as the equation of a point. Moreover, we have been hitherto accustomed to find that two *different* equations always had *different* geometrical significations, but here we should have innumerable equations, all purporting to be the equation of the same point; for it is obviously immaterial what the values of p and q are, provided only that they give imaginary values for the roots, that is to say, provided that p^2 be less than $4q$. We think it, therefore, much preferable to make our language correspond exactly to the language of algebra; and as we do not say that the equation above has *no* roots when p^2 is less than $4q$, but that it has two *imaginary* roots, so we shall not say that, in this case, it represents *no* right lines, but that it represents two *imaginary* right lines. In short the equation $x^2 - pxy + qy^2 = 0$ being *always* reducible to the form $(x - ay)(x - by) = 0$, we shall always say that it represents two right lines drawn through the origin; but when a and b are real, we shall say that these lines are real; when a and b are equal, that the lines coincide; and when a and b are imaginary, that the lines are imaginary. It may seem to the student a matter of indifference which mode of speaking we adopt; we shall find, however, as we proceed, that we should lose sight of many important analogies by refusing to adopt the language here recommended.

Similar remarks apply to the equation

$$Ax^2 + Bxy + Cy^2 = 0,$$

which can be reduced to the form $x^2 - pxy + qy^2 = 0$, by dividing by the coefficient of x^2. This equation will always represent two right lines through the origin; these lines will be real if $B^2 - 4AC$ be positive, as at once appears from solving the equation; they will coincide if $B^2 - 4AC = 0$; and they will be imaginary if $B^2 - 4AC$ be negative. So, again, the same language is used if we meet with equal or imaginary roots in the solution of the general homogeneous equation of the n^{th} degree.

70. *To find the angle contained by the lines represented by the equation* $x^2 - pxy + qy^2 = 0$.

Let this equation be equivalent to $(x - ay)(x - by) = 0$, then the tangent of the angle between the lines is (Art. 40) $\frac{a-b}{1+ab}$, but the product of the roots of the given equation $= q$, and their difference $= \sqrt{(p^2 - 4q)}$. Hence

$$\tan\phi = \frac{\sqrt{(p^2-4q)}}{1+q}.$$

If the equation had been given in the form

$$Ax^2 + Bxy + Cy^2 = 0,$$

it will be found that

$$\tan\phi = \frac{\sqrt{(B^2-4AC)}}{A+C}.$$

COR.—The lines will cut at right angles, or $\tan\phi$ will become infinite, if $q = -1$ in the first case, or if $A + C = 0$ in the second.

Ex. Find the angle between the lines

$x^2 + xy - 6y^2 = 0$. *Ans.* 45°.

$x^2 - 2xy\sec\theta + y^2 = 0$. *Ans.* θ.

* If the axes be oblique, we should find, in like manner,

$$\tan\phi = \frac{\sin\omega\sqrt{(B^2-4AC)}}{A+C-B\cos\omega}.$$

71. *To find the equation which will represent the lines bisecting the angles between the lines represented by the equation*

$$Ax^2 + Bxy + Cy^2 = 0.$$

Let these lines be $x - ay = 0$, $x - by = 0$; let the equation of the bisector be $x - \mu y = 0$, and we seek to determine μ. Now (Art. 22) μ is the tangent of the angle made by this bisector with the axis of y, and it is plain that this angle is half the sum of the angles made with this axis by the lines themselves. Equating, therefore, tangent of twice this angle to tangent of sum, we get

$$\frac{2\mu}{1-\mu^2} = \frac{a+b}{1-ab};$$

but, from the theory of equations,

$$a + b = -\frac{B}{A}, \qquad ab = \frac{C}{A};$$

therefore

$$\frac{2\mu}{1-\mu^2} = -\frac{B}{A-C},$$

or

$$\mu^2 - 2\frac{A - C}{B}\mu - 1 = 0.$$

This gives us a quadratic to determine μ, one of whose roots will be the tangent of the angle made with the axis of y by the *internal* bisector of the angle between the lines, and the other the tangent of the angle made by the *external* bisector. We can find the combined equation of both lines by substituting in the last quadratic for μ its value $= \frac{x}{y}$, and we get

$$x^2 - 2\frac{A - C}{B}xy - y^2 = 0,*$$

and the form of this equation shows that the bisectors cut each other at right angles (Art. 70).

The student may also obtain this equation by forming (Art. 43) the equations of the internal and external bisectors of the angle between the lines $x - ay = 0$, $x - by = 0$, and multiplying them together, when he will have

$$\frac{(x - ay)^2}{1 + a^2} = \frac{(x - by)^2}{1 + b^2},$$

and then clearing of fractions, and substituting for $a + b$, and ab their values in terms of A, B, C, the equation already found is obtained.

72. *To find the condition that the general equation of the second degree should represent two right lines.*

Let the general equation be

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0:$$

write it in the form

$$Ax^2 + (By + D)x + Cy^2 + Ey + F = 0,$$

and solving this quadratic for x, the roots are found to be

$$x = -\frac{By + D \pm \sqrt{\{(B^2 - 4AC)y^2 + 2(BD - 2AE)y + D^2 - 4AF\}}}{2A}.$$

* It is remarkable that the roots of this last equation will always be real, even if the roots of the equation $Ax^2 + Bxy + Cy^2 = 0$ be imaginary, which leads to the curious result, that a pair of imaginary lines may have a pair of real lines bisecting the angle between them. It is the existence of such relations between real and imaginary lines which makes the consideration of the latter profitable.

Now this value cannot be reduced to the form $x = my + n$, unless the quantity under the radical be a perfect square. The condition that this should be the case is

$$(B^2 - 4AC)(D^2 - 4AF) = (BD - 2AE)^2,$$

or expanding and dividing by 4A,

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0;$$

which is the required condition that the equation of the second degree should represent right lines.

Ex. 1. Verify that the following equation represents right lines, and find the lines.

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0.$$

Ans. Solving for x as in the text, the lines are found to be

$$x - y - 1 = 0, \quad x - 4y + 2 = 0.$$

Ex. 2. Verify that the following equation represents right lines:

$$(\alpha x + \beta y - r^2)^2 = (\alpha^2 + \beta^2 - r^2)(x^2 + y^2 - r^2).$$

Ex. 3. What lines are represented by the equation

$$x^2 - xy + y^2 - x - y + 1 = 0?$$

Ans. The imaginary lines $x + \theta y + \theta^2 = 0$, $x + \theta^2 y + \theta = 0$, where θ is one of the imaginary cube roots of 1.

Ex. 4. Determine B, so that the following equation may represent right lines:

$$x^2 + Bxy + y^2 - 5x - 7y + 6 = 0.$$

Ans. Substituting these values of the coefficients in the general condition, we get for B the quadratic, $6B^2 - 35B + 50 = 0$, whose roots are $\frac{10}{3}$ and $\frac{5}{2}$.

* 73. The method used in the preceding Article, though the most simple in the case of the equation of the second degree, is not applicable to equations of higher degrees: we therefore give another solution of the same problem. It is required to ascertain whether the given equation of the second degree can be identical with the product of the equations of two right lines

$$(\alpha x + \beta y - 1)(\alpha' x + \beta' y - 1) = 0;$$

multiply out this product, and equate the coefficient of each term to the corresponding coefficient in the general equation of the second degree, having previously divided the latter by F, so as to make the absolute term in each equation = 1. We thus obtain five equations: four of them enable us to determine the four unknown quantities, α, α', β, β', in terms of the coefficients of the general equation; and then these values being substituted in the fifth give the condition required. The five equations actually are

$$\alpha\alpha' = \frac{A}{F}, \quad \alpha + \alpha' = -\frac{D}{F}, \quad \beta\beta' = \frac{C}{F}, \quad \beta + \beta' = -\frac{E}{F}, \quad \alpha\beta' + \alpha'\beta = \frac{B}{F}.$$

From the first four we can at once form two quadratic equations for determining α, α', β, β', as indeed we might have otherwise inferred from the consideration that these quantities are the reciprocals of the intercepts made by the lines on the axes; and that the intercepts made by the locus on the axes are found (by making alternately $x = 0$, $y = 0$, in the general equation) from the equations

$$Ax^2 + Dx + F = 0, \qquad Cy^2 + Ey + F = 0.$$

Now if the locus meet the axes in the points L, L′; M, M′; it is plain that if it represent right lines at all, these must be either the pair LM, L′M′, or else LM′, L′M, whose equations are

$$(\alpha x + \beta y - 1)(\alpha' x + \beta' y - 1) = 0, \text{ or } (\alpha x + \beta' y - 1)(\alpha' x + \beta y - 1) = 0.$$

Multiplying out, we see that $\frac{B}{F}$ might not only have the value given before $\alpha\beta' + \beta\alpha'$, but also might be $\alpha\beta + \alpha'\beta'$. The sum of those quantities

$$= (\alpha + \alpha')(\beta + \beta') = \frac{DE}{F^2},$$

and their product

$$= \alpha\alpha'(\beta^2 + \beta'^2) + \beta\beta'(\alpha^2 + \alpha'^2) = \frac{A}{F}\frac{(E^2 - 2CF)}{F^2} + \frac{C}{F}\frac{(D^2 - 2AF)}{F^2};$$

hence $\frac{B}{F}$ is given by the quadratic

$$\frac{B^2}{F^2} - \frac{DE}{F^2}\cdot\frac{B}{F} + \frac{AE^2 + CD^2 - 4ACF}{F^3} = 0,$$

which, cleared of fractions, is the condition already obtained.

Ex. To determine B so that $x^2 + Bxy + y^2 - 5x - 7y + 6 = 0$ may represent right lines.

The intercepts on the axes are given by the equations

$$x^2 - 5x + 6 = 0, \qquad y^2 - 7y + 6 = 0,$$

whose roots are $x = 2$, $x = 3$; $y = 1$, $y = 6$. Forming, then, the equation of the lines joining the points so found, we see that if the equation represent right lines, it must be of one or other of the forms

$$(x + 2y - 2)(2x + y - 6) = 0, \quad (x + 3y - 3)(3x + y - 6) = 0,$$

whence, multiplying out, B is determined.

* 74. *To find how many conditions must be satisfied in order that the general equation of the n^{th} degree may represent right lines.*

We proceed as in the last Article; we compare the general equation, having first by division made the absolute term = 1, with the product of the n right lines

$$(\alpha x + \beta y - 1)(\alpha' x + \beta' y - 1)(\alpha'' x + \beta'' y - 1) \text{ \&c.} = 0.$$

Let the number of terms in the general equation be N; then from a comparison of coefficients we obtain N − 1 equations (the absolute term being already the same in both); $2n$ of these equations are employed in determining the $2n$ unknown quantities α, α', &c., whose values being substituted in the remaining equations afford $N - 1 - 2n$ conditions. Now if we write the general equation

$$\begin{gathered} A \\ + Bx + Cy \\ + Dx^2 + Exy + Fy^2 \\ + Gx^3 + Hx^2y + Kxy^2 + Ly^3 \\ + \text{\&c.}, \end{gathered}$$

it is plain that the number of terms is the sum of the arithmetic series

$$N = 1 + 2 + 3 + \cdots (n + 1) = \frac{(n+1)(n+2)}{1 \cdot 2};$$

hence

$$N - 1 = \frac{n(n+3)}{1 \cdot 2}; \quad N - 1 - 2n = \frac{n(n-1)}{1 \cdot 2}.$$

* 75. *To find how many conditions must be fulfilled in order that the general equation of the n^{th} degree should represent n right lines, each passing through a given point.*

We should now compare the general equation with the equation

$$\{y - y' - m(x - x')\} \{y - y'' - m'(x - x'')\} \text{ \&c.} = 0.$$

There are now but the n unknown quantities, m, m', &c., to be determined; hence the number of conditions is

$$N - 1 - n = \frac{n(n+1)}{1 \cdot 2}.$$

* 76. *To find the number of conditions which must be fulfilled in order that the general equation may represent n right lines, all passing through the same point.*

We now compare the general equation with

$$\{y - y' - m(x - x')\}\ \{y - y' - m'(x - x')\}\ \&c. = 0.$$

Beside the n unknowns m, m', &c., there are also the two $x'y'$ to be determined; hence the number of conditions

$$= N - 1 - (n + 2) = \frac{n(n+1)}{1 \cdot 2} - 2.$$

CHAPTER VI.

THE CIRCLE.

77. Before we proceed to the general investigation of the curves represented by the general equation of the second degree, it seems desirable that we should examine the equation of the *circle*, which ranks next to that of the right line in simplicity.

To find the equation of the circle whose centre is the point (ab) *and radius is* r.

Expressing (Art. 5) that the distance of any point from the centre is equal to the radius, we at once obtain the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

If the axes be oblique, we have (Art. 6)

$$(x - a)^2 + (y - b)^2 + 2(x - a)(y - b)\cos\omega = r^2;$$

but we shall seldom use oblique axes in questions relating to circles.

Cor. 1.—The equation to rectangular axes of the circle whose centre is the origin is

$$x^2 + y^2 = r^2.$$

Cor. 2.—Let the axis of x be a diameter, and the axis of y a perpendicular at its extremity, then the co-ordinates of the centre are obviously $(r, 0)$, and on substituting these values for a and b, the equation of the circle becomes

$$x^2 + y^2 = 2rx.$$

The two forms just mentioned are the simplest which the equation of the circle can be made to assume by a particular choice of axes; and are those which most frequently occur in practice.

78. By comparing the equations found in the last Article with the general equation of the second degree,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we can ascertain the conditions that this latter equation should represent a circle.

If the axes be rectangular, it is evident that B must = 0 and A = C, in order that when we divide by A the equation may be capable of being put into the form

$$(x-a)^2 + (y-b)^2 = r^2, \text{ or } x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0.$$

If the axes be oblique, we must compare the general equation with the equation

$$(x-a)^2 + (y-b)^2 + 2(x-a)(y-b)\cos\omega = r^2,$$

and we find that in this case the general equation will represent a circle, if $A = C$, and $\frac{B}{A} = 2\cos\omega$.

If the general equation of the second degree, referred to rectangular axes, fulfil the conditions $B = 0$, $A = C$, *we can find the radius of the circle represented by it, and also the co-ordinates of its centre*, thus fully determining the circle, both in magnitude and position; for, comparing the equations,

$$x^2 + y^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0,$$

and $$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0,$$

we get $$\frac{D}{A} = -2a, \quad \frac{E}{A} = -2b, \quad \frac{F}{A} = a^2 + b^2 - r^2;$$

and, therefore,

$$a = -\frac{D}{2A}, \quad b = -\frac{E}{2A}, \quad r^2 = \frac{D^2 + E^2 - 4AF}{4A^2};$$

and the general equation is equivalent to

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A^2}.$$

The rule, then, for bringing the equation of any circle to the form $(x - a)^2 + (y - b)^2 = r^2$, may be expressed as follows: "By division make the coefficients of x^2 and $y^2 = 1$, transpose F, and then complete the squares by adding to both sides the sum of the squares of half the coefficients of x and y."

Ex. 1. Find the co-ordinates of the centre and the radius of

$$x^2 + y^2 - 5x - 4y = 7. \qquad \textit{Ans.} \left(\frac{5}{2}, 2\right), \frac{\sqrt{(69)}}{2}.$$

Ex. 2. Reduce to the form of Art. 77,

$$x^2 + y^2 - 2x - 4y = 20.$$

Ans. $(x - 1)^2 + (y - 2)^2 = 5^2$.

Ex. 3. Find the line joining the centres of the circles

$$x^2 + y^2 = 2y; \quad x^2 + y^2 = 2x.$$

Ans. $x + y = 1$.

Since F does not occur in the values just found for the co-ordinates of the centre, we learn that *two circles will be concentric if their equations only differ in their constant term.*

79. We consider in this Article the effect of two or three particular suppositions on the general equation.

(1.) If $F = 0$ the origin is on the curve. For then the equation is satisfied by the values $x = 0$, $y = 0$; that is, by the co-ordinates of the origin. The same argument proves that *if an equation of any degree want the absolute term, the curve represented by it passes through the origin.*

(2.) If $D^2 + E^2 = 4AF$; it appears from Art. 78 that the radius of the circle vanishes, and that the equation may be reduced to the form

$$(x - a)^2 + (y - b)^2 = 0.$$

It is plain, that this equation can be satisfied by the co-ordinates of no point save those of the point $(x = a, y = b)$; hence it has been common to say, that the equation just written is the equation of this point. We object to this mode of expression for the reasons given, Art. 69, and prefer to call it the equation *of an infinitely small circle*, having for centre the point (ab). We have seen (Art. 69) that it may also be considered as the equation of *two imaginary right lines* passing through the point (ab), since it can be resolved into the factors

$$(x-a)+(y-b)\sqrt{(-1)}=0, \text{ and } (x-a)-(y-b)\sqrt{(-1)}=0.$$

These remarks, of course, apply, in like manner, to the equation

$$x^2+y^2=0,$$

which is a particular case of the above.

(3.) If D^2+E^2 be less than $4AF$, the radius of the circle becomes imaginary, and the equation, being equivalent to one of the form

$$(x-a)^2+(y-b)^2+r^2=0,$$

cannot be satisfied by any real values of the co-ordinates x and y.

80. *To find the co-ordinates of the points in which a given right line meets a given circle.*

Let the equation of the circle be $x^2+y^2=r^2$, and that of the right line $x\cos a+y\sin a=p$. These two equations are sufficient (Art. 15) to determine the x and y of the intersection. For example, finding the values of y from both, and equating them to each other, we get for determining x, the equation

$$\frac{p-x\cos a}{\sin a}=\sqrt{(r^2-x^2)},$$

or, reducing $\quad x^2-2px\cos a+p^2-r^2\sin^2 a=0;$

hence, $\quad x=p\cos a\pm\sin a\sqrt{(r^2-p^2)},$

and, in like manner,

$$y=p\sin a\mp\cos a\sqrt{(r^2-p^2)}.$$

(The reader may satisfy himself, by substituting these values in the given equations, that the − in the value of y corresponds to the + in the value of x, and *vice versâ*.)

Since we obtained a quadratic to determine x, and since every quadratic has two roots, we must, in order to make our language conform to the language of algebra, assert that *every* line meets a circle in *two* points.

81. Let us, however, examine separately the three cases of this solution:

First. If p, which is the distance of the line from the centre, be less than the radius, we get two *real* values for x and y, and the line meets the circle in two real points.

Secondly. Let $p = r$, or the distance of the line from the centre = the radius. In this case it is evident geometrically that the line is a tangent to the circle, and our analysis points to the same conclusion, since the two values of x in this case become *equal*, as do likewise the two values of y. Consequently, the points answering to these two values, which are in general different, will in this case coincide. We shall, therefore, not say that the tangent meets the circle in only one point, but rather that it meets it in two coincident points; just as we do not say that the equation for this case

$$x^2 - 2rx\cos a + r^2\cos^2 a = 0,$$

has only one root, but rather that it has two equal roots. And, in general, we define the tangent to any curve as the line joining two indefinitely near points on it.

Thirdly. Let p be greater than r. In this case it is usual to say, that the line does not meet the circle at all. Analysis, however, though it fails to furnish us with real values for x and y, yet supplies us with *imaginary* values. We shall, therefore, find it more consistent to say that in this case the line meets the circle in two imaginary points. By an imaginary point we mean nothing more than a point, one or both of whose co-ordinates are imaginary. It is a purely analytical conception. We do not attempt to represent it geometrically. But the neglect of those imaginary points would lead to as great a want of generality in our reasonings, and to as much inconvenience in our language, as if, only paying attention to the real roots of equations, we were to deny that every equation has as many roots as it has dimensions, or to assert that the equation

$$x^2 - 2px\cos a = r^2\sin^2 a - p^2$$

has no root at all when p is greater than r. We shall presently meet with many cases in which the line joining two imaginary points is real, and enjoys all the geometrical properties of the corresponding line in the case where the points are real.

82. We should proceed as in Art. 80, if it were required to find the points where the line $ax + by + c = 0$ meets the circle

$$Ax^2 + Ay^2 + Dx + Ey + F = 0.$$

Eliminating either variable between the equations, we have a quadratic to determine the other; and if this quadratic have equal roots, the line touches the circle. We only think it necessary to notice particularly the case where the given line is one of the axes of co-ordinates. By making alternately $y = 0$, $x = 0$ in the equation of the circle, we find that the points in which it is met by the axes are determined by the quadratics

$$Ax^2 + Dx + F = 0, \qquad Ay^2 + Ey + F = 0.$$

The axis of x will be a tangent when the first quadratic has equal roots, that is, when

$$D^2 = 4AF,$$

and the axis of y when $E^2 = 4AF$.

When seeking to determine the position of a circle represented by a given equation, it is often as convenient to do so by finding the intercepts which it makes on the axes, as by finding its centre and radius. For a circle is known when three points on it are known; the determination, therefore, of the four points where the circle meets the axes serves completely to fix its position.

Ex. 1. Find the co-ordinates of intersection of $x^2 + y^2 = 65$; $3x + y = 25$.
Ans. (7, 4) and (8, 1).

Ex. 2. Find intersections of $(x - c)^2 + (y - 2c)^2 = 25c^2$; $4x + 3y = 35c$.
Ans. The line touches at the point $(5c, 5c)$.

Ex. 3. Find the points where the axes are cut by $x^2 + y^2 - 5x - 7y + 6 = 0$.
Ans. $x = 3$, $x = 2$; $y = 6$, $y = 1$.

Ex. 4. What is the equation of the circle which touches the axes at distances from the origin $= a$? *Ans.* $x^2 + y^2 - 2ax - 2ay + a^2 = 0$.

Ex. 5. When will $y = mx + b$ touch $x^2 + y^2 = r^2$? *Ans.* When $b^2 = r^2(1 + m^2)$.

Ex. 6. Find the tangent from the origin to $A(x^2 + y^2) + Dx + Ey + F = 0$. The points where any line through the origin ($y = mx$) meets the circle are given by the equation

$$A(m^2 + 1)x^2 + (D + Em)x + F = 0.$$

If the line touches, this quadratic will have equal roots,

$$\text{or } (D + Em)^2 = 4AF(m^2 + 1),$$

which gives a quadratic for determining m.

Ex. 7. Find the tangents from the origin to $x^2 + y^2 - 6x - 2y + 8 = 0$.

Ans. $x - y = 0$, $7x + y = 0$. ? $7y + x = 0$

83. *To find the equation of the tangent at the point $x'y'$ to a given circle.*

The tangent having been defined (Art. 81) as the line joining two indefinitely near points on the curve, its equation will be found by first forming the equation of the line joining any two points $(x'y', x''y'')$ on the curve, and then making $x' = x''$ and $y' = y''$ in that equation.

To apply this to the circle: first, let the centre be the origin, and, therefore, the equation of the circle $x^2 + y^2 = r^2$.

The equation of the line joining any two points $(x'y')$ and $(x''y'')$ is (Art. 29),

$$\frac{y - y'}{x - x'} = \frac{y' - y''}{x' - x''};$$

now if we were to make in this equation $y' = y''$ and $x' = x''$, the right-hand member would assume the indeterminate form of $\frac{0}{0}$. The cause of this is, that we have not yet introduced the condition, that the two points $(x'y', x''y'')$ are *on the circle*. By the help of this condition we shall be able to write the equation in a form which will not become indeterminate when the two points are made to coincide. For, since

$$r^2 = x'^2 + y'^2 = x''^2 + y''^2, \text{ we have } x'^2 - x''^2 = y''^2 - y'^2,$$

and, therefore,
$$\frac{y' - y''}{x' - x''} = -\frac{x' + x''}{y' + y''}.$$

Hence the equation of the chord becomes

$$\frac{y - y'}{x - x'} = -\frac{x' + x''}{y' + y''}.$$

And if we *now* make $x' = x''$ and $y' = y''$, we find for the equation of the tangent,

$$\frac{y - y'}{x - x'} = -\frac{x'}{y'},$$

or, reducing, and remembering that $x'^2 + y'^2 = r^2$, we get finally

$$xx' + yy' = r^2.$$

If we were now to transform the equations to a new origin, so that the co-ordinates of the centre should become a, b, we must substitute (Art. 8) $x - a$, $x' - a$, $y - b$, $y' - b$ for x, x', y, y', respectively: the equation of the circle would become

$$(x - a)^2 + (y - b)^2 = r^2,$$

and that of the tangent

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2;$$

a form easily remembered, from its similarity to the equation of the circle.

We might have obtained the equation of the tangent

$$xx' + yy' = r^2$$

in another way, by forming the equation of a line through the point $x'y'$, perpendicular to the radius, whose equation is easily seen to be $y'x = x'y$. We have preferred, however, the method actually adopted, both because it is the same as that which we shall employ in the case of other curves, and also because we wish the learner to perceive that all the properties of the circle can be deduced from its equation without a previous acquaintance with the geometrical theory of the curve; as in the present instance, where the equation just found may be used to *prove* that the tangent to a circle is perpendicular to the radius.

84. *To find the equation of the tangent to the circle whose equation referred to any axes is*

$$Ax^2 + Bxy + Ay^2 + Dx + Ey + F = 0,$$

where $B = 2A \cos\omega$. We form, as in Art. 83, the equation of the line joining two points, and then by the help of the conditions that $x'y'$, $x''y''$ are points *on the circle*, we can get an expression for $\frac{y' - y''}{x' - x''}$, which will not become indeterminate when the two points coincide. We have the two conditions

$$Ax'^2 + Bx'y' + Ay'^2 + Dx' + Ey' + F = 0,$$
$$Ax''^2 + Bx''y'' + Ay''^2 + Dx'' + Ey'' + F = 0.$$

Subtracting one from the other

$$A(x'^2 - x''^2) + B(x'y' - x''y'') + A(y'^2 - y''^2) + D(x' - x'') + E(y' - y'') = 0.$$

Now $\qquad x'y' - x''y'' = x'(y' - y'') + y''(x' - x'').$

Hence, dividing by $x' - x''$, and solving for $\frac{y' - y''}{x' - x''}$, we find

$$\frac{y' - y''}{x' - x''} = -\frac{A(x' + x'') + By'' + D}{A(y' + y'') + Bx' + E}.$$

The equation of the chord is, therefore,

$$\frac{y - y'}{x - x'} = -\frac{A(x' + x'') + By'' + D}{A(y' + y'') + Bx' + E}.$$

If the points $x'y'$, $x''y''$ coincide, we have the equation of the *tangent*

$$\frac{y - y'}{x - x'} = -\frac{2Ax' + By' + D}{2Ay' + Bx' + E},$$

or, reducing, and remembering that $x'y'$ satisfies the equation of the curve,

$$(2Ax' + By' + D)x + (2Ay' + Bx' + E)y + Dx' + Ey' + 2F = 0.$$

Ex. 1. Find the tangent at the point (5, 4) to $(x - 2)^2 + (y - 3)^2 = 10$.

Ans. $3x + y = 19$.

Ex. 2. What is the equation of the chord joining the points $x'y'$, $x''y''$ on the circle $x^2 + y^2 = r^2$? *Ans.* $(x' + x'')x + (y' + y'')y = r^2 + x'x'' + y'y''$.

Ex. 3. Find the condition that $Ax + By + C = 0$ should touch $(x - a)^2 + (y - b)^2 = r^2$.

Ans. $\frac{Aa + Bb + C}{\sqrt{(A^2 + B^2)}} = r$; since the perpendicular on the line from ab is equal to r.

85. *To find the points of contact of tangents drawn to a circle from a given point.*

Let the given point be $x'y'$, and let the co-ordinates of the point of contact which we are seeking be x'', y''. Then (Art. 83) the equation of the tangent will be

$$xx'' + yy'' = r^2;$$

but by hypothesis this line passes through the point $x'y'$, hence we get the condition

$$x'x'' + y'y'' = r^2;$$

and since the point $x''y''$ is on the circle, we have also the condition

$$x''^2 + y''^2 = r^2.$$

These two conditions are sufficient to determine the co-ordinates x'', y''. Solving the equations, we get

$$x'' = \frac{r^2x' \pm ry' \sqrt{(x'^2 + y'^2 - r^2)}}{x'^2 + y'^2}, \quad y'' = \frac{r^2y' \mp rx' \sqrt{(x'^2 + y'^2 - r^2)}}{x'^2 + y'^2}.$$

Hence, from every point may be drawn *two* tangents to a circle. These tangents will be real when $x'^2 + y'^2$ is $> r^2$, or the point outside the circle; they will be imaginary when $x'^2 + y'^2$ is $< r^2$, or the point inside the circle; and they will coincide when $x'^2 + y'^2 = r^2$, or the point on the circle.

86. *To find the equation of the line joining the points of contact of tangents from any point.* That is, to form the equation of the line joining the two points whose co-ordinates were found in the last article. It will not, however, be necessary to set about this in the usual manner, if we attend to the remark at the end of Art. 29. We saw in the last article that the co-ordinates of each point of contact were connected with those of the given point by the relation

$$x'x'' + y'y'' = r^2.$$

The equation, therefore, of the line joining the points of contact, must be

$$x'x + y'y = r^2, \text{ (by § 29)}$$

for this is the equation of a right line, and is satisfied for each point of contact. In fact, since the co-ordinates of the points of contact were found by solving for x and y from the equations

$$xx' + yy' = r^2; \quad x^2 + y^2 = r^2;$$

the geometrical meaning of these equations is, that these points are the intersections of the circle $x^2 + y^2 = r^2$ with the right line $xx' + yy' = r^2$.

We see, then, that whether the tangents from $x'y'$ be real or imaginary, the line joining their points of contact will be the real line $xx' + yy' = r^2$, which we shall call the *polar* of $x'y'$ with regard to the circle. This line is evidently perpendicular to the line $(x'y - y'x = 0)$, which joins $x'y'$ to the centre; and its distance from the centre (Art. 25) is $\frac{r^2}{\sqrt{(x'^2 + y'^2)}}$. Hence, the polar of any point P is constructed geometrically by joining it to the centre

C, taking on the joining line a point M, such that $CM \cdot CP = r^2$, and erecting a perpendicular to CP at M. We see, also, that the equation of the polar is similar in form to that of the tangent, only that in the former case the point $x'y'$ is not supposed to be necessarily on the circle: if, however, $x'y'$ be on the circle, then its polar is the tangent at that point.

87. *To find the equation of the polar of $x'y'$ with regard to the circle*

$$Ax^2 + Bxy + Ay^2 + Dx + Ey + F.$$

We adopt the same process as in Art. 86. The equation of the tangent at any point $x''y''$ may be written (Art. 84)

$$(2Ax + By + D)x'' + (2Ay + Bx + E)y'' + Dx + Ey + 2F = 0.$$

This expresses a relation between the co-ordinates of any point xy on the tangent, and those of the point of contact $x''y''$. Let us then suppose the former to be known, and the latter to be unknown; let us denote the known co-ordinates by the accented letters $x'y'$, and the unknown co-ordinates by the unaccented letters xy, and the relation just written becomes

$$(2Ax' + By' + D)x + (2Ay' + Bx' + E)y + Dx' + Ey' + 2F = 0,$$

the equation of a line on which both points of contact must lie, and therefore the equation of the line joining them. It is still similar in form to the equation of the tangent.

Cor.—The polar of *the origin* is

$$Dx + Ey + 2F = 0.$$

Ex. 1. Find the polar of (4, 4) with regard to $(x - 1)^2 + (y - 2)^2 = 13$.
Ans. $3x + 2y = 20$.

Ex. 2. Find the polar of (4, 5) with regard to $x^2 + y^2 - 3x - 4y = 8$.
Ans. $5x + 6y = 48$.

Ex. 3. Find the pole of $Ax + By + C = 0$ with regard to $x^2 + y^2 = r^2$.
Ans. $\left(-\frac{Ar^2}{C}, -\frac{Br^2}{C}\right)$, as appears from comparing the given equation with $xx' + yy' = r^2$.

Ex. 4. Find the pole of $3x + 4y = 7$ with regard to $x^2 + y^2 = 14$. *Ans.* (6, 8).

Ex. 5. Find the pole of $2x + 3y = 6$ with regard to $(x - 1)^2 + (y - 2)^2 = 12$.
Ans. (– 11, – 16).

88. *To find the length of the tangent drawn from any point to the circle, whose equation is*

$$(x-a)^2+(y-b)^2-r^2=0.$$

The square of the distance of any point from the centre

$$=(x-a)^2+(y-b)^2;$$

and since this square exceeds the square of the tangent by the square of the radius, the square of the tangent from any point is found by substituting the co-ordinates of that point for x and y in the first member of the equation of the circle

$$(x-a)^2+(y-b)^2-r^2=0.$$

Since the general equation to rectangular co-ordinates

$$Ax^2+Ay^2+Dx+Ey+F=0,$$

when divided by A, is (Art. 78) equivalent to one of the form

$$(x-a)^2+(y-b)^2-r^2=0,$$

we learn that the square of the tangent to a circle whose equation is given in its most general form is found by dividing by the coefficient of x^2, and then substituting in the equation the co-ordinates of the given point.

The square of the tangent from the origin is found by making x and $y=0$, and is, therefore, = the absolute term in the equation of the circle, divided by A.

The same reasoning is applicable if the axes be oblique.

* 89. *To find the ratio in which the line joining two given points, $x'y'$, $x''y''$, is cut by a given circle.*

We proceed precisely as in Art. 38. The co-ordinates of any point on the line must (Art. 7) be of the form

$$\frac{lx''+mx'}{l+m},\quad \frac{ly''+my'}{l+m}.$$

Substituting these values in the equation of the circle

$$x^2+y^2-r^2=0,$$

and arranging, we have to determine the ratio $l:m$, the quadratic

$$l^2(x''^2+y''^2-r^2)+2lm(x'x''+y'y''-r^2)+m^2(x'^2+y'^2-r^2)=0.$$

The values of $l : m$ being determined from this equation, we have at once the co-ordinates of the points where the right line meets the circle. The symmetry of the equation makes this method sometimes more convenient than that used Art. 80.

If $x''y''$ lie on the polar of $x'y'$, we have $x'x'' + y'y'' - r^2 = 0$ (Art. 86) and the factors of the preceding equation must be of the form $l + \mu m$, $l - \mu m$; the line joining $x'y'$, $x''y''$ is therefore cut internally and externally in the same ratio, and we deduce the well-known theorem, *any line drawn through a point is cut harmonically by the point, the circle, and the polar of the point.*

* 90. *To find the equation of the tangents from a given point to a given circle.*

We have already (Art. 85) found the co-ordinates of the points of contact; substituting, therefore, these values in the equation $xx'' + yy'' - r^2 = 0$, we have for the equation of one tangent

$$r(xx' + yy' - x'^2 - y'^2) + (xy' - yx')\sqrt{(x'^2 + y'^2 - r^2)} = 0,$$

and for that of the other,

$$r(xx' + yy' - x'^2 - y'^2) - (xy' - yx')\sqrt{(x'^2 + y'^2 - r^2)} = 0.$$

These two equations multiplied together give the equation of the pair of tangents in a form free from radicals. The preceding article enables us, however, to obtain this equation in a still more simple form. For the equation which determines $l : m$ will have equal roots if the line joining $x'y'$, $x''y''$ touch the given circle; if then $x''y''$ be any point on either of the tangents through $x'y'$, its co-ordinates must satisfy the condition

$$(x'^2 + y'^2 - r^2)(x^2 + y^2 - r^2) = (xx' + yy' - r^2)^2.$$

This, therefore, is the equation of the pair of tangents through the point $x'y'$. It is not difficult to prove that this equation is identical with that obtained by the method first indicated.

91. *To find the equation of a circle passing through three given points.*

We have only to write down the general equation

$$x^2 + y^2 + Dx + Ey + F = 0,$$

and then substituting in it, successively, the co-ordinates of each of the given points, we have three equations to determine the three unknown quantities D, E, F.

Ex. 1. Find the circle through the origin, and through (2, 3) and (3, 4).

Here $F = 0$, and we have

$$13 + 2D + 3E = 0, \quad 25 + 3D + 4E = 0, \quad \text{whence } D = -23, \; E = 11.$$

Ex. 2. Find the circle through (1, 2), (1, 3), (2, 5).

We have $5 + D + 2E + F = 0$, $10 + D + 3E + F = 0$, $29 + 2D + 5E + F = 0$,

whence $$D = -9, \; E = -5, \; F = 14.$$

Ex. 3. Find the circle through (2, − 3), (3, − 4), (− 2, − 1).

Ans. $D = 8$, $B = 20$, $C = 31$.

Ex. 4. Find the circle making intercepts a and b on the axis of x.

Ans. $D = -(a + b)$, $F = ab$, E indeterminate.

Ex. 5. Taking the same axes as in Art. 48, Ex. 1, find the equation of the circle through the origin and through the middle points of sides.

Ans. $2p(x^2 + y^2) - p(s - s')x - (p^2 + ss')y = 0.$

The circle, therefore, also passes through the middle point of base.

*92. *To express the equation of the circle through three points $x'y'$, $x''y''$, $x'''y'''$ in terms of the co-ordinates of those points.*

We have to substitute in

$$x^2 + y^2 + Dx + Ey + F = 0$$

the values of D, E, F derived from

$$\begin{aligned} (x'^2 + y'^2) + Dx' + Ey' + F &= 0, \\ (x''^2 + y''^2) + Dx'' + Ey'' + F &= 0, \\ (x'''^2 + y'''^2) + Dx''' + Ey''' + F &= 0. \end{aligned}$$

The result of thus eliminating D, E, F between these four equations will be found to be

$$\begin{aligned} &(x^2 + y^2)\{x'(y'' - y''') + x''(y''' - y') + x'''(y' - y'')\} \\ -&(x'^2 + y'^2)\{x''(y''' - y) + x'''(y - y'') + x(y'' - y''')\} \\ +&(x''^2 + y''^2)\{x'''(y - y') + x(y' - y''') + x'(y''' - y)\} \\ -&(x'''^2 + y'''^2)\{x(y' - y'') + x'(y'' - y) + x''(y - y')\} = 0, \end{aligned}$$

as may be seen by multiplying each of the four equations by the quantities which multiply $(x^2 + y^2)$ &c. in the last-written equation, and adding them together, when D, E, F will be found to vanish identically.

If it were required to find the condition that four points should lie on a circle, we have only to write x_4, y_4 for x and y

in the last equation. It is easy to see that the following is the geometrical interpretation of the resulting condition. If A, B, C, D be any four points on a circle, and O any fifth point taken arbitrarily, and if we denote by BCD the area of the triangle BCD, &c., then

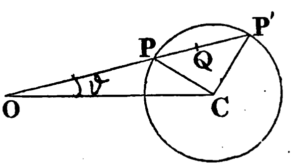
$$OA^2 \cdot BCD + OC^2 \cdot ABD = OB^2 \cdot ACD + OD^2 \cdot ABC.$$

93. We shall conclude this chapter by showing how to find the *polar equation* of a circle.

We may either obtain it by substituting for x, $\rho\cos\theta$, and for y, $\rho\sin\theta$ (Art. 12), in either of the equations of the circle already given,

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \text{ or } (x-a)^2 + (y-b)^2 = r^2,$$

or else we may find it independently, from the definition of the circle, as follows:



Let O be the pole, C the centre of the circle, and OC the fixed axis; let the distance OC = d, and let OP be any radius vector, and, therefore, = ρ, and the angle POC = θ, then we have

$$PC^2 = OP^2 + OC^2 - 2OP \cdot OC \cos POC,$$

that is, $$r^2 = \rho^2 + d^2 - 2\rho d\cos\theta,$$

or $$\rho^2 - 2d\rho\cos\theta + d^2 - r^2 = 0.$$

This, therefore, is the polar equation of the circle.

If the fixed axis did not coincide with OC, but made with it any angle a, the equation would be, as in Art. 44,

$$\rho^2 - 2d\rho\cos(\theta - a) + d^2 - r^2 = 0.$$

If we suppose the pole on the circle, the equation will take a simpler form, for then $r = d$, and the equation will be reduced to

$$\rho = 2r\cos\theta,$$

a result which we might have also obtained at once geometrically from the property that the angle in a semicircle is right; or else by substituting for x and y their polar values in the equation (Art. 77, Cor. 2), $$x^2 + y^2 = 2rx.$$

CHAPTER VII.

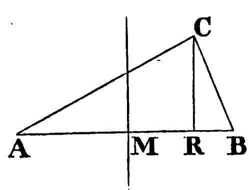
THEOREMS AND EXAMPLES ON THE CIRCLE.

94. HAVING in the last chapter shown how to form the equations of the circle and of the most remarkable lines related to it, we proceed in this chapter to illustrate these equations by Examples, and to apply them to the establishment of some of the principal properties of the circle. Having sufficiently shown, in Chapter III., how in general to apply the analytical method to the solution of problems, we do not think it necessary to enter into the subject here with equal minuteness, and shall feel ourselves at liberty to suppress many details which can easily be supplied by the reader who has worked out the examples there given.

We commence with some Examples of *circular loci*, which will serve as examples of the method of determining the position of a circle from its equation, if the learner will in each case, by Art. 78, determine the co-ordinates of the centre and the radius; or else find, by Art. 82, the points where the circle meets the axes.

Ex. 1. Given base and vertical angle of a triangle, to find the locus of vertex.

Let us take the base for axis of x, and a perpendicular through its middle point for axis of y; let the co-ordinates of the vertex be x, y, and let the base $= 2c$. Then the tangent of the base angle CAB will be $\frac{CR}{AR}$, or $\frac{y}{c+x}$, and of CBR $= \frac{CR}{BR}$, or $\frac{y}{c-x}$.



Hence we can find the tangent of the sum of the base angles, and make it $=$ − the tangent of C, the given vertical angle, or

$$\frac{\frac{y}{c+x}+\frac{y}{c-x}}{1-\frac{y^2}{c^2-x^2}} = -\tan C,$$

and, reducing this equation, the equation of the locus will be found to be

$$x^2 + y^2 - 2cy \cot C - c^2 = 0,$$

which represents a circle which passes through the extremities of the base, whose radius is $\frac{c}{\sin C}$, and centre (0, cot C). The centre will therefore be above, on, or below the base, according as C is acute, right, or obtuse.

Ex. 2. To solve the last example, the axes having any position.

Let the co-ordinates of the extremities of base be $x'y'$, $x''y''$. Let the equation of one side be

$$y - y' = m(x - x'),$$

then the equation of the other side, making with this the angle C, will be (Art. 42)

$$(1 + m \tan C)(y - y'') = (m - \tan C)(x - x'').$$

Eliminating m, the equation of the locus is

$$\tan C\{(y - y')(y - y'') + (x - x')(x - x'')\} + x(y' - y'') - y(x' - x'') + x'y'' - y'x'' = 0,$$

which reduces to the equation of the last example if $y' = y'' = 0$; $x' = +c$, $x'' = -c$.

If C be a right angle, the equations of the sides are

$$y - y' = m(x - x');\ m(y - y'') + (x - x'') = 0,$$

and that of the locus

$$(y - y')(y - y'') + (x - x')(x - x'') = 0.$$

Ex. 3. Given base and vertical angle, to find the locus of the intersection of perpendiculars of the triangle.

The equations of the perpendiculars to the sides are

$$m(y - y'') + (x - x'') = 0,\ (m - \tan C)(y - y') + (1 + m \tan C)(x - x') = 0.$$

Eliminating m, the equation of the locus is

$$\tan C\{(y - y')(y - y'') + (x - x')(x - x'')\} = x(y' - y'') - y(x' - x'') + x'y'' - y'x'';$$

an equation which only differs from that of the last article by the sign of tan C, and which is therefore the locus we should have found for the vertex had we been given the same base and a vertical angle equal to the supplement of the given one.

Ex. 4. Given base and ratio of sides of a triangle, find locus of vertex.

With the same axes as in Ex. 1, if ratio be $m : n$, we find, for equation of locus,

$$m^2\{y^2 + (c - x)^2\} = n^2\{y^2 + (c + x)^2\}.$$

Hence the locus is a circle, whose centre is on the axis of x, at a distance from origin $= \frac{m^2 + n^2}{m^2 - n^2}c$; whose radius $= \frac{2mn}{m^2 - n^2}c$; and which meets the base at the points

$$x = \frac{m + n}{m - n}c, \text{ and } x = \frac{m - n}{m + n}c.$$

Since the co-ordinates of the extremities of the base are $x = \pm c$, these (Art. 7) are the two points where the base is cut in the ratio $m : n$.

Ex. 5. Given base of a triangle, and m times square of one side, $\pm n$ times square of other; find the locus of vertex.

Ans. A circle whose centre is $\left(\frac{m \mp n}{m \pm n}c, 0\right)$.

Ex. 6. Find the locus of a point the square of whose distance from a given point is proportional to its distance from a given right line.

Ex. 7. A line of constant length moves between two fixed right lines, and perpendiculars to the lines are raised at its extremities; find the locus of their intersection.

Ex. 8. In general, given any number of points, to find locus of a point such that m' times square of its distance from the first + m'' times square of its distance from the second + &c., = a constant: or (adopting the notation used in p. 48) such that $\Sigma(mr^2)$ may be constant.

The square of the distance of any point xy from $x'y'$ is

$$(x-x')^2+(y-y')^2.$$

Multiply this by m', and add it to the corresponding terms found by expressing the distance of the point xy from the other points $x''y''$, &c. If we adopt the notation of p. 48 we may write, for the equation of the locus,

$$\Sigma(m)x^2+\Sigma(m)y^2-2\Sigma(mx')x-2\Sigma(my')y+\Sigma(mx'^2)+\Sigma(my'^2)=C.$$

Hence the locus will be a circle, the co-ordinates of whose centre will be

$$x=\frac{\Sigma(mx')}{\Sigma(m)},\quad y=\frac{\Sigma(my')}{\Sigma(m)},$$

that is to say, the centre will be the point which, in p. 48, was called the centre of mean position of the given points.

If we investigate the value of the radius of this circle, we shall find

$$R^2\Sigma(m)=\Sigma(mr^2)-\Sigma(m\rho^2),$$

where $\Sigma(mr)^2 = C$ = sum of m times square of distance of each of the given points from any point on the circle, and $\Sigma(m\rho^2)$ = sum of m times square of distance of each point from the centre of mean position.

95. We shall next give one or two examples involving the problem of Art. 80, to find the co-ordinates of the points where a given line meets a given circle.

Ex. 1. To find the locus of the middle points of chords of a given circle, drawn parallel to a given line.

Let the equation of any of the parallel chords be

$$x\cos\alpha+y\sin\alpha-p=0,$$

where α is, by hypothesis, given, and p is indeterminate; the abscissæ of the points where this line meets the circle are (Art. 80) found from the equation

$$x^2-2px\cos\alpha+p^2-r^2\sin^2\alpha=0.$$

Now, if the roots of this equation be x' and x'', the x of the middle point of the chord will (Art. 7) be $\frac{x'+x''}{2}$, or, from the theory of equations, will $= p\cos\alpha$. In like manner, the y of the middle point will equal $p\sin\alpha$. Hence the equation of the locus is $\frac{y}{x}=\tan\alpha$, that is, a right line drawn through the centre perpendicular to the system of parallel chords; since α is the angle made with the axis of x by a perpendicular to the chord

$$x\cos\alpha+y\sin\alpha-p=0.$$

Ex. 2. To find the condition that the intercept made by the circle on the line

$$x \cos a + y \sin a = p$$

should subtend a right angle at the point $x'y'$.

We found (Art. 94, Ex. 2) the condition that the lines joining the points $x''y''$, $x'''y'''$ to xy should be at right angles to each other; viz.:

$$(x - x'')(x - x''') + (y - y'')(y - y''') = 0.$$

Let $x''y''$, $x'''y'''$ be the points where the line meets the circle, then, by the last example,

$$x'' + x''' = 2p \cos a, \quad x''x''' = p^2 - r^2 \sin^2 a, \quad y'' + y''' = 2p \sin a, \quad y''y''' = p^2 - r^2 \cos^2 a.$$

Putting in these values, the required condition is

$$x'^2 + y'^2 - 2px' \cos a - 2py' \sin a + 2p^2 - r^2 = 0.$$

Ex. 3. To find the locus of the middle point of a chord which subtends a right angle at a given point.

If x and y be the co-ordinates of the middle point, we have, by Ex. 1,

$$p \cos a = x, \quad p \sin a = y, \quad p^2 = x^2 + y^2,$$

and, substituting these values, the condition found in the last example becomes

$$(x - x')^2 + (y - y')^2 + x^2 + y^2 = r^2.$$

Ex. 4. To find the locus of the foot of a perpendicular from $x'y'$ on a chord which subtends a right angle at that point.

The co-ordinates of the foot of perpendicular are determined by the equations

$$x \cos a + y \sin a = p; \quad (x - x') \sin a - (y - y') \cos a = 0;$$

whence, if we write for shortness,

$$R^2 = (x - x')^2 + (y - y')^2,$$

we have $\quad R \sin a = y - y', \quad R \cos a = x - x', \quad Rp = x^2 + y^2 - xx' - yy';$

but the condition in Ex. 2 may be written

$$0 = x'^2 + y'^2 - r^2 + 2p(p - x' \cos a - y' \sin a) = x'^2 + y'^2 - r^2 + 2p\{(x - x') \cos a + (y - y') \sin a\};$$

but $\quad (x - x') \cos a + (y - y') \sin a = R;$

hence $\quad x'^2 + y'^2 - r^2 + 2(x^2 + y^2 - xx' - yy') = 0;$

or, the locus is the same as that found in the last example.

Ex. 5. Given a line and a circle, to find a point such that if any chord be drawn through it, and perpendiculars let fall from its extremities on the given line, the rectangle under these perpendiculars may be constant.

Take the given line for axis of y, and let the axis of x be the perpendicular on it from the centre of the given circle, whose length we shall call p. Then the equation of the circle is (Art. 77)

$$y^2 + (x - p)^2 = r^2.$$

Again, if the co-ordinates of the sought point be x', y', the equation of any line through it will be

$$(y - y') = m(x - x'), \text{ or } y = mx + y' - mx'.$$

Substitute this value of y in the equation of the circle, and we shall get, to determine the x of the points where the line meets the circle,

$$(1+m^2)x^2 + \{2m(y'-mx') - 2p\}x + (y'-mx')^2 + p^2 - r^2 = 0.$$

But x is the perpendicular on the given line, and the product of the two perpendiculars (by the theory of equations)

$$= \frac{(y'-mx')^2 + p^2 - r^2}{1+m^2}.$$

This will not be independent of m, unless the numerator be divisible by $1+m^2$, and it will be found that this cannot be the case unless $y'=0$ and $x'^2 = p^2 - r^2$. Hence there are two such points situated on the axis of x, and at a distance from the origin = the tangent drawn from it to the given circle.

Ex. 6. If any chord be drawn through a fixed point on a diameter of a circle, and its extremities joined to either end of the diameter, the joining lines will cut off on the tangent, at the other end of the diameter, portions whose rectangle is constant.

Let us take the diameter for axis of x, and either extremity of it for origin, then (Art. 77, Cor. 2) the equation of the circle will be $x^2+y^2=2rx$, and that of any chord through a fixed point on the diameter will be $y=m(x-x')$. By combining these equations we can determine the co-ordinates of the extremities of the chord. We can, however, without solving for these co-ordinates, obtain directly from the equations the equation of the lines joining these extremities to the origin. For if, by combining the equations, we can obtain a homogeneous function of the second degree, it will be, by Art. 68, the equation of two right lines drawn through the origin, and it evidently must be satisfied by the co-ordinates of the points which satisfy the two given equations.

Write these equations thus,

$$x^2+y^2=2rx, \text{ and } mx' = mx - y,$$

and, multiplying them together, we get

$$mx'(x^2+y^2) = 2rx(mx-y).$$

This being homogeneous in x and y, is the required equation of the joining lines. It may be written thus,

$$mx'.y^2 + 2r.xy + m(x'-2r)x^2 = 0.$$

This equation enables us to find the values of y corresponding to any value of x, and we see that the product of these values will be $\frac{x'-2r}{x'}x^2$, and, therefore, independent of m. The intercepts made on a perpendicular at the extremity of the diameter are found by making $x=2r$ in the preceding equation, and their product is $4r^2\frac{x'-2r}{x'}$, which will be constant as long as x' is constant.

96. We shall next obtain one or two of the properties of the polar of a point from its equation (Art. 86).

If any chord be drawn through a fixed point and tangents at its extremities: to find the locus of their intersection.

Let any point on the locus be XY, then the chord joining points of contact of tangents passing through XY is

$$Xx + Yy = r^2;$$

but by hypothesis, this line passes through the point $x'y'$, therefore,

$$Xx' + Yy' = r^2;$$

this is the relation connecting the co-ordinates of the point XY, its locus, therefore, is the line

$$xx' + yy' = r^2,$$

or the polar of the point $x'y'$.

The proposition just proved may be stated otherwise, thus:

If one point lie on the polar of a second point, the second point will lie on the polar of the first point.

For the condition that $x'y'$ should lie on the polar of $x''y''$ is

$$x'x'' + y'y'' = r^2.$$

But this is also the condition that the point $x''y''$ should lie on the polar of $x'y'$.

97. *Given any point* O, *and any two lines through it; join both directly and transversely the points in which these lines meet a circle; then, if the direct lines intersect each other in* P *and the transverse in* Q, *the line* PQ *will be the polar of the point* O, *with regard to the circle.*

Take the two fixed lines for axes, and let the intercepts made on them by the circle be a and a', b and b'. Then

$$\frac{x}{a} + \frac{y}{b} - 1 = 0, \quad \frac{x}{a'} + \frac{y}{b'} - 1 = 0,$$

will be the equations of the direct lines; and

$$\frac{x}{a'} + \frac{y}{b} - 1 = 0, \quad \frac{x}{a} + \frac{y}{b'} - 1 = 0,$$

the equations of the transverse lines. Now, the equation of the line PQ will be

$$\frac{x}{a} + \frac{x}{a'} + \frac{y}{b} + \frac{y}{b'} - 2 = 0,$$

for (see Art. 36) this line passes through the intersection of

$$\frac{x}{a} + \frac{y}{b} - 1, \quad \frac{x}{a'} + \frac{y}{b'} - 1,$$

and also of

$$\frac{x}{a} + \frac{y}{b'} - 1, \quad \frac{x}{a'} + \frac{y}{b} - 1.$$

If the equation of the circle be

$$Ax^2 + Bxy + Ay^2 + Dx + Ey + F = 0,$$

a and a' are determined from the equation $Ax^2 + Dx + F = 0$ (Art. 82), therefore,

$$\frac{1}{a} + \frac{1}{a'} = -\frac{D}{F}, \text{ and } \frac{1}{b} + \frac{1}{b'} = -\frac{E}{F}.$$

Hence, equation of PQ is

$$Dx + Ey + 2F = 0;$$

but we saw (Art. 87) that this was the equation of the polar of the origin O. Hence it appears, that if the point O were given, and the two lines through it were *not* fixed, the *locus* of the points P and Q would be the polar of the point O.

98. *Given any two points* A *and* B, *and their polars, with respect to a circle whose centre is* O: *let fall a perpendicular* AP *from* A *on the polar of* B, *and a perpendicular* BQ *from* B *on the polar of* A; *then* $\frac{OA}{AP} = \frac{OB}{BQ}$.

The equation of the polar of A $(x'y')$ is $xx' + yy' - r^2 = 0$; and BQ, the perpendicular on this line from B$(x''y'')$, is (Art. 27)

$$\frac{x'x'' + y'y'' - r^2}{\sqrt{(x'^2 + y'^2)}}.$$

Hence, since $\sqrt{(x'^2 + y'^2)} = OA$, we find

$$OA \cdot BQ = x'x'' + y'y'' - r^2.$$

and, for the same reason,

$$OB \cdot AP = x'x'' + y'y'' - r^2.$$

Hence

$$\frac{OA}{AP} = \frac{OB}{BQ}.$$

99. *Given a circle and a triangle* ABC, *if we take the polars with respect to the circle of the three vertices of the triangle, we shall form a new triangle* A′B′C′ (*where* A′ *is the pole of* BC, B′ *the pole of* AC, *and* C′ *the pole of* AB), *then the lines* AA′, BB′, CC′, *will all pass through the same point.*

The equation of the line joining the point $x'y'$ to the intersection of the two lines $xx'' + yy'' - r^2 = 0$ and $xx''' + yy''' - r^2 = 0$ is (Art. 36)

$$\text{AA}' \quad (x'x''' + y'y''' - r^2)(xx'' + yy'' - r^2) - (x'x'' + y'y'' - r^2)(xx''' + yy''' - r^2) = 0.$$

In like manner,

$$\text{BB}' \quad (x'x'' + y'y'' - r^2)(xx''' + yy''' - r^2) - (x''x''' + y''y''' - r^2)(xx' + yy' - r^2) = 0;$$

and CC′

$$(x''x''' + y''y''' - r^2)(xx' + yy' - r^2) - (x'x''' + y'y''' - r^2)(xx'' + yy'' - r^2) = 0;$$

and by Art. 37 these lines must pass through the same point.

The following is a particular case of the theorem just proved. *If a circle be inscribed in a triangle, and each vertex of the triangle joined to the point of contact of the circle with the opposite side, the three joining lines will meet in a point.*

Ex. Prove, by Art. 38, that the three points of intersection of AB and A′B′, of AC and A′C′, and of BC and B′C′, lie in one right line.

100. In working out questions on the circle it is often convenient, instead of denoting the position of a point on the curve by its *two* co-ordinates $x'y'$, to express both these in terms of a single independent variable. Thus, let θ' be the angle which the radius to $x'y'$ makes with the axis of x, then $x' = r\cos\theta'$, $y' = r\sin\theta'$, and on substituting these values our formulæ will generally become simplified.

The equation of the tangent at the point $x'y'$ will by this substitution become

$$x\cos\theta' + y\sin\theta' = r;$$

and the equation of the chord joining $x'y'$, $x''y''$, which (Art. 83) is

$$x(x' + x'') + y(y' + y'') = r^2 + x'x'' + y'y'',$$

will, by a similar substitution, become

$$x\cos\tfrac{1}{2}(\theta'+\theta'')+y\sin\tfrac{1}{2}(\theta'+\theta'')=r\cos\tfrac{1}{2}(\theta'-\theta''),$$

θ' and θ'' being the angles which radii drawn to the extremities of the chord make with the axis of x.

This equation might also have been obtained directly from the general equation of a right line (Art. 25),

$$x\cos\alpha+y\sin\alpha=p,$$

for the angle which the perpendicular on the chord makes with the axis is plainly half the sum of the angles made with the axis by radii to its extremities; and the perpendicular on the chord

$$=r\cos\tfrac{1}{2}(\theta'-\theta'').$$

Ex. 1. To find the co-ordinates of the intersection of tangents at two given points on the circle. The tangents being

$$x\cos\theta'+y\sin\theta'=r,\quad x\cos\theta''+y\sin\theta''=r,$$

the co-ordinates of their intersection are

$$x=r\frac{\cos\frac{1}{2}(\theta'+\theta'')}{\cos\frac{1}{2}(\theta'-\theta'')},\quad y=r\frac{\sin\frac{1}{2}(\theta'+\theta'')}{\cos\frac{1}{2}(\theta'-\theta'')}.$$

Ex. 2. To find the locus of the intersection of tangents at the extremities of a chord whose length is constant.

Making the substitution of this article in

$$(x'-x'')^2+(y'-y'')^2=\text{constant},$$

it reduces to $\cos(\theta'-\theta'')=$ const., or $\theta'-\theta''=$ const. If the given length of the chord be $2r\sin\delta$, then $\theta'-\theta''=2\delta$. The co-ordinates then found in the last example fulfil the condition

$$(x^2+y^2)\cos^2\delta=r^2.$$

Ex. 3. What is the locus of a point where a chord of a constant length is cut in a given ratio?

Writing down (Art. 7) the co-ordinates of the point where the chord is cut in a given ratio, it will be found that they satisfy the condition $x^2+y^2=$ const.

Ex. 4. The diagonals of a hexagon circumscribing a circle meet in a point.

Let the angles made with the axis by radii to the points of contact be 2α, 2β, 2γ, 2δ, 2ϵ, 2ϕ; then the equation of the line joining the intersection of the tangents at 2α, 2β, to that of the tangents at 2δ, 2ϵ, will be $\frac{1}{\sin(\alpha-\delta)}\{x\cos(\alpha+\delta)+y\sin(\alpha+\delta)-r\cos(\alpha-\delta)\}+\frac{1}{\sin(\beta-\epsilon)}\{x\cos(\beta+\epsilon)+y\sin(\beta+\epsilon)-r\cos(\beta-\epsilon)\}=0$; which, when added to the other two equations of like form, vanishes identically.

101. We have seen that the tangent to any circle $x^2 + y^2 = r^2$ has an equation of the form

$$x \cos\theta + y \sin\theta = r;$$

and it would appear, in like manner, that the equation to the tangent to $(x-a)^2 + (y-b)^2 = r^2$ may be written

$$(x-a)\cos\theta + (y-b)\sin\theta = r:$$

conversely, then, if the equation of any right line contain an indeterminate θ in the form

$$(x-a)\cos\theta + (y-b)\sin\theta = r,$$

that right line will touch the circle

$$(x-a)^2 + (y-b)^2 = r^2.$$

Ex. 1. If a chord of a constant length be inscribed in a circle, it will always touch another circle. For, in the equation of the chord

$$x \cos\tfrac{1}{2}(\theta' + \theta'') + y \sin\tfrac{1}{2}(\theta' + \theta'') = r \cos\tfrac{1}{2}(\theta' - \theta'')$$

by the last article, $\theta' - \theta''$ is known, and $\theta' + \theta''$ indeterminate; the chord, therefore, always touches the circle

$$x^2 + y^2 = r^2 \cos^2\delta.$$

Ex. 2. Given any number of points, if a right line be such that m' times the perpendicular on it from the first point, $+ m''$ times the perpendicular from the second, + &c., be constant, the line will always touch a circle.

This only differs from the question, p. 48, in that the sum, in place of being = 0, is constant. Adopting then the notation of that Article, instead of the equation there found,

$$\{x\Sigma(m) - \Sigma(mx')\}\cos\alpha + \{y\Sigma(m) - \Sigma(my')\}\sin\alpha = 0,$$

we have only to write

$$\{x\Sigma m - \Sigma(mx')\}\cos\alpha + \{y\Sigma(m) - \Sigma(my')\}\sin\alpha = \text{const.}$$

Hence this line must always touch the circle

$$\left(x - \frac{\Sigma(mx')}{\Sigma(m)}\right)^2 + \left(y - \frac{\Sigma(my')}{\Sigma(m)}\right)^2 = \text{const.},$$

whose centre is the centre of mean position of the given points.

102. We shall conclude this Chapter with some examples of the use of polar co-ordinates.

Ex. 1. If through a fixed point any chord of a circle be drawn, the rectangle under its segments will be constant (Euclid, III. 35, 36).

Take the fixed point for the pole, and the polar equation is (Art. 93)

$$\rho^2 - 2\rho d\cos\theta + d^2 - r^2 = 0;$$

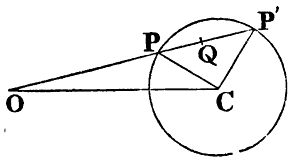
the roots of which are evidently OP, OP′, the values of the radius vector answering to any given value of θ or POC.

Now, by the theory of equations, OP . OP′, the product of these roots will $= d^2 - r^2$, a quantity *independent* of θ, and therefore constant, whatever be the direction in which the line OP is drawn. If the point O be outside the circle, it is plain that $d^2 - r^2$ must be = the square of the tangent.

Ex. 2. If through a fixed point O any chord of a circle be drawn, and OQ taken an arithmetic mean between the segments OP, OP′; to find the locus of Q.

We have OP + OP′, or the sum of the roots of the quadratic in the last example $= 2d\cos\theta$; but OP + OP′ = 2OQ, therefore,

$$OQ = d\cos\theta.$$



Hence the polar equation of the locus is

$$\rho = d\cos\theta.$$

Now it appears from the final equation in Art. 93, that this is the equation of a circle described on the line OC as diameter.

The question in this example might have been otherwise stated: "To find the locus of the middle points of chords which all pass through a fixed point."

Ex. 3. If the line OQ had been taken an *harmonic* mean between OP and OP′, to find the locus of Q.

That is to say, $OQ = \dfrac{2OP \,.\, OP'}{OP + OP'}$, but $OP \,.\, OP' = d^2 - r^2$, and $OP + OP' = 2d\cos\theta$, therefore, the polar equation of the locus is

$$\rho = \frac{d^2 - r^2}{d\cos\theta}, \quad \text{or } \rho\cos\theta = \frac{d^2 - r^2}{d}.$$

This is the equation of a right line (Art. 44) perpendicular to OC, and at a distance from $O = d - \dfrac{r^2}{d}$, and, therefore, at a distance from $C = \dfrac{r^2}{d}$. Hence (Art. 86) the locus is the *polar* of the point O.

We can, in like manner, solve this and similar questions when the equation is given in the form

$$Ax^2 + Ay^2 + Dx + Ey + F = 0,$$

for, transforming to polar co-ordinates, the equation becomes

$$\rho^2 + \left(\frac{D}{A}\cos\theta + \frac{E}{A}\sin\theta\right)\rho + \frac{F}{A} = 0,$$

and, proceeding precisely as in this example, we find, for the locus of harmonic means,

$$\rho = \frac{-2F}{D\cos\theta + E\sin\theta},$$

and, returning to rectangular co-ordinates, the equation of the locus is

$$Dx + Ey + 2F = 0,$$

the same as the equation of the polar obtained already (Art. 87).

Ex. 4. Given a point and a right line; if OQ be taken inversely as OP, the radius vector to the right line, find the locus of Q.

Ex. 5. Given vertex and vertical angle of a triangle and rectangle under sides; if one base angle describe a right line or a circle, find locus described by the other base angle.

Take the vertex for pole; let the lengths of the sides be ρ and ρ', and the angles they make with the axis θ and θ', then we have $\rho\rho' = k^2$ and $\theta - \theta' = C$.

The student must write down the polar equation of the locus which one base angle is said to describe; this will give him a relation between ρ and θ; then, writing for ρ, $\frac{k^2}{\rho'}$, and for θ, $C + \theta'$, he will find a relation between ρ' and θ', which will be the polar equation of the locus described by the other base angle.

This example might be solved in like manner, if the *ratio* of the sides, instead of their rectangle, had been given.

Ex. 6. Through the intersection of two circles a right line is drawn; find the locus of the middle point of the portion intercepted between the circles.

The equations of the circles will be of the form,

$$\rho = 2r\cos(\theta - \alpha); \qquad \rho = 2r'\cos(\theta - \alpha');$$

and the equation of the locus will be

$$\rho = r\cos(\theta - \alpha) + r'\cos(\theta - \alpha');$$

which also represents a circle.

Ex. 7. If through any point O, on the circumference of a circle, any three chords be drawn, and on each, as diameter, a circle be described, these three circles (which, of course, all pass through O) will intersect in three other points, which lie in one right line. (See *Cambr. Math. Jour.*, I. 169.)

Take the fixed point O for pole, then if d be the diameter of the original circle, its polar equation will be (Art. 93)

$$\rho = d\cos\theta.$$

In like manner, if the diameter of one of the other circles make an angle α with the fixed axis, its length will be $= d\cos\alpha$, and the equation of this circle will be

$$\rho = d\cos\alpha \, . \, \cos(\theta - \alpha).$$

The equation of another circle will, in like manner, be

$$\rho = d\cos\beta \, . \, \cos(\theta - \beta).$$

To find the polar co-ordinates of the point of intersection of these two, we should seek what value of θ would render

$$\cos\alpha \, . \, \cos(\theta - \alpha) = \cos\beta \, . \cos(\theta - \beta),$$

and it is easy to find that θ must $= \alpha + \beta$, and the corresponding value of $\rho = d\cos\alpha\cos\beta$.

Similarly, the polar co-ordinates of the intersection of the first and third circles are

$$\theta = \alpha + \gamma, \quad \text{and } \rho = d\cos\alpha\cos\gamma.$$

Now, to find the polar equation of the line joining these two points, take the general equation of a right line, $\rho\cos(k-\theta)=p$ (Art. 44), and substitute in it successively these values of θ and ρ, and we shall get two equations to determine p and k. We shall get

$$p = d\cos\alpha\cos\beta\cos(k-\overline{\alpha+\beta}) = d\cos\alpha\cos\gamma\cos(k-\overline{\alpha+\gamma}).$$

Hence

$$k=\alpha+\beta+\gamma, \quad \text{and } p = d\cos u\cos\beta\cos\gamma.$$

The symmetry of these values shows that it is the same right line which joins the intersections of the first and second, and of the second and third circles, and, therefore, that the three points are in a right line.

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* CHAPTER VIII.

APPLICATION OF ABRIDGED NOTATION TO THE EQUATION OF THE CIRCLE.

103. If we have an equation of the second degree expressed in the abridged notation explained in Chap. iv., and if we desire to know whether it represents a circle, we have only to transform to x and y co-ordinates, by substituting for each abbreviation (α) its equivalent ($x\cos\alpha + y\sin\alpha - p$); and then to examine whether the coefficient of xy in the transformed equation vanishes, and whether the coefficients of x^2 and of y^2 are equal. This is sufficiently illustrated in the examples which follow.

When will the locus of a point be a circle if the product of perpendiculars from it on two opposite sides of a quadrilateral be in a given ratio to the product of perpendiculars from it on the other two sides?

Let $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ be the equations of the four sides of the quadrilateral, then the equation of the locus is at once written down $\alpha\gamma = k\beta\delta$, which represents a curve of the second degree passing through the angles of the quadrilateral; since it is satisfied by any of the four suppositions,

$$\alpha=0,\ \beta=0;\quad \alpha=0,\ \delta=0;\quad \beta=0,\ \gamma=0;\quad \beta=0,\ \delta=0.$$

Now, in order to ascertain whether this equation represents a circle, write it at full length

$$(x \cos \alpha + y \sin \alpha - p)(x \cos \gamma + y \sin \gamma - p_{\prime\prime})$$
$$= k(x \cos \beta + y \sin \beta - p_{\prime})(x \cos \delta + y \sin \delta - p_{\prime\prime\prime}).$$

Multiplying out, equating the coefficient of x^2 to that of y^2, and putting that of $xy = 0$, we obtain the conditions

$$\cos(\alpha + \gamma) = k \cos(\beta + \delta); \quad \sin(\alpha + \gamma) = k \sin(\beta + \delta).$$

Squaring these equations, and adding them, we find $k = \pm 1$; and if this condition be fulfilled, we must have

$$\alpha + \gamma = \beta + \delta, \quad \text{or else} = 180^\circ + \beta + \delta;$$

whence $\alpha - \beta = \delta - \gamma, \quad \text{or } 180 + \delta - \gamma.$

Recollecting that $\alpha - \beta$ is the angle between the perpendiculars from the origin on the lines α and β, and is, therefore, the supplement of that angle between α and β, in which the origin lies, we see that this condition will be fulfilled if the quadrilateral formed by $\alpha\beta\gamma\delta$ be inscribable in a circle (Euclid, III. 22). And it will be seen on examination that when the origin is within the quadrilateral we are to take $k = -1$, and the angle (in which the origin lies) between α and β is supplemental to that between γ and δ; but that we are to take $k = +1$, when the origin is without the quadrilateral, and the opposite angles are equal.

104. *When will the locus of a point be a circle, if the square of its distance from the base of a triangle be in a constant ratio to the product of its distances from the sides?*

Let the sides of the triangle be α, β, γ, and the equation of the locus is $\alpha\beta = k\gamma^2$. If now we look for the points where the line α meets this locus, by making in it $\alpha = 0$, we obtain the perfect square $\gamma^2 = 0$. Hence α meets the locus in two coincident points, that is to say (Art. 83), it touches the locus at the point $\alpha\gamma$. Similarly, β touches the locus at the point $\beta\gamma$. Hence α and β are both tangents, and γ their chord of contact. Now, to ascertain whether the locus is a circle, writing at full length as in the last article, and applying the tests of Art. 78, we obtain the conditions

$$\cos(\alpha + \beta) = k \cos 2\gamma; \quad \sin(\alpha + \beta) = k \sin 2\gamma,$$

whence (as in the last article) we get $k = 1$, $\alpha - \gamma = \gamma - \beta$, or the triangle is isosceles. Hence we may infer that *if from any point of a circle perpendiculars be let fall on any two tangents and on their chord of contact, the square of the last will be equal to the rectangle under the other two.*

Ex. When will the locus of a point be a circle if the sum of the squares of the perpendiculars from it on the sides of any triangle be constant.

The locus is $\alpha^2 + \beta^2 + \gamma^2 = c^2$: and the conditions that this should represent a circle are

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0; \quad \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0.$$

$$\cos 2\alpha = -2\cos(\beta+\gamma)\cos(\beta-\gamma); \quad \sin 2\alpha = -2\sin(\beta+\gamma)\cos(\beta-\gamma).$$

Squaring and adding, $1 = 4\cos^2(\beta-\gamma)$; $\beta - \gamma = 60°$.

And so, in like manner, each of the other two angles of the triangle are proved to be 60°, or the triangle must be equilateral.

105. *To obtain the equation of the circle circumscribing the triangle formed by the lines* $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

Any equation of the form

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

denotes a curve of the second degree circumscribing the given triangle, since it is satisfied by any of the suppositions

$$\alpha = 0, \beta = 0; \quad \beta = 0, \gamma = 0; \quad \gamma = 0, \alpha = 0.$$

The conditions that it should represent a circle are found, by the same process as in Art. 103, to be

$$l\cos(\beta+\gamma) + m\cos(\gamma+\alpha) + n\cos(\alpha+\beta) = 0,$$
$$l\sin(\beta+\gamma) + m\sin(\gamma+\alpha) + n\sin(\alpha+\beta) = 0.$$

Eliminating successively m and n between the equations, we get

$$\frac{m}{l} = \frac{\sin(\gamma-\alpha)}{\sin(\beta-\gamma)}; \quad \frac{n}{l} = \frac{\sin(\alpha-\beta)}{\sin(\beta-\gamma)}.$$

Now, if C be the angle contained by the sides α, β, then

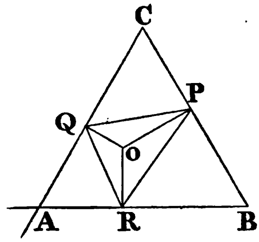
$$\sin C = \sin(\alpha - \beta), \text{ \&c.}$$

(since $\alpha - \beta$ is the angle between the perpendiculars on those sides), hence the equation of the circle circumscribing a triangle is,

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0.$$

106. The geometrical interpretation of the equation just found deserves attention. If from any point O we let fall perpendicu-

lars OP, OQ, on the lines α, β, then (Art. 53) α, β are the lengths of these perpendiculars; and since the angle between them is the supplement of C, the quantity $\alpha\beta \sin C$ is double the area of the triangle OPQ. In like manner, $\gamma\alpha \sin B$ and $\beta\gamma \sin A$ are double the triangles OPR, OQR. Hence the quantity



$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C$$

is double the area of the triangle PQR, and the equation found in the last article asserts, that if the point O be taken on the circumference of the circumscribing circle, the area PQR will vanish, that is to say (Art. 31, Cor. 2), the three points P, Q, R will lie on one right line.

If it were required to find the locus of a point from which, if we let fall perpendiculars on the sides of a triangle, and join their feet, the triangle PQR so formed should have a constant magnitude, the equation of the locus would be

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = \text{const.},$$

and, since this only differs from the equation of the circumscribing circle in the constant part, it is (Art. 78) the equation of a circle concentric with the circumscribing circle.

107. From the equation

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0,$$

we can find the equations of the tangents to the circle at the vertices of the triangle. Put the equation into the form

$$\gamma (\beta \sin A + \alpha \sin B) + \alpha\beta \sin C = 0,$$

and we saw (in Art. 105) that γ meets the circle in the two points where it meets the lines α and β, since, if we make $\gamma = 0$ in the equation of the circle, that equation will be reduced to $\alpha\beta = 0$. Now, for the very same reason, the two points in which the line $\beta \sin A + \alpha \sin B$ meets the circle, are the two points where *it* meets the lines α and β. But these two points coincide, since $\beta \sin A + \alpha \sin B$ passes through the point $\alpha\beta$. Hence, since the line $\beta \sin A + \alpha \sin B$ meets the circle in two coincident points, it is (Art. 83) a tangent at the point $\alpha\beta$.

We saw (Art. 63) that $\alpha \sin A + \beta \sin B$ is the equation of a parallel to the base (γ) drawn through the vertex $\alpha\beta$. Hence, by Art. 57, the tangent $\alpha \sin B + \beta \sin A$ makes the same angle with one side that the base makes with the other (Euclid, III. 32).

From the forms of the equations of the three tangents,

$$\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} = 0, \quad \frac{\beta}{\sin B} + \frac{\gamma}{\sin C} = 0, \quad \frac{\gamma}{\sin C} + \frac{\alpha}{\sin A} = 0,$$

it appears, that the three points in which they intersect each the opposite side are in one right line, whose equation is

$$\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} + \frac{\gamma}{\sin C} = 0.$$

It will be found that the equations of the lines joining the vertices of the inscribed triangle to those of the circumscribed are

$$\frac{\alpha}{\sin A} - \frac{\beta}{\sin B} = 0, \quad \frac{\beta}{\sin B} - \frac{\gamma}{\sin C} = 0, \quad \frac{\gamma}{\sin C} - \frac{\alpha}{\sin A} = 0;$$

and these meet in a point (Art. 36).

108. We shall next show how to obtain the equation of the circle *inscribed* in the triangle α, β, γ. The equation

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0,$$

represents a curve of the second degree, touching each of the lines α, β, γ; for if we seek the point where any side (γ) cuts the figure, making $\gamma = 0$, we obtain the perfect square,

$$l^2\alpha^2 + m^2\beta^2 - 2lm\alpha\beta = 0;$$

the roots of this equation being equal, we infer that the two points coincide in which γ cuts the figure, and therefore (Art. 83) that γ is a tangent.

In the same manner it can be proved that the sides α and β touch the curve represented by the preceding equation.

This equation may also be written in a convenient form,

$$l^{\frac{1}{2}}\alpha^{\frac{1}{2}} + m^{\frac{1}{2}}\beta^{\frac{1}{2}} + n^{\frac{1}{2}}\gamma^{\frac{1}{2}} = 0;$$

for if we clear this equation of radicals, we shall find it to be identical with that just written.

For the simplest method of obtaining the particular values of l, m, n, for which the preceding equation represents *a circle*, I am indebted to Dr. Hart, who derives the equation of the inscribed circle from that of the circumscribed, as follows: Join the points of contact of the circle inscribed in a triangle; let the equations of the sides of the triangle so formed be $\alpha' = 0$, $\beta' = 0$, $\gamma' = 0$, and its angles A′, B′, C′; then (Art. 105) the equation of the circle must be

$$\beta'\gamma' \sin A' + \gamma'\alpha' \sin B' + \alpha'\beta' \sin C' = 0.$$

Now we have proved (Art. 104) that for any point on the circle

$$\alpha'^2 = \beta\gamma; \quad \beta'^2 = \gamma\alpha; \quad \gamma'^2 = \alpha\beta,$$

and it is easy to see that

$$A' = 90° - \tfrac{1}{2}A; \quad B' = 90° - \tfrac{1}{2}B; \quad C' = 90° - \tfrac{1}{2}C.$$

Substituting these values, the equation of the circle becomes

$$\alpha^{\frac{1}{2}} \cos \tfrac{1}{2}A + \beta^{\frac{1}{2}} \cos \tfrac{1}{2}B + \gamma^{\frac{1}{2}} \cos \tfrac{1}{2}C = 0.$$

The general equation will, therefore, represent a circle if l, m, n, be proportional to

$$\cos^2 \tfrac{1}{2}A, \quad \cos^2 \tfrac{1}{2}B, \quad \cos^2 \tfrac{1}{2}C.$$

It can be proved, in like manner, that the equation of the circle touching the side a, and the sides b and c *produced*, is

$$\alpha^{\frac{1}{2}} \cos \tfrac{1}{2}A + \beta^{\frac{1}{2}} \sin \tfrac{1}{2}B + \gamma^{\frac{1}{2}} \sin \tfrac{1}{2}C = 0.$$

109. Since the general equation given in the last article may be written in the form

$$n\gamma(n\gamma - 2l\alpha - 2m\beta) + (l\alpha - m\beta)^2 = 0,$$

it follows that the line $(l\alpha - m\beta)$, which obviously passes through the point $\alpha\beta$, passes also through the point where γ meets the curve. The three lines, then, which join the points of contact of the sides with the opposite angles of the circumscribing triangle are

$$l\alpha - m\beta = 0, \quad m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0,$$

and these obviously meet in a point.

The very same proof which showed that γ touches the curve shows also that $n\gamma - 2l\alpha - 2m\beta$ touches the curve, for when this quantity is put = 0, we have the perfect square $(l\alpha - m\beta)^2 = 0$; hence this line meets the curve in two coincident points, that is, touches the curve, and $l\alpha - m\beta$ passes through the point of contact. Hence, if the vertices of the triangle be joined to the points of contact of opposite sides, and at the points where the joining lines meet the circle again, tangents be drawn, their equations are

$$2l\alpha + 2m\beta - n\gamma = 0, \quad 2m\beta + 2n\gamma - l\alpha = 0, \quad 2n\gamma + 2l\alpha - m\beta = 0.$$

Hence we infer that the three points, where each of these tangents meets the opposite side, lie in one right line,

$$l\alpha + m\beta + n\gamma = 0,$$

for this line passes through the intersection of the first line with γ, of the second with α, and of the third with β.

CHAPTER IX.

PROPERTIES OF A SYSTEM OF TWO OR MORE CIRCLES.

110. *To find the equation of the chord of intersection of two circles.*

If $S = 0$, $S' = 0$, be the equations of two circles, then any equation of the form $S - kS' = 0$ will be the equation of a figure passing through their points of intersection (Art. 36).

Let us write down the equations

$$S = (x - a)^2 + (y - b)^2 - r^2 = 0,$$
$$S' = (x - a')^2 + (y - b')^2 - r'^2 = 0,$$

and it is evident that the equation $S - kS' = 0$ will in general represent a circle, since the coefficient of $xy = 0$, and that of x^2 = that of y^2. There is one case, however, where it will represent a right line, namely, when $k = 1$. The terms of the second degree then vanish, and the equation becomes

$$S - S' = 2(a' - a)x + 2(b' - b)y + r'^2 - r^2 + a^2 - a'^2 + b^2 - b'^2 = 0.$$

This is, therefore, the equation of the right line passing through the points of intersection of the two circles.

111. The points of intersection of the two circles are found by seeking, as in Art. 80, the points in which the line $S - S'$ meets either of the given circles. These points will be real, coincident, or imaginary, according to the nature of the roots of the resulting equation; but it is remarkable that, whether the circles meet in real or imaginary points, the equation of the chord of intersection, $S - S' = 0$, always represents a real line, having important geometrical properties in relation to the two circles. This is in conformity with our assertion (Art. 81), that the line joining two points may preserve its existence and its properties when those points have become imaginary.

In order to avoid the harshness of calling the line $S - S' = 0$ the chord of intersection in the case where the circles do not *geometrically* appear to intersect, it has been called* the *radical axis* of the two circles.

112. One of the most remarkable properties of this line is found by examining the geometric meaning of the equation $S - S' = 0$. We saw (Art. 88) that if the co-ordinates of any point xy be substituted in S, it represents the square of the tangent drawn to the circle S, from the point xy. So also S' is the square of the tangent drawn to the circle S', and the equation $S - S' = 0$ asserts, that *if from any point on the radical axis tangents be drawn to the two circles, these tangents will be equal.*

The line $(S - S')$ possesses this property whether the circles meet in real points or not. When the circles do not meet in real points, the position of the radical axis is determined geometrically by cutting the line joining their centres, so that the difference of the squares of the parts may = the difference of the squares of the radii, and erecting a perpendicular at this point; as is evident, since the tangents from this point must be equal to each other.

If it were required to find the locus of a point whence tangents to two circles have *a given ratio*, it appears, from Art. 88, that the equation of the locus will be

$$S - k^2S' = 0,$$

* By M. Gaultier of Tours (*Journal de l'École Polytechnique*, Cahier xvi.; 1813).

which (Art. 110) represents a circle passing through the real or imaginary points of intersection of S and S′. When the circles S and S′ do not intersect in real points, we may express the relation which they bear to the circle $S - k^2S'$ by saying that the three circles have a common radical axis.

113. From the form of the equation of the radical axis of two circles, we at once derive the following theorem:

Given any three circles, if we take the radical axis of each pair of circles, these three lines will meet in a point, and this point is called the radical centre *of the three circles.*

For the equations of the three radical axes are

$$S - S' = 0, \quad S' - S'' = 0, \quad S'' - S = 0,$$

which, by Art. 37, meet in a point.

From this theorem we immediately derive the following:

If several circles pass through two fixed points, their chord of intersection with a fixed circle will pass through a fixed point.

For, imagine one circle through the two given points to be fixed, then its chord of intersection with the given circle will be fixed; and its chord of intersection with any variable circle drawn through the given points will plainly be the fixed line joining the two given points. These two lines determine, by their intersection, a fixed point through which the chord of intersection of the variable circle with the first given circle must pass.

Ex. 1. Find the radical axis of

$$x^2 + y^2 - 4x - 5y + 7 = 0; \quad x^2 + y^2 + 6x + 8y - 9 = 0.$$

Ans. $10x + 13y = 16$.

Ex. 2. Find the radical centre of

$$(x-1)^2 + (y-2)^2 = 7; \ (x-3)^2 + y^2 = 5; \ (x+4)^2 + (y+1)^2 = 9.$$

Ans. $\left(-\frac{1}{16}, -\frac{25}{16}\right)$.

114. A system of circles having a common radical axis possesses many remarkable properties which are more easily investigated by taking the radical axis for the axis of y, and the line joining the centres for the axis of x. Then the equation of any circle will be

$$x^2 + y^2 - 2kx + \delta^2 = 0,$$

where δ^2 is the same for all the circles of the system, and the equations of the different circles are obtained by giving different values to k.

For it is evident (Art. 78) that the centre is on the axis of x, at the variable distance k, and if we take any two circles,

$$x^2 + y^2 - 2k'x + \delta^2 = 0,$$
$$x^2 + y^2 - 2k''x + \delta^2 = 0,$$

and subtract one equation from the other, their chord of intersection will be $x = 0$, or the axis of y.

When we give to δ^2 the sign +, the radical axis will meet the circles in imaginary points, and when we give the sign −, in real points.

115. *If several circles pass through two fixed points, the polar of a given point, with regard to any of them, will always pass through a fixed point.*

The equation of the polar of $x'y'$ with regard to

$$x^2 + y^2 - 2kx + \delta^2 = 0$$

is (Art. 87) $\qquad xx' + yy' - k(x + x') + \delta^2 = 0;$

therefore, since this line involves the indeterminate k in the first degree, this line will always pass through the intersection of $xx' + yy' + \delta^2 = 0$, and $x + x' = 0$.

116. *There can always be found two points, however, such that their polars, with regard to any of the circles, will not only pass through a fixed point, but will be altogether fixed.*

This will happen when $xx' + yy' + \delta^2 = 0$, and $x + x' = 0$, represent the same right line, for this right line would then be the polar whatever the value of k. But that this should be the case we must have $\quad y' = 0$ and $x'^2 = \delta^2$, or $x' = \pm\,\delta$.

The two points whose co-ordinates have been just found have many remarkable properties in the theory of these circles, and are such that the polar of either of them, with regard to any of the circles, is a line drawn through the other perpendicular to the line of centres.

The equation of the circle may be written in the form

$$y^2 + (x - k)^2 = k^2 - \delta^2,$$

which evidently cannot represent a real circle if k^2 be less than δ^2; and if $k^2 = \delta^2$, then the equation will be of Class II. (Art. 79), and will represent a circle of infinitely small radius, the co-ordinates of whose centre are $y = 0$, $x = \pm\delta$. Hence the points just found may themselves be considered as circles of the system, and have, accordingly, been termed by Poncelet* the *limiting* points of the system of circles.

117. If from any point on the radical axis we draw tangents to all these circles, the locus of the points of contact must be a circle, since we proved (Art. 112) that all these tangents were equal. It is evident, also, that this circle cuts any of the given system at right angles, since its radii are tangents to the given system. The equation of this circle can be readily found.

The square of the tangent from any point ($x = 0$, $y = h$) to the circle

$$x^2 + y^2 - 2kx + \delta^2 = 0,$$

being found by substituting these co-ordinates in this equation, is $h^2 + \delta^2$; and the circle whose centre is the point ($x = 0$, $y = h$), and whose radius squared $= h^2 + \delta^2$, must have for its equation

$$x^2 + (y - h)^2 = h^2 + \delta^2,$$

or

$$x^2 + y^2 - 2hy = \delta^2.$$

Hence, whatever be the point taken on the radical axis (i. e. whatever the value of h may be), still this circle will always pass through the fixed points ($y = 0$, $x = \pm\delta$) found in the last Article. And we infer that all circles which cut the given system at right angles pass through the limiting points of the system.

Ex. 1. If the polar of A with respect to the system pass through the fixed point B, prove that the semicircle described on AB passes through the limiting points.

Ex. 2. The square of the tangent from any point of one circle to another is in a constant ratio to the perpendicular from that point upon their radical axis.

Ex. 3. To find the angle (α) at which two circles intersect.

Let the radii of the circles be R, r, and let D be the distance between their centres, then

$$D^2 = R^2 + r^2 - 2Rr\cos\alpha.$$

Since the angle at which the circles intersect is equal to that between the radii to the point of intersection.

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* *Traité des Propriétés Projectives*, p. 41.

Ex. 4. If a moveable circle cut two fixed circles at constant angles, it will cut all circles having the same radical axis at constant angles.

Let the equations of the two fixed circles be $S = 0$, $S' = 0$, and their radii r, r'; then the co-ordinates of the centre of the moveable circle fulfil the relations,

$$R^2 - 2Rr\cos\alpha = S, \qquad R^2 - 2Rr'\cos\beta = S',$$

since $D^2 - r^2$ = the square of the tangent to the first fixed circle = S (Art. 88). Then, we have

$$R^2 - 2R\frac{kr\cos\alpha + lr'\cos\beta}{k+l} = \frac{kS + lS'}{k+l},$$

which is precisely the condition that the moveable circle should cut the circle $kS + lS'$ at the constant angle γ; where $(k + l)r''\cos\gamma = kr\cos\alpha + lr'\cos\beta$, r'' being the radius of the circle $kS + lS'$

Ex. 5. A circle which cuts two fixed circles at constant angles will also touch two fixed circles.

For we can determine the ratio $k : l$, so that γ shall = 0, or $\cos\gamma = 1$. It will easily be found that if D be the distance between the centres of S and S′,

$$(k + l)^2 r''^2 = (k + l)(kr^2 + lr'^2) - klD^2.$$

Substituting this value for r'' in the equation of the last example, we get a quadratic to determine $k : l$.

118. *To draw a common tangent to two circles.*

Let their equations be

$$(x - a)^2 + (y - b)^2 = r^2 \qquad (S),$$

and

$$(x - a')^2 + (y - b')^2 = r'^2 \qquad (S').$$

We saw (Art. 83) that the equation of a tangent to (S) was

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2;$$

or, as in Art. 100, writing

$$\frac{x' - a}{r} = \cos\alpha, \quad \frac{y' - b}{r} = \sin\alpha,$$

$$(x - a)\cos\alpha + (y - b)\sin\alpha = r.$$

In like manner, any tangent to (S′) is

$$(x - a')\cos\beta + (y - b)\sin\beta = r'.$$

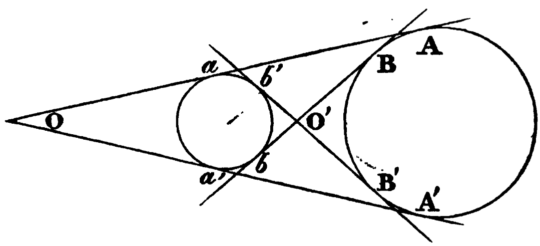
Now, if we seek the conditions necessary that these two equations should represent *the same* right line; first, from comparing the ratio of the coefficients of x and y, we get $\tan\alpha = \tan\beta$, whence β either = α, or = $180° + \alpha$. If either of these conditions be fulfilled, we must equate the absolute terms, and we find, in the first case,

$$(a - a')\cos\alpha + (b - b')\sin\alpha + r - r' = 0,$$

and in the second case,

$$(a - a') \cos \alpha + (b - b') \sin \alpha + r + r' = 0.$$

Either of these equations would give us a quadratic to determine α. The two roots of the first equation would correspond



to the direct or exterior common tangents, Aa, A$'a'$; the roots of the second equation would correspond to the transverse or interior tangents, Bb, B$'b'$.

If we wished to find the co-ordinates of the point of contact of the common tangent with the circle (S), we must substitute, in the equation just found, for $\cos \alpha$, its value, $\dfrac{x' - a}{r}$, and for $\sin \alpha$, $\dfrac{y' - b}{r}$, and we find

$$(a - a')(x' - a) + (b - b')(y' - b) + r(r - r') = 0;$$

or else,

$$(a - a')(x' - a) + (b - b')(y' - b) + r(r + r') = 0.$$

The first of these equations, combined with the equation (S) of the circle, will give a quadratic, whose roots will be the co-ordinates of the points A and A$'$, in which the direct common tangents touch the circle (S); and it will appear, as in Art. 86, that

$$(a' - a)(x - a) + (b' - b)(y - b) = r(r - r')$$

is the equation of AA$'$, the chord of contact of direct common tangents. So, likewise,

$$(a' - a)(x - a) + (b' - b)(y - b) = r(r + r')$$

is the equation of the chord of contact of transverse common tangents. If the origin be the centre of the circle (S), then a and $b = 0$; and we find, for the equation of the chord of contact,

$$a'x + b'y = r(r \mp r').$$

Ex. 1. Find the common tangents to the circles

$$x^2 + y^2 - 4x - 2y + 4 = 0, \quad x^2 + y^2 + 4x + 2y - 4 = 0.$$

The chords of contact of common tangents with the first circle are

$$2x + y = 6, \quad 2x + y = 3.$$

The first chord meets the circle in the points (2, 2), $\left(\frac{14}{5}, \frac{2}{5}\right)$, the tangents at which are

$$y = 2, \; 4x - 3y = 10,$$

and the second chord meets the circle in the points (1, 1), $\left(\frac{7}{5}, \frac{1}{5}\right)$, the tangents at which are

$$x = 1, \quad 3x + 4y = 5.$$

119. The points O and O′, in which the direct or transverse tangents intersect, are (for a reason explained in the next Article) called the *centres of similitude* of the two circles.

Their co-ordinates are easily found, for O is the pole, with regard to circle (S), of the chord AA′, whose equation is

$$\frac{(a' - a)\,r}{r - r'}(x - a) + \frac{(b' - b)\,r}{r - r'}(y - b) = r^2.$$

Comparing this equation with the equation of the polar of the point $x'y'$,

$$(x' - a)(x - a) + (y' - b)(y - b) = r^2,$$

we get

$$x' - a = \frac{(a' - a)\,r}{r - r'}, \text{ or } x' = \frac{a'r - ar'}{r - r'},$$

$$y' - b = \frac{(b' - b)\,r}{r - r'}, \text{ or } y' = \frac{b'r - br'}{r - r'}.$$

So, likewise, the co-ordinates of O′ are found to be

$$x = \frac{a'r + ar'}{r + r'}, \text{ and } y = \frac{b'r + br'}{r + r'}.$$

These values of the co-ordinates indicate (see Art. 7) that the centres of similitude are the points where the line joining the centres is cut externally and internally in the ratio of the radii.

Ex. Find the common tangents to the circles

$$x^2 + y^2 - 6x - 8y = 0, \quad x^2 + y^2 - 4x - 6y = 3.$$

The equation of the pair of tangents through $x'y'$ to

$$(x - a)^2 + (y - b)^2 = r^2$$

is found (Art. 90) to be

$$\{(x'-a)^2+(y'-b)^2-r^2\}\,\{(x-a)^2+(y-b)^2-r^2\} = \{(x-a)(x'-a)+(y-b)(y'-b)-r^2\}^2.$$

Now, the co-ordinates of the exterior centre of similitude are found to be $(-2, -1)$, and hence the pair of tangents through it is

$$25(x^2+y^2-6x-8y)=(5x+5y-10)^2; \text{ or } xy+x+2y+2=0; \text{ or } (x+2)(y+1)=0.$$

As the given circles intersect in real points, the other pair of common tangents become imaginary; but their equation is found, by calculating the pair of tangents through the other centre of similitude $\left(\frac{22}{9}, \frac{31}{9}\right)$, to be

$$40x^2+xy+40y^2-199x-278y+722=0.$$

120. *Every right line drawn through the intersection of common tangents is cut similarly by the two circles.*

It is evident that if on the radius vector to any point P there be taken a point Q, such that OP = m times OQ, then the x and y of the point P will be respectively m times the x and y of the point Q; and that, therefore, if P describe any curve, the locus of Q is found by substituting mx, my for x and y in the equation of the curve described by P.

Now, if the common tangents be taken for axes, and if we denote Oa by a, OA by a', the equations of the two circles are (Art. 82, Ex. 4)

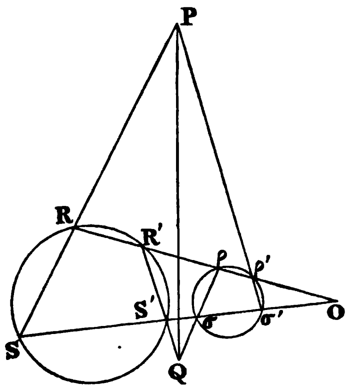
$$x^2+y^2+2xy\cos\omega-2ax-2ay+a^2=0,$$
$$x^2+y^2+2xy\cos\omega-2a'x-2a'y+a'^2=0.$$

But the second equation is what we should have found if we had substituted $\frac{ax}{a'}$, $\frac{ay}{a'}$, for x, y, in the first equation; and it therefore represents the locus formed by producing each radius vector to the first circle in the ratio $a:a'$.

Cor.—Since the rectangle $O\rho\cdot O\rho'$ is constant (see fig. next page), and since we have proved OR to be in a constant ratio to $O\rho$, it follows that the rectangle $OR\cdot O\rho' = OR'\cdot O\rho$ is constant, however the line be drawn through O.

121. *If through a centre of similitude we draw any two lines meeting the first circle in the points* R, R′, S, S′, *and the second in the points* ρ, ρ', σ, σ', *then the chords* RS, $\rho\sigma$; R′S′, $\rho'\sigma'$; *will be parallel, and the chords* RS, $\rho'\sigma'$; R′S′, $\rho\sigma$; *will meet on the radical axis of the two circles.*

Take OR, OS for axes, then we saw (Art. 120) that $OR = mO\rho$, $OS = mO\sigma$, and that if the equation of the circle $\rho\sigma\rho'\sigma'$ be



$$Ax^2 + Bxy + Ay^2 + Dx + Ey + F = 0,$$

that of the other will be

$$Ax^2 + Bxy + Ay^2 + m(Dx + Ey) + m^2F = 0,$$

and, therefore, the equation of the radical axis will be (Art. 111)

$$Dx + Ey + (m + 1)F = 0.$$

Now let the equations of $\rho\sigma$ and of $\rho'\sigma'$ be

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a'} + \frac{y}{b'} = 1,$$

then the equations of RS and of R'S' must be

$$\frac{x}{ma} + \frac{y}{mb} = 1, \quad \frac{x}{ma'} + \frac{y}{mb'} = 1.$$

It is evident, from the form of the equations, that RS is parallel to $\rho\sigma$; and RS and $\rho'\sigma'$ must intersect on the line

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 1 + m,$$

or, as in Art. 97, on

$$Dx + Ey + (1 + m)F = 0,$$

the radical axis of the two circles.

A particular case of this theorem is, that the tangents at R and ρ are parallel, and that those at R and ρ' meet on the radical axis.

122. *Given three circles; the line joining a centre of similitude of the first and second to a centre of similitude of the first and third will pass through a centre of similitude of the second and third.*

Form the equation of the line joining the points

$$\left(\frac{ra' - ar'}{r - r'}, \frac{rb' - br'}{r - r'}\right), \quad \left(\frac{ra'' - ar''}{r - r''}, \frac{rb'' - br''}{r - r''}\right),$$

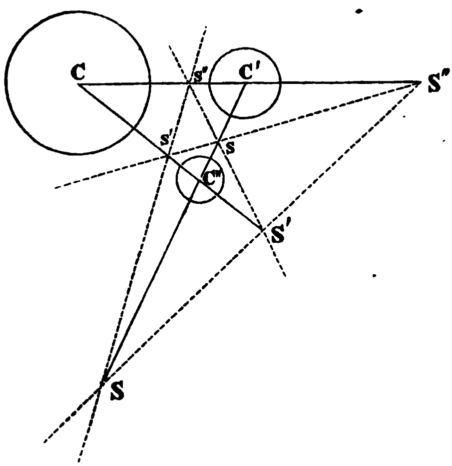
(Art. 119), and we get (see Ex. 7, p. 24),

$$\{r(b'-b'')+r'(b''-b)+r''(b-b')\}\,x$$
$$-\{r(a'-a'')+r'(a''-a)+r''(a-a')\}\,y$$
$$+r(b'a''-b''a')+r'(b''a-ba'')+r''(ba'-b'a)=0.$$

Now the symmetry of this equation sufficiently shows, that the line it represents must pass through the third centre of similitude,

$$x=\frac{r'a''-r''a'}{r'-r''},\quad y=\frac{r'b''-r''b'}{r'-r''}.$$

This line is called an *axis of similitude* of the three circles.



Since for each pair of circles there are two centres of similitude, there will be in all *six* for the three circles, and these will be distributed along *four* axes of similitude, as represented in the figure. The equations of the other three will be found by changing the signs of either r, or r', or r'', in the equation just given.

123. *If a circle* (Σ) *touch two others* (S *and* S′) *the line joining the points of contact will pass through a centre of similitude of* S *and* S′.

For when two circles touch, one of their centres of similitude will coincide with the point of contact, and, by the theorem proved in the last article, the line joining a centre of similitude of S and Σ, to a centre of similitude of S′ and Σ must pass through a centre of similitude of S and S′.

If Σ touch S and S′, either both externally or both internally, the line joining the points of contact will pass through the *external* centre of similitude of S and S′. If Σ touch one externally and the other internally, the line joining the points of contact will pass through the *internal* centre of similitude.

*124. We shall conclude this chapter by investigating the problem: *To describe a circle to touch three given circles.*

Let the equations of the three circles be

$$(x-a)^2+(y-b)^2-r^2=0, \text{ or } S=0,$$
$$(x-a')^2+(y-b')^2-r'^2=0, \text{ or } S'=0,$$
$$(x-a'')^2+(y-b'')^2-r''^2=0, \text{ or } S''=0.$$

We can determine the position of the centre of the touching circle from the condition, that the distance between the centres of any two touching circles must equal the sum of their radii.

Now the square of the distance of any point from the centre of (S)

$$=(x-a)^2+(y-b)^2=S+r^2.$$

Hence we get the condition

$$S+r^2=(R+r)^2;$$

and, in like manner,

$$S'+r'^2=(R+r')^2,$$

and

$$S''+r''^2=(R+r'')^2.$$

[These equations, evidently, apply to the case of *external* contact. If the contact with any of the circles be *internal*, the distance between the centres will then = the *difference* of the radii, and we must change the sign of r or r' or r'' in the preceding formulæ. As this gives rise to the following different possible combinations of signs,

$$\begin{array}{llllllllll} r, & + & + & + & + & - & - & - & -, \\ r', & + & + & - & - & + & + & - & -, \\ r'', & + & - & + & - & + & - & + & -, \end{array}$$

there may be *eight* circles touching the three given circles.]

If now we eliminate R from the preceding formulæ, we shall get two equations which will enable us to determine the co-ordinates of the centre of the touching circle.

Subtract the equations, and we get

$$S-S'=2R(r-r'), \text{ and } S-S''=2R(r-r''),$$

or

$$\frac{S-S'}{r-r'}=\frac{S-S''}{r-r''}.$$

This is the equation of the line joining the centre of the touching circle to the *radical centre* (Art. 113). It may be written in the more symmetrical form

$$(r' - r'')\,S + (r'' - r)\,S' + (r - r')\,S'' = 0.$$

If we now write for S, &c., their values, the coefficient of x in this equation is found to be

$$-2\,\{a(r' - r'') + a'(r'' - r) + a''(r - r')\},$$

and of y to be

$$-2\,\{b(r' - r'') + b'(r'' - r) + b''(r - r')\}.$$

Now if we compare these coefficients with the coefficients in the equation of the axis of similitude (Art. 122), we arrive at the conclusion (see Art. 40) that *the centre of the circle touching three others lies on the perpendicular let fall from their radical centre on the axis of similitude.*

We saw that eight circles can be drawn to touch three given circles, and as the three circles have four axes of similitude, the centres of the touching circles will lie, a pair on each of the perpendiculars let fall from the radical centre on the four axes of similitude.

Two circles answer to each axis of similitude; for the equation of an axis of similitude (Art. 122) remains unaltered, if we change in it the signs of *all* the radii. Hence the axis answering to the case of external contact (or $+r + r' + r''$) must also answer to the case of internal contact (or $-r - r' - r''$); and similarly for the other axes of similitude.

125. From the three equations found in the last article we can obtain another relation between the co-ordinates of the centre of the touching circle. This relation, however, will be of the second degree, and, though sufficient for the *algebraical* solution of the problem, does not enable us to represent the results in an *elementary geometrical* manner. To remove this inconvenience M. Gergonne proposed to seek the co-ordinates not of the *centre* of the touching circle, but of its *point of contact* with one of the given circles. We have already one relation connecting these co-ordinates, since the point lies on a given circle; therefore, if we can find another relation between them, it will suffice completely to determine the point.*

* Gergonne, *Annales des Mathématiques*, vol. vii. p. 289.

Let us for simplicity take for origin the centre of the circle, the point of contact with which we are seeking, that is to say, let us take $a = 0$, $b = 0$, then if A and B be the co-ordinates of the centre of Σ, the sought circle, we have seen in the last article, that they fulfil the relations

$$S - S' = 2R(r - r'), \quad S - S'' = 2R(r - r'').$$

But if x and y be the co-ordinates of the point of contact of Σ with S, we have from similar triangles

$$A = \frac{x(R+r)}{r}, \quad B = \frac{y(R+r)}{r}.$$

Now if in the equation of any right line we substitute mx, my for x and y, the result will evidently be the same as if we multiplied the whole equation by m and subtract $(m-1)$ times the absolute term. Hence, remembering that the absolute term in $S - S'$ is (Art. 110) $r'^2 - r^2 - a'^2 - b'^2$, the result of making the above substitutions for A and B in $(S - S') = 2R(r - r')$ is

$$\frac{R+r}{r}(S - S') + \frac{R}{r}(a'^2 + b'^2 + r^2 - r'^2) = 2R(r - r'),$$

or $$(R + r)(S - S') = R\{(r - r')^2 - a'^2 - b'^2\}.$$

Similarly $$(R + r)(S - S'') = R\{(r - r'')^2 - a''^2 - b''^2\}.$$

Eliminating R, the point of contact is determined as one of the intersections of the circle S with the right line

$$\frac{S - S'}{a'^2 + b'^2 - (r - r')^2} = \frac{S - S''}{a''^2 + b''^2 - (r - r'')^2}.$$

126. To complete the geometrical solution of the problem it is necessary to show how to construct the line whose equation has been just found. It obviously passes through the radical centre of the circles; and a second point on it is found as follows. Write at full length for $S - S'$ (Art. 110), and the equation is

$$\frac{2a'x + 2b'y + r'^2 - r^2 - a'^2 - b'^2}{a'^2 + b'^2 - (r - r')^2} = \frac{2a''x + 2b''y + r''^2 - r^2 - a''^2 - b''^2}{a''^2 + b''^2 - (r - r'')^2}.$$

Add 1 to both sides of the equation, and we have

$$\frac{a'x + b'y + (r' - r)r}{a'^2 + b'^2 - (r - r')^2} = \frac{a''x + b''y + (r'' - r)r}{a''^2 + b''^2 - (r - r'')^2},$$

showing that the above line passes through the intersection of

$$a'x + b'y + (r' - r)r = 0, \qquad a''x + b''y + (r'' - r)r = 0.$$

But the first of these lines (Art. 118) is the chord of common tangents of the circles S and S′; or, in other words (Art. 119), is the polar with regard to S of the centre of similitude of these circles. And in like manner the second line is the polar of the centre of similitude of S and S″; therefore (since the intersection of any two lines is the pole of the line joining their poles) the intersection of the lines

$$a'x + b'y + (r' - r)r = 0, \qquad a''x + b''y + (r'' - r)r = 0$$

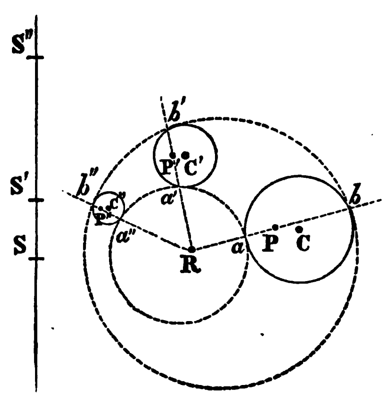
is the pole of the axis of similitude of the three circles, with regard to the circle S.

Hence we obtain the following construction:

Drawing any of the four axes of similitude of the three circles, take its pole with respect to each circle, and join the points so found (P, P′, P″) with the radical centre; then, if the joining lines meet the circles in the points

$$(a, b;\ a', b';\ a'', b''),$$

the circle through a, a', a'' will be one of the touching circles, and that through b, b', b'' will be another. Repeating this process with the other three axes of similitude, we can determine the other six touching circles.



127. It is useful to show how the preceding results may be derived without algebraical calculations.

(1.) By Art. 123 the lines ab, $a'b'$, $a''b''$ meet in a point, viz., the centre of similitude of the circles $aa'a''$, $bb'b''$.

(2.) In like manner $a'a''$, $b'b''$ intersect in S, the centre of similitude of C′, C″.

(3.) Hence (Art. 121) the transverse lines $a'b'$, $a''b''$ intersect on the radical axis of C′, C″. So again $a''b''$, ab, intersect on the radical axis of C″, C. Therefore the point R (the centre of similitude of $aa'a''$, $bb'b''$) must be the radical centre of the circles C, C′, C″.

(4.) In like manner, since $a'b'$, $a''b''$ pass through a centre of similitude of $aa'a''$, $bb'b''$; therefore (Art. 121) $a'a''$, $b'b''$ meet on the radical axis of these two circles. So again the points S′ and S″ must lie on the same radical axis; *therefore* SS′S″, *the axis of similitude of the circles* C, C′, C″, *is the radical axis of the circles* $aa'a''$, $bb'b''$.

(5.) Since $a''b''$ passes through the centre of similitude of $aa'a''$, $bb'b''$, therefore (Art. 121) the tangents to these circles where it meets them intersect on the radical axis SS′S″. But this point of intersection must plainly be the pole of $a''b''$ with regard to the circle C″. Now since the pole of $a''b''$ lies on SS′S″, therefore (Art. 96) the pole of SS′S″ with regard to C″ lies on $a''b''$. Hence $a''b''$ is constructed by joining the radical centre to the pole of SS′S″ with regard to C″.

(6.) Since the centre of similitude of two circles is on the line joining their centres, and the radical axis is perpendicular to that line, we learn (as in Art. 124) that the line joining the centres of $aa'a''$, $bb'b''$ passes through R, and is perpendicular to SS′S″.

Ex. To describe a circle cutting three given circles at given angles.

By the help of (Ex. 5, Art. 117) this is reduced to the problem of the present article; or else the three equations

$$R^2 - 2Rr\cos\alpha = S, \quad R^2 - 2Rr'\cos\beta = S', \quad R^2 - 2Rr''\cos\gamma = S'',$$

may be discussed directly as in Art. 124.

CHAPTER X.

PROPERTIES COMMON TO ALL CURVES OF THE SECOND DEGREE, DEDUCED FROM THE GENERAL EQUATION.

128. The most general form of the equation of the second degree is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, C, D, E, F are all constants.

The nature of the curve represented by this equation will vary according to the particular values of these constants. Thus we saw (Chap. v.), that in some cases this equation might represent two right lines, and (Chap. vi.) that for other values of the constants it might represent a circle. It is our object in this chapter to classify the different curves which can be represented by equations of the general form just written, and to obtain some of the properties which are common to them all.*

Five relations between the coefficients are sufficient to determine a curve of the second degree. It is true that the general equation contains *six* constants, but it is plain that the nature of the curve does not depend on the *absolute magnitude* of these coefficients, since, if we multiply or divide the equation by any constant, it will still represent the same curve. We may, therefore, divide the equation by F, so as to make the absolute term = 1, and there will then remain but five constants to be determined.

Thus, for example, a conic section can be described *through five points*. Substituting in the equation the co-ordinates of each point $(x'y')$ through which the curve must pass, we obtain five relations between the coefficients, viz.,

$$\frac{A}{F}x'^2 + \frac{B}{F}x'y' + \frac{C}{F}y'^2 + \frac{D}{F}x' + \frac{E}{F}y' + 1 = 0, \text{ \&c.},$$

which will enable us to determine the five quantities, $\frac{A}{F}$, &c.

129. We shall in this chapter often have occasion to use the method of transformation of co-ordinates; and it will be useful to find what the general equation becomes when transformed to parallel axes through a new origin $(x'y')$. We form the new equation by substituting $x + x'$ for x, and $y + y'$ for y (Art. 8), and we get

$$A(x+x')^2 + B(x+x')(y+y') + C(y+y')^2 + D(x+x') + E(y+y') + F = 0.$$

* We shall prove hereafter, that the section made by any plane in a cone standing on a circular base is a curve of the second degree, and, conversely, that there is no curve of the second degree which may not be considered as a *conic section*. It was in this point of view that these curves were first examined by geometers. We mention the property here, because we shall often find it convenient to use the terms "conic section" or "conic," instead of the longer appellation, "curve of the second degree."

Arranging this equation according to the powers of the variables, we find that the coefficients of x^2, xy, and y^2, will be, as before, A, B, C; that

the new D, $D' = 2Ax' + By' + D$;
the new E, $E' = 2Cy' + Bx' + E$;
the new F, $F' = Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F$.

Hence, *if the equation of a curve of the second degree be transformed to parallel axes through a new origin, the coefficients of the highest powers of the variables will remain unchanged, while the new absolute term will be the result of substituting in the original equation the co-ordinates of the new origin.**

130. *Every right line must meet a curve of the second degree in two real, coincident, or imaginary points.*

Let us first consider the case of lines which pass through the origin. The truth of the proposition will then easily appear by transformation to polar co-ordinates. If the angle between the axes be ω, then for a line making angles α, β, with the axes, we saw (Art. 12) that $x \sin\omega = \rho \sin\alpha$, $y \sin\omega = \rho \sin\beta$, or, as we shall write for shortness, $x = m\rho$, $y = n\rho$. Making these substitutions in the general equation, we have, to determine the length of the radius vector to either of the points where the line (whose equation obviously is $my = nx$) meets the curve, the quadratic,

$$(Am^2 + Bmn + Cn^2)\rho^2 + (Dm + En)\rho + F = 0.$$

Since this equation always gives *two* values for ρ, we see, as in Art. 81, that every line through the origin will meet the curve in two real, coincident, or imaginary points.

The case of a line not passing through the origin is reduced to the former, by transferring the origin to any point on the line. The equation will then become

$$Ax^2 + Bxy + Cy^2 + D'x + E'y + F' = 0;$$

where D', E', F' have the values found in the last article, and the distances from the new origin of the points where any line through it meets the curve, are the *two* roots of a quadratic equation, precisely similar in form to that already given.

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* This is equally true for equations of any degree, as can be proved in like manner.

131. The next articles will be occupied with a discussion of the different forms assumed by the quadratic just found for ρ, according to the different values we may give the ratio $m:n$. The reader will better understand the method we pursue if he bear in mind the following elementary principles. Suppose that we have to discuss any quadratic,

$$a\rho^2 + b\rho + c = 0,$$

its solution may be written in either of the following equivalent forms,

$$\rho = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} = \frac{2c}{-b \mp \sqrt{(b^2 - 4ac)}};$$

the latter being the form in which the solution would have presented itself had we divided the given equation by ρ^2, and solved it for the reciprocal of ρ.

I. If we have $c = 0$, the quadratic is divisible by ρ, and one of its roots is $\rho = 0$, the other being $= -\frac{b}{a}$. If we had not only $c = 0$, but also $b = 0$, then the quadratic would be divisible by ρ^2, and *both* its roots would $= 0$.

II. If we have $a = 0$, then one of the roots of the equation is $\rho = \infty$. For if we had written the equation

$$c\left(\frac{1}{\rho}\right)^2 + b\left(\frac{1}{\rho}\right) + a = 0,$$

it appears from the last case that when $a = 0$ the two roots are $\frac{1}{\rho} = 0$, $\frac{1}{\rho} = -\frac{b}{c}$, to which values correspond $\rho = \infty$, $\rho = -\frac{c}{b}$. The same thing may be seen by making $a = 0$ in the general form of the solution. If not only $a = 0$, but also $b = 0$, both the roots $= \infty$.

III. If $b = 0$, the roots of the quadratic are equal with opposite signs.

IV. If we have $b^2 = 4ac$, the two roots are equal, and we may write either $\rho = -\frac{b}{2a}$ or $= -\frac{2c}{b}$. If b^2 be greater than $4ac$, the roots of the quadratic are real; if b^2 be less than $4ac$, the roots are imaginary.

132. Let us now apply these principles to the equation which determines the points where the line ($my = nx$) meets the curve, viz.
$$(Am^2 + Bmn + Cn^2)\rho^2 + (Dm + En)\rho + F = 0.$$

1. Let $F = 0$. In this case one of the values of ρ is $= 0$, or the origin is one of the points where the line meets the curve (see also Art. 79). The other value is
$$\rho = -\frac{Dm + En}{Am^2 + Bmn + Cn^2}.$$

If, however, we have not only $F = 0$, but also the line be drawn in such a direction that $Dm + En = 0$, then the second value of ρ is also $= 0$: the line ($my = nx$) meets the curve in two coincident points at the origin, or, in other words, is a tangent at the origin. Multiplying by ρ the equation $Dm + En = 0$, and remembering that $m\rho = x$, $n\rho = y$, we find the equation of the tangent at the origin, viz.
$$Dx + Ey = 0.$$

Ex. 1. Find the tangent at the origin to
$$5x^2 + 7xy + y^2 - x + 2y = 0. \qquad \textit{Ans.} \quad x = 2y.$$

Ex. 2. The point (1, 1) is on the curve
$$3x^2 - 4xy + 2y^2 + 7x - 5y - 3 = 0;$$
transform the equation to parallel axes through this point, and find the tangent at it.
Ans. $9x - 5y = 0$ referred to the new axes, or $9(x-1) - 5(y-1) = 0$ referred to the old.

133. *To find the equation of the tangent at any point $x'y'$ on the curve.*

Transform to parallel axes through $x'y'$, and (Art. 129) F' will vanish, since $x'y'$ is on the curve. The equation of the tangent will then be $D'x + E'y = 0$ referred to the new axes, or $D'(x - x') + E'(y - y') = 0$ referred to the old. Write for D' and E' the values found in Art. 129, and the equation of the tangent is
$$(2Ax' + By' + D)(x - x') + (Bx' + 2Cy' + E)(y - y') = 0,$$
which may be written in a simpler form by adding to both sides the identity
$$2Ax'^2 + 2Bx'y' + 2Cy'^2 + 2Dx' + 2Ey' + 2F = 0,$$
when the equation of the tangent becomes
$$(2Ax' + By' + D)x + (Bx' + 2Cy' + E)y + Dx' + Ey' + 2F = 0.$$

This equation might also have been found by the method pursued in Art. 84.

Ex. Find the tangent at (2, 1) to

$$3x^2 + 4xy + 5y^2 - 7x - 8y - 3 = 0.$$

Ans. $9x + 10y = 28$.

134. II. Let us next consider the case in which one value of ρ may become infinite. We have seen (Art. 131) that this will be the case when the coefficient of ρ^2 vanishes in the quadratic which determines ρ; or, in other words, when

$$Am^2 + Bmn + Cn^2 = 0.$$

If then $m:n$ be taken so as to satisfy this relation, the line $(my = nx)$ will meet the curve in one infinitely distant point: the other value of ρ will in general remain finite, and will

$$= -\frac{F}{Dm + En}.$$

Since two values of $m:n$ can in general be found, which will render $Am^2 + Bmn + Cn^2 = 0$, *there can be drawn through the origin two real, coincident, or imaginary lines, which will meet the curve at an infinite distance, and each of these lines will only meet the curve in* one *other point.* If we multiply by ρ^2 the equation $Am^2 + Bmn + Cn^2 = 0$, and substitute for $m\rho$ and $n\rho$ their values x and y, we obtain for the equation of these two lines,

$$Ax^2 + Bxy + Cy^2 = 0.$$

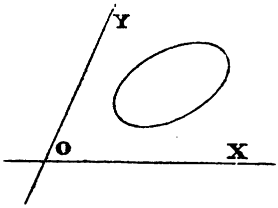
We may prove, by the transformation of co-ordinates, as in Art. 130, that there are two directions in which lines can be drawn through *any* point to meet the curve at infinity; and, since it was proved, in Art. 129, that the coefficients A, B, C were unaltered by transformation, we obtain for every point the very same quadratic, $Am^2 + Bmn + Cn^2 = 0$, to determine those directions. Hence, *if through any point two real lines can be drawn to meet the curve at infinity, parallel lines through any other point will meet the curve at infinity.**

* This, indeed, is evident geometrically, since parallel lines may be considered as passing through the same point at infinity.

135. The most important question we can ask, concerning the *form* of the curve represented by any equation, is, whether it be limited in every direction, or whether it extend in any direction to infinity. We have seen, in the case of the circle, that an equation of the second degree may represent a limited curve, while the case where it represents right lines shows us that it may also represent loci extending to infinity. It is necessary, therefore, to find a test whereby we may distinguish which class of locus is represented by any particular equation of the second degree.

With such a test we are at once furnished by the last article. For if the curve be limited in every direction, *no* radius vector drawn from the origin to the curve can have an infinite value; but we found in the last Article, that, in order that the radius vector should become infinite, we must have $Am^2 + Bmn + Cn^2 = 0$.

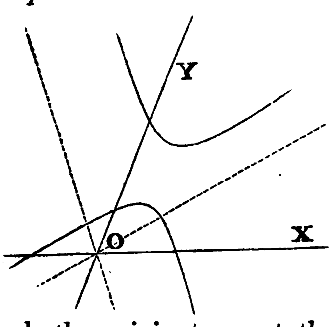
(1.) If now we suppose $B^2 - 4AC$ to be negative, the roots of this equation will be imaginary, and *no* real value of $m : n$ can be found which will render $Am^2 + Bmn + Cn^2 = 0$. In this case, therefore, no real line can be drawn to meet the curve at infinity, and *the curve will be limited in every direction.* We shall show, in the next chapter, that its form is that represented in the figure. A curve of this class is called an *Ellipse.*

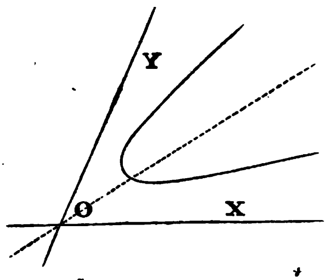


(2.) If $B^2 - 4AC$ be *positive*, the roots of the equation

$$Am^2 + Bmn + Cn^2 = 0$$

will be real; consequently, there are two real values of $m : n$ which will render infinite the radius vector to one of the points where the line $(my = nx)$ meets the curve. Hence, two real lines $(Ax^2 + Bxy + Cy^2 = 0)$ can, in this case, be drawn through the origin to meet the curve at infinity. A curve of this class is called an *Hyperbola*, and we shall show, in the next chapter, that its form is that represented in the figure.





(3.) If $B^2 - 4AC = 0$, the roots of the equation $Am^2 + Bmn + Cn^2 = 0$ will then be equal, and, therefore, the two directions in which a right line can be drawn to meet the curve at infinity will in this case coincide. A curve of this class is called a *Parabola*, and we shall (Chap. XII.) show that its form is that here represented.

136. In applying to Examples the principles just laid down, the following are some of the particular cases which most frequently present themselves:—

(1.) The circle is a particular form of the *ellipse*, for, since in the most general form of the equation of the circle $C = A$, $B = 2A \cos\omega$ (Art. 78), we have

$$B^2 - 4AC = -4A^2 \sin^2\omega,$$

and, therefore, always negative.

(2.) If $B = 0$, the curve will be an ellipse if A and C have the same sign; but an hyperbola if they have different signs.

(3.) If either A or $C = 0$, and B not $= 0$, the quantity $B^2 - 4AC$ will reduce to B^2, which being essentially positive, the curve is an hyperbola.

In the case where $A = 0$ the axis of x is itself one of the lines which meet the curve at infinity; and where $C = 0$, the axis of y; these lines being in general given by the equation

$$Ax^2 + Bxy + Cy^2 = 0.$$

(4.) If either A or C be $= 0$, and at the same time $B = 0$, then $B^2 - 4AC = 0$, and the curve is a parabola.

(5.) In general the curve will be a parabola, if the three first terms form a perfect square.

Ex. Determine the species of each of the following curves:

$3x^2 + 4xy + 5y^2 - 2x - 7y - 4 = 0.$ *Ans.* Ellipse.

$2x^2 + xy - y^2 + 3x + y = 0.$ *Ans.* Hyperbola.

$x^2 - 2xy + y^2 - x - y - 1 = 0.$ *Ans.* Parabola.

$\frac{x^2}{a^2} - \frac{2xy}{ab} + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0.$ *Ans.* Parabola.

137. III. Let us next examine the case where the value of $m:n$ is such that the quadratic (Art. 130) which determines ρ has its roots equal with opposite signs. This will be the case (Art. 131) when $Dm + En = 0$.

The points answering to the equal and opposite values of ρ are equidistant from the origin, and on opposite sides of it; therefore, the chord represented by the equation $Dx + Ey = 0$ is bisected at the origin.

Hence, *through any given point can in general be drawn one chord, which will be bisected at that point.*

138. There is one case, however, where more chords than one can be drawn, so as to be bisected, through a given point.

If, in the general equation, we had $D = 0$, $E = 0$, then the quantity $Dm + En$ would be $= 0$, whatever were the value of $m:n$; and we see, as in the last Article, that in this case *every* chord drawn through the origin would be bisected. The origin would then be called the *centre* of the curve. Now, although for any origin, taken arbitrarily, the quantities D and E are not $= 0$, yet we see, that if the curve have a centre, by taking this point for our origin, the quantities D and E will vanish; or, conversely, that if the axes be transformed to any new origin, so that the coefficients of x and y may vanish, then will the new origin be a centre of the curve.

In order to determine whether it be possible, by transformation of co-ordinates, to make the new D and $E = 0$, we have only to refer to the formulæ given in Art. 129, whence we find, that the co-ordinates of the new origin must fulfil the conditions

$$2Ax' + By' + D = 0, \qquad 2Cy' + Bx' + E = 0.$$

These *two* equations are sufficient to determine x' and y', and, *being linear*, can be satisfied by only *one* value of x and y; hence,

Conic sections have in general one, *and* only *one centre.*

Its co-ordinates are found, by solving the above equations, to be

$$x = -\frac{BE - 2CD}{B^2 - 4AC}, \quad y = -\frac{BD - 2AE}{B^2 - 4AC}.$$

In the ellipse and hyperbola $B^2 - 4AC$ is always finite (Art. 135); but in the parabola $B^2 - 4AC = 0$, and the co-ordinates of the centre become infinite. The ellipse and hyperbola are hence often classed together as *central* curves, while the parabola is called a *non-central* curve. The student must be careful, however, to remember that, strictly speaking, *every* curve of the second degree has a centre, although in the case of the parabola this centre is situated at an infinite distance.

139. *To find the locus of the middle points of chords, parallel to a given line, of a curve of the second degree.*

We saw (Art. 137) that a chord through the origin $my = nx$ is bisected if $Dm + En = 0$. Now, transforming the origin to any point, it appears, in like manner, that a parallel chord will be bisected at the new origin if m times the new $D + n$ times the new $E = 0$, or (Art. 129)

$$m(2Ax' + By' + D) + n(Bx' + 2Cy' + E) = 0.$$

This, therefore, is a relation which must be satisfied by the coordinates of the new origin, if it be the middle point of a chord parallel to $my = nx$. Hence the middle point of any parallel chord must lie on the right line

$$m(2Ax + By + D) + n(Bx + 2Cy + E) = 0,$$

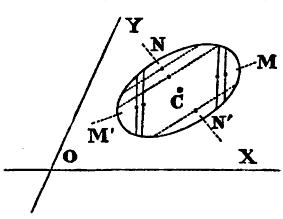
which is, therefore, the required locus.

Every right line bisecting a system of parallel chords is called a *diameter*, and the lines which it bisects are called its *ordinates*.

The form of the equation shows (Art. 36) that every diameter must pass through the intersection of the two lines

$$2Ax + By + D = 0, \text{ and } 2Cy + Bx + E = 0;$$

but, these being the equations by which we determined the coordinates of the centre (Art. 138), we infer, that *every diameter passes through the centre of the curve.*



Since

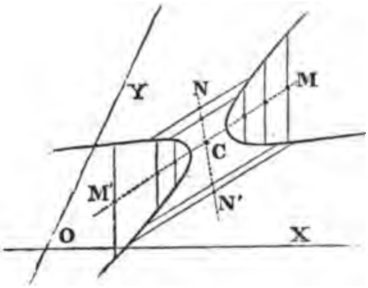
$$m(2Ax + By + D) + n(Bx + 2Cy + E) = 0$$

is the equation of the diameter bisecting chords parallel to $my = nx$, it appears, by making m and n alternately $= 0$, that $2Ax + By + D = 0$

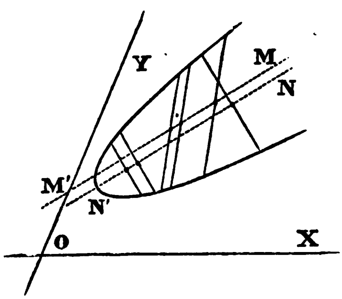
is the equation of the diameter bisecting chords parallel to the axis of x, and that

$$2Cy + Bx + E = 0$$

is the equation of the diameter bisecting chords parallel to the axis of y.



In the parabola $B^2 = 4AC$, or $\frac{2A}{B} = \frac{B}{2C}$, and hence the line $2Ax + By + D = 0$ is parallel to the line $2Cy + Bx + E = 0$; consequently, *all diameters of a parabola are parallel to each other.* This, indeed, is evident, since we have proved that all diameters of any conic section must pass through the centre, which, in the case of the parabola, is at an infinite distance; and since parallel right lines may be considered as meeting in a point at infinity.*



The familiar example of the circle will sufficiently illustrate to the beginner the nature of the diameters of curves of the second degree. He must observe, however, that diameters do not in general, as in the case of the circle, cut their ordinates at right angles. In the parabola, for instance, the direction of the diameter being invariable, while that of the ordinates may be any whatever, the angle between them *may take any possible value.*

140. *The direction of the diameters of a parabola is the same as that of the line through the origin which meets the curve at an infinite distance.*

For the lines through the origin which meet the curve at infinity are (Art. 134) $Ax^2 + Bxy + Cy^2 = 0$,

* Hence, given any conic section, we can find its centre geometrically. For if we draw any two parallel chords, and join their middle points, we have one diameter. In like manner we can find another diameter. Then, if these two diameters be parallel, the curve is a parabola, but if not, the point of intersection is the centre.

or, writing for B its value, $\sqrt{(4AC)}$,

$$(\sqrt{A}x + \sqrt{C}y)^2 = 0.$$

But the diameters are parallel to $2Ax + By = 0$ (by the last Article), which, if we write for B the same value, $\sqrt{(4AC)}$, will also reduce to

$$\sqrt{A}x + \sqrt{C}y = 0.$$

Hence every diameter of the parabola meets the curve once at infinity, and, therefore, can only meet it in one finite point.

141. *If two diameters of a conic section be such, that one of them bisects all chords parallel to the other, then, conversely, the second will bisect all chords parallel to the first.*

The equation of the diameter which bisects chords parallel to $my = nx$ is (Art. 139)

$$(2Am + Bn)x + (Bm + 2Cn)y + Dm + En = 0.$$

If then this be parallel to $m'y = n'x$, we must have

$$\frac{m'}{n'} = -\frac{Bm + 2Cn}{2Am + Bn},$$

or $$2Amm' + B(m'n + mn') + 2Cnn' = 0.$$

But the symmetry of the equation shows that it is also the condition that the line $my = nx$ should be parallel to the diameter bisecting the chord $m'y = n'x$.

Diameters so related, that each bisects every chord parallel to the other, are called *conjugate diameters.**

If in the general equation $B = 0$, the axes will be parallel to a pair of conjugate diameters.

For the diameter bisecting chords parallel to the axis of x will, in this case, become $2Ax + D = 0$, and will, therefore, be parallel to the axis of y. In like manner, the diameter bisecting chords parallel to the axis of y will, in this case, be $2Cy + E = 0$, and will, therefore, be parallel to the axis of x.

142. IV. Lastly, let us discuss the case, when the equation which determines ρ has equal roots. When this is the case, the

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* It is evident that none but central curves can have conjugate diameters, since in the parabola the direction of all diameters is the same.

line $my = nx$ will meet the curve in two coincident points, and will therefore touch it. Now (Art. 131) the equation

$$(Am^2 + Bmn + Cn^2)\rho^2 + (Dm + En)\rho + F = 0$$

will have equal roots if

$$(Dm + En)^2 = 4F(Am^2 + Bmn + Cn^2).$$

Since this gives us a quadratic to determine $m : n$, we see that through the origin can always be drawn two real, coincident, or imaginary tangents. Multiplying by ρ^2, the equation just found, and substituting x and y for $m\rho$, $n\rho$, we obtain the equation of the pair of tangents through the origin, viz.,

$$(D^2 - 4AF)x^2 + 2(DE - 2BF)xy + (E^2 - 4CF)y^2 = 0.$$

It is only necessary to notice particularly the case where these two tangents coincide. If we apply the condition that the equation just obtained should have equal roots, we get

$$(D^2 - 4AF)(E^2 - 4CF) = (DE - 2BF)^2,$$

or $$4F(AE^2 + CD^2 + FB^2 - BDE - 4ACF) = 0.$$

This will be satisfied, if $F = 0$, that is, if the origin be on the curve. Hence, *any point on the curve may be considered as the intersection of two coincident tangents*, just as any tangent may be considered as the line joining two coincident points.

The equation will also have equal roots if

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0.$$

Now we obtained this equation (p. 67) as the condition that the equation of the second degree should represent two right lines. To explain why we should here meet with this equation again, it must be remarked that by a tangent we mean in general a line which meets the curve in two coincident points; if then the curve reduce to two right lines, the only line which can meet the locus in two coincident points is the line drawn to the point of intersection of these right lines, and since *two* tangents can always be drawn to a curve of the second degree, both tangents must in this case coincide with the line to the point of intersection.

143. *To find the equation of the line joining the points of contact of tangents through the origin.*

We have seen in the last article that if $m' : n'$ be either of the roots of

$$(D^2 - 4AF)m^2 + 2(DE - 2BF)mn + (E^2 - 4CF)n^2 = 0,$$

the line $m'y = n'x$ will touch the curve, and the quadratic

$$(Am'^2 + Bm'n' + Cn'^2)\rho^2 + (Dm' + En')\rho + F = 0$$

will have equal roots. But (Art. 131) when $a\rho^2 + b\rho + c = 0$ has equal roots, the common value of the equal roots is $-\frac{2c}{b}$. The value, therefore, of the radius vector to the point of contact is

$$\rho = -\frac{2F}{Dm' + En'}, \quad \text{or } Dm'\rho + En'\rho + 2F = 0.$$

The co-ordinates, then, of either point of contact satisfy the relation

$$Dx + Ey + 2F = 0,$$

which is the equation of the line required. This is the equation of a real line, whether the tangents through the origin be real or imaginary. We shall call it, as in the case of the circle, the *polar* of the origin, and, conversely, we shall call the origin the *pole* of this line.

144. *To find the equation of the polar of any point $x'y'$.*

If we transform the equation to parallel axes through $x'y'$, the polar of the new origin is $D'x + E'y + 2F' = 0$, or, transforming back to the old origin by writing $x - x'$ for x, and $y - y'$ for y,

$$D'(x - x') + E'(y - y') + 2F' = 0.$$

Writing for D', E', F' their values (Art. 129), and reducing as in Art. 133, we find for the equation of the polar

$$(2Ax' + By' + D)x + (Bx' + 2Cy' + E)y + Dx' + Ey' + 2F = 0.$$

Comparing this with the equation found in Art. 133, we see that the polar of any point on the curve is the tangent at that point.

145. The polar of the origin $(Dx + Ey + 2F = 0)$ is parallel to the chord $(Dx + Ey = 0)$ drawn through the origin so as to be bisected, which evidently is an ordinate to the diameter passing through the origin. Hence, *the polar of any point is parallel to the ordinates of the diameter passing through that point.* This includes, as a particular case: *The tangent at the extremity of any*

diameter is parallel to the ordinates of that diameter. Or, again, in the case of central curves, since the ordinates of any diameter are parallel to the conjugate diameter, we infer that, *The polar of any point on a diameter of a central curve is parallel to the conjugate diameter.*

146. *If any point* $(x''y'')$ *be taken on the polar of* $(x'y')$, *its polar must pass through* $(x'y')$.

For, the condition that $(x''y'')$ should lie on the polar of $(x'y')$ is (Art. 144),

$$(2Ax' + By' + D)x'' + (2Cy' + Bx' + E)y'' + Dx' + Ey' + 2F = 0.$$

But this may be arranged

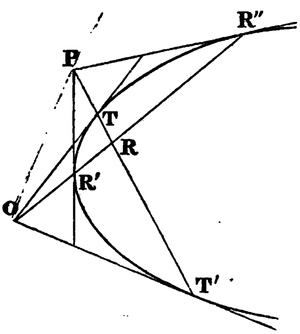
$$(2Ax'' + By'' + D)x' + (2Cy'' + Bx'' + E)y' + Dx'' + Ey'' + 2F = 0,$$

and is, therefore, also the condition that $(x'y')$ should lie on the polar of $(x''y'')$.

The form of the equation of the polar indicates (see Art. 50) that, *if any point move along a fixed right line, its polar must always pass through a fixed point*, namely, as appears from this article, the pole of the fixed line.

The theorem of this article may also be stated thus: *The intersection of any two lines is the pole of the line joining their poles;* or, conversely: *The line joining any two points is the polar of the intersection of the polars of these points.* For the polars of any two points on the polar of $x'y'$ intersect in $x'y'$.

147. *If on any radius vector through the origin,* OR *be taken an harmonic mean between* OR′ *and* OR″: *to prove that* R *lies on the polar of* O.



We found (Art. 130) that OR′, OR″ were determined by the quadratic

$$(Am^2 + Bmn + Cn^2)\rho^2 + (Dm + En)\rho + F = 0.$$

Hence, by the theory of equations,

$$\frac{2}{OR} = \frac{1}{OR'} + \frac{1}{OR''} = -\frac{Dm + En}{F}.$$

In order to find the locus of R we must write x and y for $m \cdot OR$ and $n \cdot OR$, and the equation of the locus is

$$Dx + Ey + 2F = 0,$$

that is, the equation of the polar of the origin.

Hence, *any line drawn through a point is cut harmonically by the point, the curve, and the polar of the point.**

148. *If two lines be drawn through any point, and the points joined where they meet a curve of the second degree, the joining lines will intersect on the polar of that point.*

The proof given (p. 90) of this property in the case of the circle will apply, word for word, to conics in general, since no use was made of the equality of the coefficients of x^2 and y^2.

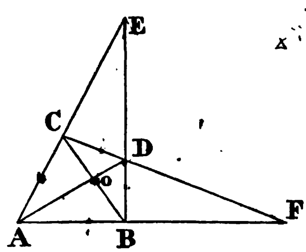
If through a point O *any line* OR *be drawn, the tangents at* R′ *and* R″ *will meet on the polar of* O.

This is a particular case of the preceding theorem, namely, where the two lines are supposed to coincide; or else it follows immediately from Art. 146, since the pole of any chord through O must lie on TT′; and by the pole of the line we mean the intersection of tangents at the points where it meets the curve.†

149. *If any line* (OR) *be drawn through a point* (O), *and* (P) *the pole of that line, be joined to* O, *then the lines* OP, OR *will form an harmonic pencil with the tangents from* O.

For, since OR is the polar of P, PTRT′ is cut harmonically, therefore OP, OT, OR, OT′, form an harmonic pencil.

Ex. 1. If a quadrilateral, ABCD, be inscribed in a conic section, any of the points E, F, O, is the pole of the line joining the other two.



Since EC, ED, are two lines drawn through the point E, and CD, AB, one pair of lines joining the points where they meet the conic, these lines must intersect on the polar of E; so must also AD and CB; therefore, the line OF is the polar of E. In like manner it can be proved that EF is the polar of O, and EO the polar of F.

Ex. 2. To draw a tangent to a given conic section from a point outside, with the help of the ruler only.

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* For an enumeration of some of the particular cases included in this theorem, the reader is referred to the section (Chap. xv.) on the anharmonic properties of conics.

† From this property the polar of a point might have been *defined* as the locus of the intersection of tangents at the extremities of any chord passing through the point. This definition applies, whether the point be within or without the conic.

Draw any two lines through the given point E, and complete the quadrilateral as in the figure, then the line OF will meet the conic in two points, which, being joined to E, will give the two tangents required.

Ex. 3. If a quadrilateral be circumscribed about a conic section, any diagonal is the polar of the intersection of the other two.

We shall prove this Example, as we might have proved Ex. 1, by means of the harmonic properties of a quadrilateral. It was proved (p. 57) that EA, EO, EB, EF, are an harmonic pencil. Hence, since EA, EB, are, by hypothesis, two tangents to a conic section, and EF a line through their point of intersection, by Art. 149, EO must pass through the pole of EF; for the same reason, FO must pass through the pole of EF: this pole must therefore be O.

*150. The theorem of Art. 147 may also be proved by a process precisely similar to that employed (Art. 89). We may seek the ratio in which the line joining two points is cut by the curve. Substituting $\frac{lx'' + mx'}{l + m}$, $\frac{ly'' + my'}{l + m}$, for x and y, the ratio $l : m$ is determined by the quadratic

$$l^2(Ax''^2 + Bx''y'' + Cy''^2 + Dx'' + Ey'' + F)$$
$$+ lm\{(2Ax'' + By'' + D)x' + (Bx'' + 2Cy'' + E)y' + Dx'' + Ey'' + 2F\}$$
$$+ m^2(Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F) = 0.$$

Now if $x''y''$ be on the polar of $x'y'$ the coefficient of lm vanishes, the roots of the equation are of the form $l = \pm \mu m$, and the line joining the points is cut harmonically.

The same equation enables us to form the equation of the pair of tangents drawn from any point to the curve. For if $x''y''$ lie on either of the tangents through $x'y'$, the equation for $l : m$ must have equal roots, and $x''y''$ must therefore satisfy the equation

$$4(Ax^2 + Bxy + Cy^2 + Dx + Ey + F)(Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F)$$
$$= \{(2Ax' + By' + D)x + (Bx' + 2Cy' + E)y + Dx' + Ey' + 2F\}^2.$$

151. *If through any point O two chords be drawn, meeting the curve in the points R′, R″, S′, S″, then the ratio of the rectangles* $\frac{OR' \cdot OR''}{OS' \cdot OS''}$ *will be constant, whatever be the position of the point O, provided that the* directions *of the lines OR, OS be constant.*

For, from the equation given to determine ρ in Art. 130, it appears that

$$OR' \cdot OR'' = \frac{F}{Am^2 + Bmn + Cn^2}.$$

In like manner

$$OS' \cdot OS'' = \frac{F}{Am'^2 + Bm'n' + Cn'^2};$$

hence

$$\frac{OR' \cdot OR''}{OS' \cdot OS''} = \frac{Am'^2 + Bm'n' + Cn'^2}{Am^2 + Bmn + Cn^2}.$$

But this is a constant ratio: for A, B, C remain unaltered when the axes are transformed to any new origin (Art. 129), and m, n, m', n' depend only on the angles which the radius vector makes with the axes, and are therefore constant while the direction of this radius vector is constant.

The theorem of this Article may be otherwise stated thus: *If through two fixed points* O *and* O′ *any two parallel lines* OR *and* O′ρ *be drawn, then the ratio of the rectangles* $\frac{OR' \cdot OR''}{O'\rho' \cdot O'\rho''}$ *will be constant, whatever be the direction of these lines.*

For, these rectangles are

$$\frac{F}{Am^2 + Bmn + Cn^2}, \quad \frac{F'}{Am^2 + Bmn + Cn^2},$$

(F′ being the new absolute term when the equation is transferred to O′ as origin); the ratio of these rectangles $= \frac{F}{F'}$, and is, therefore, independent of m and n.

This theorem is the generalization of Euclid, III. 35, 36.

152. The theorem of the last Article includes under it several particular cases, which it is useful to notice separately.

I. Let O′ be the centre of the curve, then $O'\rho' = O'\rho''$ and the quantity $O'\rho' \cdot O'\rho''$ becomes the square of the semidiameter parallel to OR′. Hence, *The rectangles under the segments of two chords which intersect are to each other as the squares of the diameters parallel to those chords.*

II. Let the line OR be a tangent, then $OR' = OR''$, and the quantity $OR' \cdot OR''$ becomes the square of the tangent; and, since two tangents can be drawn through the point O, we may extract the square roots of the ratio found in the last paragraph, and infer that *Two tangents drawn through any point are to each other as the diameters to which they are parallel.*

III. Let the line OO′ be a diameter, and OR, O′ρ, parallel to its ordinates, then OR′ = OR″ and O′ρ′ = O′ρ″. Let the diameter meet the curve in the points A, B, then $\frac{OR^2}{AO \cdot OB} = \frac{O'\rho^2}{AO' \cdot O'B}$. Hence, *The squares of the ordinates of any diameter are proportional to the rectangles under the segments which they make on the diameter.*

153. There is one case in which the theorem of Article 151 becomes no longer applicable, namely, when the line OS is parallel to one of the lines which meet the curve at infinity; the segment OS″ is then infinite, and OS only meets the curve in one finite point. We propose, in the present Article, to inquire whether, in this case, the ratio $\frac{OS'}{OR' \cdot OR''}$ will be constant.

Let us, for simplicity, take the line OS for our axis of x, and OR for the axis of y. Since the axis of x is parallel to one of the lines which meet the curve at infinity, the term A will = 0 (Art. 136(3)), and the equation of the curve will be of the form

$$Bxy + Cy^2 + Dx + Ey + F = 0.$$

Making $y = 0$, the intercept on the axis of x is found to be $OS' = -\frac{F}{D}$; and making $x = 0$, the rectangle under the intercepts on the axis of y is $= \frac{F}{C}$.

Hence
$$\frac{OS'}{OR' \cdot OR''} = -\frac{C}{D}.$$

Now, if we transform the axes to any parallel axes (Art. 129), C will remain unaltered, and the new $D = By' + D$.

Hence the new ratio will be

$$-\frac{C}{By' + D}.$$

Now, if the curve be a parabola, $B = 0$, and this ratio is constant; hence, *if a line parallel to a given one meet any diameter* (Art. 140) *of a parabola, the rectangle under its segments is in a constant ratio to the intercept on the diameter.*

If the curve be a hyperbola, the ratio will only be constant while y' is constant; hence *the intercepts made by two parallel chords of a hyperbola, on a parallel to an asymptote, are proportional to the rectangles under the segments of the chords.*

*154. *To find the condition that the line $ax + by + c = 0$ should touch the conic represented by the general equation.*

Solving for y from $ax + by + c = 0$, and substituting in the general equation, the abscissæ of the points where this line meets the conic are determined by the quadratic

$$(Ab^2 - Bab + Ca^2)x^2 - (Bbc - 2Cac - Db^2 + Eab)x + Cc^2 - Ebc + Fb^2 = 0.$$

If the line touch the conic, this quadratic will have equal roots, or

$$(Bbc - 2Cac - Db^2 + Eab)^2 = 4(Ab^2 - Bab + Ca^2)(Cc^2 - Ebc + Fb^2).$$

Multiplying out, this equation becomes divisible by b^2, and may be arranged

$$(E^2 - 4CF)a^2 + (D^2 - 4AF)b^2 + (B^2 - 4AC)c^2 + 2(2AE - BD)bc$$
$$+ 2(2CD - BE)ca + 2(2BF - DE)ab = 0.$$

Miscellaneous Examples.

Ex. 1. To find the equation of the conic which makes intercepts a, a', b, b', on the axes.

The intercepts on the axes are given by the quadratics

$$x^2 - (a + a')x + aa' = 0, \quad y^2 - (b + b')y + bb' = 0,$$

but these must be what the general equation becomes when in it we make $y = 0$, $x = 0$; hence the equation is

$$bb'x^2 + Bxy + aa'y^2 - bb'(a + a')x - aa'(b + b')y + aa'bb' = 0,$$

where B is still indeterminate.

Ex. 2. To find the equation of the parabola which touches the axes at points $x = a$, $y = b$.

In the preceding make $a = a'$, $b = b'$, and determine B by the condition $B^2 = 4AC$, and we find

$$b^2x^2 - 2abxy + a^2y^2 - 2b^2ax - 2a^2by + a^2b^2 = 0.$$

We give the sign $-$ to the coefficient of xy, since if we gave the sign $+$ it would not represent a parabola, but the square of the line $bx + ay - ab = 0$.

Ex. 3. Given four points on a conic, the polar of any fixed point passes through a fixed point.

Take for axes two opposite sides of the quadrilateral formed by the points; then form by Art. 144 the polar of $x'y'$ with regard to the conic found in Ex. 1, and it will contain the indeterminate B in the first degree, and therefore passes through a fixed point.

Ex. 4. Find the locus of the centre of a conic passing through four given points.

The centre of the conic in Ex. 1 is given by the equations

$$2bb'x + By - bb'(a + a') = 0, \quad 2aa'y + Bx - aa'(b + b') = 0.$$

Eliminate the indeterminate B, and the locus is

$$2bb'x^2 - 2aa'y^2 - bb'(a + a')x - aa'(b + b')y = 0,$$

a conic passing through the intersections of each of the three pair of lines which can be drawn through the four points, and through the middle points of those lines.

CHAPTER XI.

EQUATIONS OF THE SECOND DEGREE REFERRED TO THE CENTRE AS ORIGIN.

155. In investigating the properties of the ellipse and hyperbola, we shall find our equations much simplified by choosing the centre for the origin of co-ordinates. If we transform the general equation of the second degree to the centre as origin, we saw (Art. 138) that the coefficients of x and y will $= 0$ in the transformed equation, which will be of the form

$$Ax^2 + Bxy + Cy^2 + F' = 0.$$

It is sometimes useful to know the value of F′ in terms of the coefficients of the first given equation. We saw (Art. 129) that

$$F' = Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F,$$

where x', y', are the co-ordinates of the centre. The calculation of this may be facilitated by putting F′ into the form

$$F' = \tfrac{1}{2}\{(2Ax' + By' + D)x' + (2Cy' + Bx' + E)y' + Dx' + Ey' + 2F\}.$$

The first two terms must be rendered $= 0$ by the co-ordinates of the centre, and the last (Art. 138)

$$= D \cdot \frac{2CD - BE}{B^2 - 4AC} + E \cdot \frac{2AE - BD}{B^2 - 4AC} + 2F.$$

Hence

$$F' = \frac{AE^2 + CD^2 + FB^2 - BDE - 4ACF}{B^2 - 4AC}.$$

156. If the numerator of this fraction were $= 0$, the transformed equation would be reduced to the form

$$Ax^2 + Bxy + Cy^2 = 0,$$

and would, therefore (Art. 69), represent two real or imaginary right lines, according as $B^2 - 4AC$ is positive or negative. Hence, as we have already seen, p. 67, the condition that the general equation of the second degree should represent two right lines, is

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0.$$

For it must plainly be fulfilled, in order that when we transfer the origin to the point of intersection of the right lines, the absolute term may vanish.

Ex. 1. Transform $3x^2 + 4xy + y^2 - 5x - 6y - 3 = 0$ to the centre $\left(\frac{7}{2}, -4\right)$.

Ans. $12x^2 + 16xy + 4y^2 + 1 = 0$.

Ex. 2. Transform $x^2 + 2xy - y^2 + 8x + 4y - 8 = 0$ to the centre $(-3, -1)$.

Ans. $x^2 + 2xy - y^2 = 22$.

157. We have seen (Art. 134) that the equation

$$Ax^2 + Bxy + Cy^2 = 0$$

represents the real or imaginary lines drawn through the origin to meet the curve at infinity; and that each of these lines will meet the curve in one other point, at a distance from the origin,

$$\rho = \frac{-F}{Dm + En}.$$

But if the origin be *the centre*, we have $D = 0$, $E = 0$, and this distance will *also* become infinite. Hence two lines can be drawn through the centre, which will meet the curve in *two coincident points* at infinity, and which therefore may be considered as tangents to the curve whose points of contact are at infinity. These lines are called the *asymptotes* of the curve; they are imaginary in the case of the ellipse, but real in that of the hyperbola. We shall show hereafter that though the asymptotes do not meet the curve at any finite distance, yet that the further they are produced the more nearly they approach the curve.

Since the points of contact of the two real or imaginary tangents drawn through the centre are at an infinite distance, the line joining these points of contact is altogether at an infinite distance. Hence, from our definition of poles and polars (Art. 143) *the centre may be considered as the pole of a line situated altogether at an infinite distance.* This inference may be confirmed from the equation of the polar of the origin, $Dx + Ey + 2F = 0$, which, if the centre be the origin, reduces to $F = 0$, an equation which (Art. 64) represents a line at infinity.

158. We have seen that by taking the centre for origin the coefficients D and E in the general equation can be made to vanish; but the equation can be further simplified by taking a pair of conjugate diameters for axes, since then (Art. 141) B will vanish, and the equation be reduced to the form

$$Ax^2 + Cy^2 = F.$$

It is evident, now, that any line parallel to either axis is bisected by the other, for if we give to x any value, we obtain equal and opposite values for y. Now the angle between conjugate diameters is not in general right; but we shall show that there is always *one pair* of conjugate diameters which cut each other at right angles. These diameters are called the *axes* of the curve, and the points where they meet it are called its *vertices*.

The equation of the diameter conjugate to $my = nx$ is

$$m(2Ax + By + D) + n(2Cy + Bx + E) = 0,$$

(Art. 141); and this will be perpendicular to $my = nx$ (Art. 40) if

$$(2Am + Bn)n - (Bm + 2Cn)m = 0,$$

or

$$Bm^2 - 2(A - C)mn - Bn^2 = 0;$$

or, multiplying by ρ^2, and writing x, y for $m\rho$, $n\rho$,

$$Bx^2 - 2(A - C)xy - By^2 = 0.$$

This is the equation of two real lines at right angles to each other (Art. 70); we perceive, therefore, that central curves have two, and only two, conjugate diameters at right angles to each other.

On referring to Art. 71 it will be found, that the equation which we have just obtained for the *axes* of the curve is the same as that of the lines bisecting the internal and external angles between the real or imaginary lines represented by the equation

$$Ax^2 + Bxy + Cy^2 = 0.$$

The axes of the curve, therefore, are the diameters which bisect the angles between the asymptotes; and (note, p. 66) they will be real whether the asymptotes be real or imaginary: that is to say, whether the curve be an ellipse or an hyperbola.

159. We might have obtained the results of the last Article by the method of transformation of co-ordinates, since we can thus prove directly that it is always possible to transform the

equation to a pair of rectangular axes, such that the coefficient of xy in the transformed equation may vanish. Let the original axes be rectangular; then, if we turn them round through any angle θ, we have (Art. 9) to substitute for x, $x\cos\theta - y\sin\theta$, and for y, $x\sin\theta + y\cos\theta$; the equation will therefore become

$$A(x\cos\theta - y\sin\theta)^2 + B(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + C(x\sin\theta + y\cos\theta)^2 = F;$$

or, arranging the terms, we shall have

the new $A = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$;
the new $B = 2C\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) - 2A\sin\theta\cos\theta$;
the new $C = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta$.

Now, if we put the new $B = 0$, we get the very same equation to determine $\tan\theta$, which we had, in Art. 158, to determine $m : n$. This equation gives us a simple expression for the angle made with the given axes by the axes of the curve, namely,

$$\tan 2\theta = \frac{B}{A - C}.$$

160. When it is required to transform a given equation to the form $Ax^2 + Cy^2 = F$, and to calculate numerically the value of the new coefficients, our work will be much facilitated by the following theorem: *If we transform an equation of the second degree from one set of rectangular axes to another, the quantities* $A + C$, *and* $B^2 - 4AC$, *will remain unaltered.*

The first part is proved immediately by adding the values of the new A and C (Art. 159), when we have

$$A' + C' = A + C.$$

To prove the second part, write the values in the last article,

$$2A' = A + C + B\sin 2\theta + (A - C)\cos 2\theta,$$
$$2C' = A + C - B\sin 2\theta - (A - C)\cos 2\theta.$$

Hence

$$4A'C' = (A + C)^2 - \{B\sin 2\theta + (A - C)\cos 2\theta\}^2.$$

But $$B'^2 = \{B\cos 2\theta - (A - C)\sin 2\theta\}^2;$$

therefore,

$$B'^2 - 4A'C' = B^2 + (A - C)^2 - (A + C)^2 = B^2 - 4AC.$$

When, therefore, we want to form the equation transformed to the *axes*, we have the new B = 0,

$$A' + C' = A + C, \quad 4A'C' = 4AC - B^2.$$

Having, therefore, the sum and the product of A′ and C′, we can form the quadratic which determines these quantities.

Ex. 1. Find the axes of the ellipse $14x^2 - 4xy + 11y^2 = 60$, and transform the equation to them.

The axes are (Art. 158) $4x^2 + 6xy - 4y^2 = 0$, or $(2x - y)(x + 2y) = 0$.

We have $A' + C' = 25$; $4A'C' = 600$; $A' = 10$; $C' = 15$; and the transformed equation is $2x^2 + 3y^2 = 12$.

Ex. 2. Transform the hyperbola $11x^2 + 84xy - 24y^2 = 156$ to the axes.

$$A' + C' = -13, \quad A'C' = -2028; \quad A' = 39, \quad C' = -52.$$

Transformed equation is $3x^2 - 4y^2 = 12$.

Ex. 3. Transform $Ax^2 + Bxy + Cy^2 = F$ to the axes.

Ans. $(A + C - R)x^2 + (A + C + R)y^2 = 2F$: where $R^2 = B^2 + (A - C)^2$.

*161. Having proved that the quantities A + C, and $B^2 - 4AC$ remain unaltered when we transform from one rectangular system to another, let us now inquire what these quantities become if we transform to an oblique system. We may retain the old axis of x, and if we take an axis of y inclined to it at an angle ω, then (Art. 9) we are to substitute $x + y\cos\omega$ for x, and $y\sin\omega$ for y. We shall then have

$$A' = A, \quad B' = 2A\cos\omega + B\sin\omega,$$
$$C' = A\cos^2\omega + B\cos\omega\sin\omega + C\sin^2\omega.$$

Hence, it easily follows

$$\frac{A' + C' - B\cos\omega}{\sin^2\omega} = A + C, \quad \frac{B'^2 - 4A'C'}{\sin^2\omega} = B^2 - 4AC.$$

If, then, we transform the equation from one pair of axes to any other, the quantities $\dfrac{A + C - B\cos\omega}{\sin^2\omega}$ *and* $\dfrac{B^2 - 4AC}{\sin^2\omega}$ *remain unaltered.*

We may, by the help of this theorem, transform to the *axes* an equation given in oblique co-ordinates, for we can still express the sum and product of the new A and C in terms of the old coefficients.

Ex. 1. If $\cos\omega = \frac{3}{5}$ transform to the axes, $10x^2 + 6xy + 5y^2 = 10$.

$$A + C = \frac{285}{16},\quad AC = \frac{1025}{16},\quad A = 5,\quad C = \frac{205}{16}.$$

Ans. $16x^2 + 41y^2 = 32$.

Ex. 2. Transform to the axes, $x^2 - 3xy + y^2 + 1 = 0$, where $\omega = 60°$.

Ans. $x^2 - 15y^2 = 3$.

Ex. 3. Transform $Ax^2 + Bxy + Cy^2 = F$ to the axes.

Ans. $(A + C - B\cos\omega - R)\,x^2 + (A + C - B\cos\omega + R)\,y^2 = 2F\sin^2\omega$,
where $R^2 = \{B - (A + C)\cos\omega\}^2 + (A - C)^2\sin^2\omega$.

*162. We add the demonstration of the theorems of the last two articles given by Professor Boole (Cambridge Math. Jour., iii. 1, 106, and New Series, vi. 87).

Let us suppose that we are transforming an equation from axes inclined at an angle ω, to any other axes inclined at an angle Ω; and that, on making the substitutions of Art. 9, the quantity $Ax^2 + Bxy + Cy^2$ becomes $A'X^2 + B'XY + C'Y^2$. Now we know that the effect of the same substitution will be to make the quantity $x^2 + 2xy\cos\omega + y^2$ become $X^2 + 2XY\cos\Omega + Y^2$, since either is the expression for the square of the distance of any point from the origin. It follows, then, that

$$\begin{aligned} &Ax^2 + Bxy + Cy^2 + h\,(x^2 + 2xy\cos\omega + y^2) \\ &= A'X^2 + B'XY + C'Y^2 + h\,(X^2 + 2XY\cos\Omega + Y^2). \end{aligned}$$

And if we determine h so that the first side of the equation may be a perfect square, the second must be a perfect square also. But the condition that the first side may be a perfect square is

$$(B + 2h\cos\omega)^2 = 4\,(A + h)\,(C + h),$$

or h must be one of the roots of the equation

$$4h^2\sin^2\omega + 4\,(A + C - B\cos\omega)\,h + 4AC - B^2 = 0.$$

We get a quadratic of like form to determine the value of h, which will make the second side of the equation a perfect square; but since both sides become perfect squares for the *same* values of h, these two quadratics must be identical. Equating, then, the coefficients of the corresponding terms, we have, as before,

$$\frac{A + C - B\cos\omega}{\sin^2\omega} = \frac{A' + C' - B'\cos\Omega}{\sin^2\Omega};\quad \frac{B^2 - 4AC}{\sin^2\omega} = \frac{B'^2 - 4A'C'}{\sin^2\Omega}.$$

Ex. 1. The sum of the squares of the reciprocals of two semidiameters at right angles to each other is constant.

Let their lengths be a and b; then making alternately $x = 0$, $y = 0$, in the equation of the curve, we have $Aa^2 = F$, $Cb^2 = F$, and the theorem just stated is only the geometrical interpretation of the fact that $A + C$ is constant.

Ex. 2. The area of the triangle formed by joining the extremities of two conjugate semidiameters is constant.

The equation referred to two conjugate diameters is $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, and since $\frac{4AC - B^2}{\sin^2\omega}$ is constant, we have $a'b' \sin\omega$ constant.

Ex. 3. The sum of the squares of two conjugate semidiameters is constant.

Since $\frac{A + C - B\cos\omega}{\sin^2\omega}$ is constant, $\frac{1}{\sin^2\omega}\left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)$ is constant; and since $a'b' \sin\omega$ is constant, so must $a'^2 + b'^2$.

THE EQUATION REFERRED TO THE AXES.

163. We saw that the equation referred to the axes was of the form
$$Ax^2 + Cy^2 = F,$$
C being positive in the case of the ellipse, and negative in that of the hyperbola (Art. 136, II.)

The equation of the ellipse may be written in the following more convenient form:—

Let the intercepts made by the ellipse on the axes be $x = a$, $y = b$, then a is found by making $y = 0$ and $x = a$ in the equation of the curve, or $Aa^2 = F$, and $A = \frac{F}{a^2}$. In like manner, $C = \frac{F}{b^2}$. Substituting these values, the equation of the ellipse may be written
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since we may choose whichever axis we please for the axis of x, we shall suppose that we have chosen the axes so that a may be greater than b.

The equation of the hyperbola, which, we saw, only differs from that of the ellipse in the sign of the coefficient of y^2, may be written in the corresponding form,
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The intercept on the axis of x is evidently $= \pm a$, but that on the axis of y, being found from the equation $y^2 = -b^2$ is imaginary; the axis of y, therefore, does not meet the curve in real points.

Since we have chosen for our axis of x the axis which meets the curve in real points, we are not in this case entitled to assume that a is greater than b.

164. *To find the polar equation of the ellipse, the centre being the pole.*

Write $\rho\cos\theta$ for x, and $\rho\sin\theta$ for y, in the preceding equation, and we get

$$\frac{1}{\rho^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2},$$

an equation which we may write in any of the equivalent forms,

$$\rho^2 = \frac{a^2 b^2}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{a^2 b^2}{b^2 + (a^2 - b^2)\sin^2\theta} = \frac{a^2 b^2}{a^2 - (a^2 - b^2)\cos^2\theta}.$$

It is customary to use the following abbreviations,

$$a^2 - b^2 = c^2; \quad \frac{a^2 - b^2}{a^2} = e^2;$$

and the quantity e is called the *eccentricity* of the curve.

Dividing by a^2 the numerator and denominator of the fraction last found, we obtain the form most commonly used, viz.,

$$\rho^2 = \frac{b^2}{1 - e^2\cos^2\theta}.$$

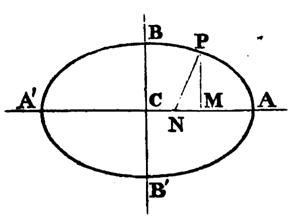
165. *To investigate the figure of the ellipse.*

The *least* value that $b^2 + (a^2 - b^2)\sin^2\theta$ can have, is when $\theta = 0$; therefore, since

$$\rho^2 = \frac{a^2 b^2}{b^2 + (a^2 - b^2)\sin^2\theta},$$

the *greatest* value of ρ is the intercept on the axis of x, and is $= a$.

Again, the *greatest* value of $b^2 + (a^2 - b^2)\sin^2\theta$, is, when $\sin\theta = 1$, or $\theta = 90°$; hence the *least* value of ρ is the intercept on the axis of y, and is $= b$. The greatest line, therefore, that can be drawn through the centre is the axis of x, and the least line, the axis of y. From this property these lines are called the axis *major* and the axis *minor* of the curve.



It is plain that the smaller θ is, the greater ρ will be; hence, *the nearer any diameter is to the axis major, the greater it will be.* The form of the curve will, therefore, be that here represented.

We obtain the same value of ρ whether we suppose $\theta = a$, or $\theta = -a$. Hence, *Two diameters which make equal angles with the axis will be equal.* And it is easy to show that the converse of this theorem is also true.

This property enables us, being given the centre of a conic, to determine its axes geometrically. For, describe any concentric circle intersecting the conic, then the semidiameters drawn to the points of intersection will be equal; and by the theorem just proved, the axes of the conic will be the lines internally and externally bisecting the angle between them.

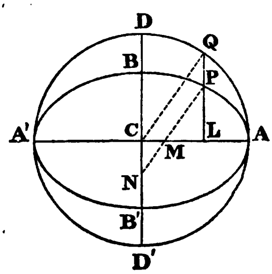
166. The equation of the ellipse can be put into another form, which will make the figure of the curve still more apparent. If we solve for y we get

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}.$$

Now, if we describe a concentric circle with the radius a, its equation will be $$y = \sqrt{(a^2 - x^2)}.$$

Hence we derive the following construction:

"*Describe a circle on the axis major, and take on each ordinate* LQ *a point* P, *such that* LP *may be to* LQ *in the constant ratio* $b : a$, *then the locus of* P *will be the required ellipse.*"



Hence the circle described on the axis major lies wholly *without* the curve. We might, in like manner, construct the ellipse, by describing a circle on the axis minor, and *increasing* each ordinate in the constant ratio $a : b$.

Hence the circle described on the axis minor lies wholly *within* the curve.

The equation of the circle is the particular form which the equation of the ellipse assumes when we suppose $b = a$.

167. *To find the polar equation of the hyperbola.*

Transforming to polar co-ordinates, as in Art. 164, we get

$$\rho^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{a^2b^2}{b^2 - (a^2+b^2)\sin^2\theta} = \frac{a^2b^2}{(a^2+b^2)\cos^2\theta - a^2}.$$

Since formulæ concerning the ellipse are altered to the corresponding formulæ for the hyperbola by changing the sign of b^2, we must, in this case, use the abbreviation c^2 for $a^2 + b^2$, and e^2 for $\frac{a^2+b^2}{a^2}$, the quantity e being called the *eccentricity* of the hyperbola. Dividing then by a^2 the numerator and denominator of the last found fraction, we obtain the polar equation of the hyperbola, which only differs from that of the ellipse in the sign of b^2, viz.,

$$\rho^2 = \frac{b^2}{e^2 \cos^2\theta - 1}.$$

168. *To investigate the figure of the hyperbola.*

The terms axis major and axis minor not being applicable to the hyperbola (Art. 163), we shall call the axis of x the *transverse* axis, and the axis of y the *conjugate* axis.

Now $b^2 - (a^2 + b^2)\sin^2\theta$, the denominator in the value of ρ^2, will plainly be greatest when $\theta = 0$, therefore, in the same case, ρ will be least; or *the transverse axis is the shortest line which can be drawn from the centre to the curve.*

As θ increases, ρ continually increases, until

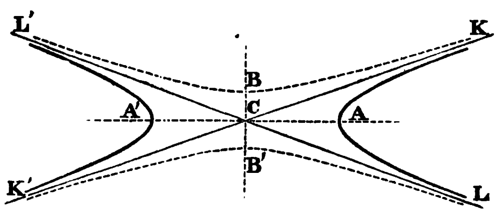
$$\sin\theta = \frac{b}{\sqrt{(a^2+b^2)}}, \quad \left(\text{or } \tan\theta = \frac{b}{a}\right),$$

when the denominator of the value of ρ becomes $= 0$, and ρ becomes infinite. After this value of θ, ρ^2 becomes negative, and the diameters cease to meet the curve in real points until

$$\sin\theta = -\frac{b}{\sqrt{(a^2+b^2)}}, \quad \left(\text{or } \tan\theta = -\frac{b}{a}\right),$$

when ρ again becomes infinite. It then decreases regularly as θ increases, until θ becomes $= 180°$, when it again receives its minimum value $= a$.

The form of the hyperbola, therefore, is that represented by the dark curve on the figure.



169. We found that the axis of y does not meet the hyperbola in real points, since we obtained the equation $y^2 = -b^2$ to determine its point of intersection with the curve.

We shall, however, still mark off on the axis of y portions, CB, CB′ $= \pm b$, and we shall find that the length CB has an important connexion with the curve, and may be conveniently called an axis of the curve. In like manner, if we obtained an equation to determine the length of any other diameter, of the form $\rho^2 = -R^2$, although this diameter cannot meet the curve, yet if we measure on it from the centre lengths $= \pm R$, these lines may be conveniently spoken of as diameters of the hyperbola.

The *locus* of the extremities of these diameters which do not meet the curve is, by changing the sign of ρ^2 in the equation of the curve, at once found to be

$$\frac{1}{\rho^2} = \frac{\sin^2\theta}{b^2} - \frac{\cos^2\theta}{a^2},$$

or

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

This is the equation of a hyperbola having the axis of y for its axis meeting it in real points, and the axis of x for the axis meeting it in imaginary points. It is represented by the dotted curve on the figure, and is called the hyperbola *conjugate* to the given hyperbola.

170. We proved (Art. 168) that the diameters answering to $\tan\theta = \pm\frac{b}{a}$ meet the curve at infinity; they are, therefore, the same as the lines called, in Art. 157, the *asymptotes* of the curve. They are the lines CK, CL on the figure, and evidently separate those diameters which meet the curve in real points from those which meet it in imaginary points. It is evident also, that two conjugate hyperbolæ have the same asymptotes. ||

The expression $\tan\theta = \pm\frac{b}{a}$ enables us, being given the axes in magnitude and position, to find the asymptotes, for, if we form a rectangle by drawing parallels to the axes through B and A, then the asymptote CK must be the diagonal of this rectangle.

Again,
$$\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}} = \frac{1}{e}.$$

But, since the asymptotes make equal angles with the axis of x, the angle which they make with each other must be $= 2\theta$. Hence, *being given the eccentricity of a hyperbola, we are given the angle between the asymptotes*, which is double the angle whose secant is the eccentricity.

Ex. To find the eccentricity of a conic given by the general equation.

We can (Art. 70) write down the tangent of the angle between the lines denoted by $Ax^2 + Bxy + Cy^2 = 0$, and thence form the expression for the secant of its half; or we may proceed by the help of Art. 160, Ex. 3.

We have $$\frac{1}{a^2} = \frac{A + C - R}{2F}, \quad \frac{1}{b^2} = \frac{A + C + R}{2F},$$

where $$R^2 = B^2 + (A - C)^2, \quad = B^2 - 4AC + (A + C)^2.$$

Hence $$\frac{1}{b^2} - \frac{1}{a^2} = \frac{R}{F}; \quad \frac{a^2 - b^2}{a^2} = \frac{2R}{A + C + R}.$$

THE TANGENT.

171. We now proceed to investigate some of the properties of the ellipse and hyperbola. We shall find it convenient to consider both curves together, for, since their equations only differ in the sign of b^2, they have many properties in common which can be proved at the same time, by considering the sign of b^2 as indeterminate. We shall, in the following Articles, use the signs which apply to the ellipse. The reader may then obtain the corresponding formulæ for the hyperbola by changing the sign of b^2.

We might deduce several of the results which follow, as particular cases of those obtained in the last chapter from the general equation, but we have thought it worth while to establish the more important equations independently.

To find the equation of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The method we pursue is identical with that used Art. 83. The co-ordinates of two points on the curve satisfy the relations

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 = \frac{x''^2}{a^2} + \frac{y''^2}{b^2};$$

hence $$\frac{x'^2 - x''^2}{y'^2 - y''^2} = -\frac{a^2}{b^2}, \quad \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\frac{x' + x''}{y' + y''}.$$

The equation of the line joining the two points is, therefore,

$$\frac{y - y'}{x - x'} = \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\frac{x' + x''}{y' + y''}.$$

That of the tangent is found by making $x' = x''$, $y' = y''$;

$$\frac{y - y'}{x - x'} = -\frac{b^2}{a^2}\frac{x'}{y'};$$

or, reducing, and remembering that $x'y'$ satisfies the equation of the curve,

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

172. *To find the equation of the line joining the points of contact of tangents through any point* $(x'y')$.

Let the co-ordinates of the point of contact of one of the tangents through $(x'y')$ be X, Y; then forming (Art. 171) the equation of the tangent at XY, and substituting in it the co-ordinates $x'y'$ which must satisfy it, we have

$$\frac{Xx'}{a^2} + \frac{Yy'}{b^2} = 1.$$

We see, therefore, that the co-ordinates of either point of contact must satisfy the equation

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1;$$

and, since this is the equation of a right line, it must represent the line joining them.

Ex. 1. To find the condition that any line $\frac{x}{m} + \frac{y}{n} = 1$ should touch the conic section

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Comparing the equations

$$\frac{x}{m} + \frac{y}{n} = 1, \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

we find

$$\frac{x'}{a^2} = \frac{1}{m}, \text{ and } \frac{y'}{b^2} = \frac{1}{n};$$

and, substituting for $x'y'$ in the equation of the curve, we have, for the required condition,

$$\frac{a^2}{m^2} + \frac{b^2}{n^2} = 1.$$

Ex. 2. To find the equation of the pair of tangents through $x'y'$ to the conic. Proceeding, as in Art. 150, we find

$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right)^2.$$

Ex. 3. To find the angle ϕ between the pair of tangents from $x'y'$ to the curve.

When an equation of the second degree represents two right lines, the three highest terms being put = 0, denote two parallel lines through the origin; hence, the angle included by the first pair of right lines depends solely on the three highest terms of the general equation. Arranging, then, the equation found in the last Example, we find, by Art. 70,

$$\tan\phi = \frac{2ab\sqrt{\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)}}{x'^2+y'^2-a^2-b^2}.$$

Ex. 4. Find the locus of a point the tangents through which intersect at right angles.

Equating to 0, the denominator in the value of $\tan\phi$, we find $x^2+y^2=a^2+b^2$, the equation of a circle concentric with the ellipse. The locus of the intersection of tangents which cut at a given angle is, in general, a curve of the fourth degree.

CONJUGATE DIAMETERS.

173. When the equation of the curve is referred to any pair of conjugate diameters, the coefficient of xy vanishes (Art. 158); and if a', b', be the lengths of these diameters, the equation may be written (as in Art. 163)

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.$$

Now it can be proved, precisely as in Art. 171, that the equation of any tangent, referred to these axes, is

$$\frac{xx'}{a'^2}+\frac{yy'}{b'^2}=1,$$

and, as in Art. 172, that the equation of the polar of any point $(x'y')$ is of the same form. The polar of any point on the axis of x is, therefore,

$$\frac{xx'}{a'^2}=1.$$

Hence, the polar of any point P is found by drawing a diameter through the point, taking CP · CP′ = to the square of the semi-diameter, and then drawing through P′ a parallel to the conjugate diameter. This includes, as a particular case, the theorem proved already (Art. 145), viz.:

The tangent at the extremity of any diameter is parallel to the conjugate diameter.

174. The theorem just stated enables us easily to find the equation, referred to the rectangular axes, of the diameter conjugate to that passing through any point $(x'y')$ on the curve.

For we have only to form the equation of a line drawn through the origin parallel to the tangent, whose equation we found (Art. 171); and we have for the equation of the conjugate diameter

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0.$$

Let θ be the angle made with the axis of x by the original diameter, then $\tan\theta$ plainly $= \frac{y'}{x'}$, and if θ' be the angle made by the conjugate diameter, this equation shows (Art. 22) that

$$\tan\theta = -\frac{b^2x'}{a^2y'}.$$

Hence

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

This relation, connecting the angles made with the axis major by a pair of conjugate diameters, enables us at once to determine whether any given pair of diameters be conjugate or not.

The corresponding relation for the hyperbola is (see Art. 171)

$$\tan\theta\tan\theta' = \frac{b^2}{a^2}.$$

175. Since, in the ellipse, $\tan\theta\tan\theta'$ is negative, if one of the angles θ, θ', be acute (and, therefore, its tangent positive), the other must be obtuse (and, therefore, its tangent negative). Hence, *conjugate diameters in the ellipse lie on different sides of the axis minor* (which answers to $\theta = 90°$).

In the hyperbola, on the contrary, $\tan\theta\tan\theta'$ is positive, therefore, θ and θ' must be either both acute or both obtuse. Hence, *in the hyperbola, conjugate diameters lie on the same side of the conjugate axis.*

In the hyperbola, if $\tan\theta$ be less, $\tan\theta'$ must be greater than $\frac{b}{a}$, but (Art. 170) the diameter answering to the angle whose tangent is $\frac{b}{a}$, is the asymptote which (by the same Article) separates those diameters which meet the curve from those which do not intersect it. Hence, *if one of two conjugate diameters meet a hyperbola in real points, the other will not.* Hence also it may be seen that each asymptote is its own conjugate.

176. *To find the co-ordinates $x''y''$ of the extremity of the diameter conjugate to that passing through $x'y'$.*

These co-ordinates are obviously found by solving for x and y between the equation of the conjugate diameter, and that of the curve, viz.,

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Substituting in the second the values of x and y, found from the first equation, and remembering that x', y' satisfy the equation of the curve, we find without difficulty

$$\frac{x''}{a} = \pm\frac{y'}{b}, \quad \frac{y''}{b} = \mp\frac{x'}{a}.$$

177. *To express the lengths of a diameter (a'), and its conjugate (b'), in terms of the abscissa of the extremity of the diameter.*

(1.) We have
$$a'^2 = x'^2 + y'^2.$$

But
$$y'^2 = \frac{b^2}{a^2}(a^2 - x'^2).$$

Hence
$$a'^2 = b^2 + \frac{a^2 - b^2}{a^2}x'^2 = b^2 + e^2x'^2.$$

(2.) Again, we have
$$b'^2 = x''^2 + y''^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2,$$

or
$$= (a^2 - x'^2) + \frac{b^2}{a^2}x'^2;$$

hence
$$b'^2 = a^2 - e^2x'^2.$$

From these values we have

$$a'^2 + b'^2 = a^2 + b^2;$$

or, *The sum of the squares of any pair of conjugate diameters of an ellipse is constant* (see Ex. 3, Art. 162).

178. In the hyperbola we must change the signs of b^2 and b'^2, and we get
$$a'^2 - b'^2 = a^2 - b^2,$$

or, *The difference of the squares of any pair of conjugate diameters of a hyperbola is constant.*

If in the hyperbola we have $a = b$, its equation becomes

$$x^2 - y^2 = a^2,$$

and it is called an *equilateral* hyperbola.

The theorem just proved shows that *every diameter of an equilateral hyperbola is equal to its conjugate.*

The asymptotes of the equilateral hyperbola being given by the equation

$$x^2 - y^2 = 0,$$

are at right angles to each other. Hence this hyperbola is often called a *rectangular* hyperbola.

The condition that the general equation of the second degree should represent an equilateral hyperbola is $A = -C$; for (Art. 70) this is the condition that the asymptotes $(Ax^2 + Bxy + Cy^2 = 0)$ should be at right angles to each other; but if the hyperbola be *rectangular* it must be *equilateral*, since (Art. 170) the tangent of half the angle between the asymptotes $= \frac{b}{a}$; therefore, if this angle $= 45°$, we have

$$b = a.$$

179. *To find the length of the perpendicular from the centre on the tangent.*

The length of the perpendicular from the origin on the line

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

is (Art. 27)

$$\frac{1}{\sqrt{\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right)}} = \frac{ab}{\sqrt{\left(\frac{b^2x'^2}{a^2} + \frac{a^2y'^2}{b^2}\right)}};$$

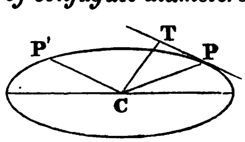
but we proved (Art. 177) that

$$b'^2 = \frac{b^2x'^2}{a^2} + \frac{a^2y'^2}{b^2};$$

hence

$$p = \frac{ab}{b'}.$$

180. *To find the angle between any pair of conjugate diameters.*



The angle between the diameters is equal to the angle between either, and the tangent parallel to the other. Now

$$\sin CPT = \frac{CT}{CP} = \frac{p}{a'}.$$

Hence $$\sin\phi \text{ (or PCP)} = \frac{ab}{a'b'}.$$

The equation $a'b'\sin\phi = ab$ proves, that *the triangle formed by joining the extremities of conjugate diameters of an ellipse or hyperbola has a constant area* (see Art. 162, Ex. 2).

181. The sum of the squares of any two conjugate diameters of an ellipse being constant, their rectangle is a maximum when they are equal, and, therefore, in this case, $\sin\phi$ is a minimum; hence the acute angle between the two *equal* conjugate diameters is less (and, consequently, the obtuse angle greater) than the angle between any other pair of conjugate diameters.

The length of the equal conjugate diameters is found by making $a' = b'$ in the equation $a'^2 + b'^2 = a^2 + b^2$, whence a'^2 is half the sum of a^2 and b^2, and in this case

$$\sin\phi = \frac{2ab}{a^2 + b^2}.$$

The angle which either of the equiconjugate diameters makes with the axis of x is found from the equation

$$\tan\theta \tan\theta' = -\frac{b^2}{a^2}.$$

by making $\tan\theta = -\tan\theta'$, for any two equal diameters make equal angles with the axis of x on opposite sides of it (Art. 165).

Hence $$\tan\theta = \frac{b}{a}.$$

It follows, therefore, from Art. 170, that if an ellipse and hyperbola have the same axes in magnitude and position, then the asymptotes of the hyperbola will coincide with the equiconjugate diameters of the ellipse.

The general equation of an ellipse, referred to two conjugate diameters (Art. 173), becomes $x^2 + y^2 = a'^2$, when $a' = b'$. We see, therefore, that, by taking the equiconjugate diameters for axes, the equation of *any* ellipse may be put into the same form as the equation of the circle, $x^2 + y^2 = r^2$, but that in the case of the ellipse the angle between these axes will be *oblique*.

182. *To express the perpendicular from the centre on the tangent in terms of the angles which it makes with the axes.*

If we proceed to throw the equation of the tangent $\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1\right)$ into the form $x \cos a + y \sin a = p$ (Art. 25), we find immediately, by comparing these equations,

$$\frac{x'}{a^2} = \frac{\cos a}{p}, \quad \frac{y'}{b^2} = \frac{\sin a}{p}.$$

Substituting in the equation of the curve the values of x', y', hence obtained, we find

$$p^2 = a^2 \cos^2 a + b^2 \sin^2 a.*$$

The equation of the tangent may, therefore, be written

$$x \cos a + y \sin a - \sqrt{(a^2 \cos^2 a + b^2 \sin^2 a)} = 0.$$

Hence, by Art. 27, the perpendicular from any point $(x'y')$ on the tangent is

$$x' \cos a + y' \sin a - \sqrt{(a^2 \cos^2 a + b^2 \sin^2 a)}.$$

Ex. To find the locus of the intersection of tangents which cut at right angles. Let p, p' be the perpendiculars on those tangents, then

$$p^2 = a^2 \cos^2 a + b^2 \sin^2 a, \quad p'^2 = a^2 \sin^2 a + b^2 \cos^2 a, \quad p^2 + p'^2 = a^2 + b^2.$$

But the square of the distance from the centre of the intersection of two lines, which cut at right angles, is equal to the sum of the squares of its distances from the lines themselves. This distance, therefore, is constant, and the required locus is a circle (see p. 151).

183. The chords which join the extremities of any diameter to any point on the curve are called *supplemental chords.*

Diameters parallel to any pair of supplemental chords are conjugate.

For if we consider the triangle formed by joining the extremities of any diameter AB to any point on the curve D; since, by elementary geometry, the line joining the middle points of two sides must be parallel to the third, the diameter bisecting AD will be parallel to BD, and the diameter bisecting BD will be parallel to AD. The same thing may be proved analytically, by forming the equations of AD and BD, and showing that the product of the tangents of the angles made by these lines with the axis is $= -\frac{b^2}{a^2}$.

* In like manner, $p^2 = a'^2 \cos^2 a + b'^2 \cos^2 \beta$, a and β being the angles the perpendicular makes with any pair of conjugate diameters.

This property enables us to draw geometrically a pair of conjugate diameters making any angle with each other. For if we describe on any diameter a segment of a circle containing the given angle, and join the points where it meets the curve to the extremities of the assumed diameter, we obtain a pair of supplemental chords inclined at the given angle, the diameters parallel to which will be conjugate to each other.

Ex. 1. Tangents at the extremities of any diameter are parallel.

Their equations are $$\frac{xx'}{a^2} + \frac{yy'}{b^2} = \pm 1.$$

This also follows from the theorem of Art. 148, and from considering that the centre is the pole of the line at infinity (Art. 157).

Ex. 2. If any variable tangent to a central conic section meet two fixed parallel tangents, it will intercept portions on them, whose rectangle is constant, and equal to the square of the semidiameter parallel to them.

Let us take for axes the diameter parallel to the tangents and its conjugate, then the equations of the curve and of the variable tangent will be

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \qquad \frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1.$$

The intercepts on the fixed tangents are found by making x alternately $= \pm a'$ in the latter equation, and we get

$$y = \frac{b'^2}{y'}\left(1 \mp \frac{x'}{a'}\right),$$

and, therefore, their product is

$$\frac{b'^4}{y'^2}\left(1 - \frac{x'^2}{a'^2}\right);$$

which, substituting for y'^2 from the equation of the curve, reduces to b'^2.

Ex. 3. The same construction remaining, the rectangle under the segments of the variable tangent is equal to the square of the semidiameter parallel to it.

For, the intercept on either of the parallel tangents is to the adjacent segment of the variable tangent as the parallel semidiameters (Art. 152); therefore, the rectangle under the intercepts of the fixed tangents is to the rectangle under the segments of the variable tangent as the *squares* of these semidiameters; and, since the first rectangle is equal to the square of the semidiameter parallel to it, the second rectangle must be equal to the square of the semidiameter parallel to *it*.

Ex. 4. If any tangent meet any two conjugate diameters, the rectangle under its segments is equal to the square of the parallel semidiameter.

Take for axes the semidiameter parallel to the tangent and its conjugate; then the equations of any two conjugate diameters being (Art. 174)

$$y = \frac{y'}{x'}x, \qquad \frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 0,$$

the intercepts made by them on the tangent are found by making $x = a'$ to be

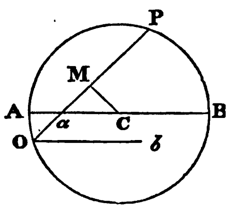
$$y = \frac{y'}{x'}a', \text{ and } y = -\frac{b'^2}{a'}\frac{x'}{y'},$$

whose rectangle is evidently $= b'^2$.

We might, in like manner, have given a purely algebraical proof of Ex. 3.

Hence, also, if the points be joined to the centre where two parallel tangents meet any tangent, the joining lines will be conjugate diameters.

Ex. 5. Given, in magnitude and position, two conjugate semidiameters, Oa, Ob, of a central conic, to determine the axes.



The following construction is founded on the theorem proved in the last Example:—Through a, the extremity of either diameter, draw a parallel to the other; it must of course be a tangent to the curve. Now, on Oa take a point P, such that the rectangle $Oa \cdot aP = Ob^2$ (on the side remote from O for the ellipse, on the same side for the hyperbola), and describe a circle through O, P, having its centre on aC, then the lines OA, OB, are the axes of the curve; for, since the rectangle $Aa \cdot aB = Oa \cdot aP = Ob^2$, the lines OA, OB are conjugate diameters, and since AB is a diameter of the circle, the angle AOB is right.

Ex. 6. Given any two semidiameters, if from the extremity of each an ordinate be drawn to the other, the triangles so formed will be equal in area.

Ex. 7. Or if tangents be drawn at the extremity of each, the triangles so formed will be equal in area.

THE NORMAL.

184. A line drawn through any point of a curve perpendicular to the tangent at that point is called the *Normal.*

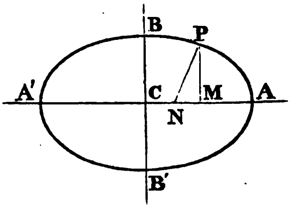
Forming, by Art. 40, the equation of a line drawn through $(x'y')$ perpendicular to $\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1\right)$, we find for the equation of the normal to a conic

$$\frac{x'}{a^2}(y - y') = \frac{y'}{b^2}(x - x'),$$

or

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = c^2,$$

c^2 being used, as in Art. 164, to denote $a^2 - b^2$.



Hence we can find the portion CN intercepted by the normal on either axis; for, making $y = 0$ in the equation just given, we find

$$x = \frac{c^2}{a^2}x', \text{ or } x = e^2x'.$$

We can thus draw a normal to an ellipse from any point on the axis, for given CN we can find x', the abscissa of the point through which the normal is drawn.

The circle may be considered as an ellipse whose eccentricity $= 0$, since $c^2 = a^2 - b^2 = 0$. The intercept CN, therefore, is constantly $= 0$ in the case of the circle, or *every normal to a circle passes through its centre.*

185. The portion MN intercepted on the axis between the normal and ordinate is called the *Subnormal.* Its length is, by the last Article,

$$x' - \frac{c^2}{a^2}x' = \frac{b^2}{a^2}x'.$$

The normal, therefore, cuts the abscissa into parts which are in a constant ratio.

If a tangent drawn at the point P cut the axis in T, the intercept MT is, in like manner, called the *Subtangent.*

Since the whole length $\text{CT} = \frac{a^2}{x'}$ (Art. 173), the subtangent

$$= \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}.$$

The length of the *normal* can also be easily found. For

$$\text{PN}^2 = \text{PM}^2 + \text{NM}^2 = y'^2 + \frac{b^4}{a^4}x'^2 = \frac{b^2}{a^2}\left(\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2\right).$$

But if b' be the semidiameter conjugate to CP, the quantity within the parentheses $= b'^2$ (Art. 177). Hence the length of the normal $\text{PN} = \frac{bb'}{a}$.

If the normal be produced to meet the axis minor it can be proved, in like manner, that its length $= \frac{ab'}{b}$. Hence, *the rectangle under the segments of the normal is equal to the square of the semiconjugate diameter.*

Again, we found (Art. 181) that the perpendicular from the centre on the tangent $= \frac{ab}{b'}$. Hence, *the rectangle under the normal and the perpendicular from the centre on the tangent, is constant and equal to the square of the semiaxis.*

Thus, too, we can express the normal in terms of the angles it makes with the axis, for

$$\text{PN} = \frac{b^2}{p} = \frac{b^2}{\sqrt{(a^2\cos^2\alpha + b^2\sin^2\alpha)}} \text{ (Art. 182)}; = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\alpha)}}.$$

Ex. 1. To draw a normal to an ellipse or hyperbola passing through a given point.

Let the point on the curve at which the normal is drawn be XY, then its equation is (Art. 184)

$$\frac{a^2x}{X} - \frac{b^2y}{Y} = c^2:$$

and if this normal passes through a fixed point $x'y'$, we have the relation

$$\frac{a^2x'}{X} - \frac{b^2y'}{Y} = c^2.$$

Hence the points on the curve, whose normals will pass through $(x'y')$ are the points of intersection of the given curve with the hyperbola

$$c^2xy = a^2x'y - b^2y'x.$$

Ex. 2. If through a given point on a conic any two lines at right angles to each other be drawn to meet the curve, the line joining their extremities will pass through a fixed point on the normal.

Let us take for axes the tangent and normal at the given point, then the equation of the curve must be of the form

$$Ax^2 + Bxy + Cy^2 + Ey = 0$$

(for $F = 0$, because the origin is on the curve, and $D = 0$ (Art. 132), because the tangent is supposed to be the axis of x, whose equation is $y = 0$).

Now, let the equation of *any* two lines through the origin be

$$x^2 + pxy + qy^2 = 0.$$

Multiply this equation by A, and subtract it from that of the curve, and we get

$$(B - Ap)\,xy + (C - Aq)\,y^2 + Ey = 0.$$

This (Art. 36) is the equation of a figure passing through the points of intersection of the lines and conic; but it may evidently be resolved into $y = 0$ (the equation of the tangent at the given point), and

$$(B - Ap)\,x + (C - Aq)\,y + E = 0,$$

which must be the equation of the chord joining the extremities of the given lines.

The point where this chord meets the normal (the axis of y) is $y = \frac{E}{Aq - C}$; but if the lines are at right angles $q = -1$ (Art. 70), and the intercept on the normal has the constant length

$$= \frac{-E}{A + C}.$$

This theorem will be equally true if the lines be drawn so as to make with the normal angles, the product of whose tangents is constant, for, in this case, q is constant: and, therefore, the intercept $\frac{E}{Aq - C}$ is constant.

Ex. 3. To find the co-ordinates of the intersection of the tangents at the points $x'y'$, $x''y''$.

The co-ordinates of the intersection of the lines

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1, \qquad \frac{x''x}{a^2} + \frac{y''y}{b^2} = 1$$

are

$$x = \frac{a^2(y' - y'')}{y'x'' - y''x'}, \qquad y = \frac{b^2(x' - x'')}{x'y'' - y'x''}.$$

These results may be written in another form, since

$$2(y'x'' - y''x') = (x' + x'')(y' - y'') - (y' + y'')(x' - x''),$$

and (Art. 171) $$\frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\frac{x' + x''}{y' + y''};$$

on making which substitutions, the preceding values become

$$x = \frac{(x' + x'')}{1 + \frac{x'x''}{a^2} + \frac{y'y''}{b^2}}, \quad y = \frac{(y' + y'')}{1 + \frac{x'x''}{a^2} + \frac{y'y''}{b^2}}.$$

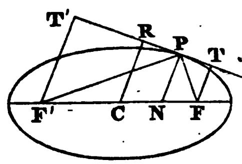
Ex. 4. To find the co-ordinates of the intersection of the normals at the points $x'y'$, $x''y''$.

Proceeding as in the last Example, we find

$$x = \frac{(a^2 - b^2)\,x'x''\,X}{a^4}, \quad y = \frac{(b^2 - a^2)\,y'y''\,Y}{b^4},$$

where X, Y are the co-ordinates of the intersection of tangents, found in the last Example.

THE FOCI.



186. If on the axis major of an ellipse we take two points equidistant from the centre, whose common distance

$$= \pm\sqrt{(a^2 - b^2)}, \text{ or } = \pm c,$$

these points are called the *foci* of the curve.

The foci of an hyperbola are two points on the transverse axis, at a distance from the centre still $= \pm c$, c being in the hyperbola

$$= \sqrt{(a^2 + b^2)}.$$

To express the distance of any point on an ellipse from the focus.

Since the co-ordinates of one focus are $(x = +c, y = 0)$, the square of the distance of any point from it

$$= (x' - c)^2 + y'^2 = x'^2 + y'^2 - 2cx' + c^2.$$

But (Art. 177)

$$x'^2 + y'^2 = b^2 + e^2x^2, \text{ and } b^2 + c^2 = a^2.$$

Hence $$FP^2 = a^2 - 2cx + e^2x;$$

and recollecting that $c = ae$, we have

$$FP = a - ex.$$

[We reject the value $(ex - a)$ obtained by giving the other sign to the square root. For, since x is less than a, and e less than 1, the quantity $ex - a$ is constantly negative, and, therefore, does not concern us, as we are now considering, not the direction, but the absolute magnitude of the radius vector FP.]

We have, similarly, the distance from the other focus

$$F'P = a + ex,$$

since we have only to write $-c$ for $+c$ in the preceding formulæ.

Hence $$FP + F'P = 2a,$$

or, *The sum of the distances of any point on an ellipse from the foci is constant and equal to the axis major.*

187. In applying the preceding proposition to the hyperbola, we obtain the same value for FP^2; but in extracting the square root we must change the sign in the value of FP, for in the hyperbola x is greater than a, and e is greater than 1.

Hence, $a - ex$ is constantly negative; the absolute magnitude, therefore, of the radius vector is

$$FP = ex - a.$$

In like manner, $$F'P = ex + a.$$

Hence $$F'P - FP = 2a.$$

Therefore, *in the hyperbola, the* difference *of the focal radii is constant, and equal to the transverse axis.*

For both curves the rectangle *under the focal radii* $= a^2 - e^2x^2$, that is (Art. 177), *is equal to the square of the semiconjugate diameter.*

188. The reader may prove the converse of the above results by seeking the locus of the vertex of a triangle, if the base and either sum or difference of sides be given.

Taking the middle point of the base ($= 2c$) for origin, the equation is

$$\sqrt{\{y^2 + (c + x)^2\}} \pm \sqrt{\{y^2 + (c - x)^2\}} = 2a,$$

which, when cleared of radicals, becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

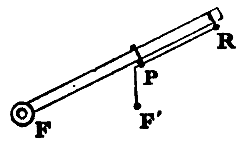
Now, if the *sum* of the sides be given, since the sum must always be greater than the base, a is greater than c, therefore the coefficient of y^2 is positive, and the locus an *ellipse*.

If the *difference* be given, a is less than c; the coefficient of y^2 is negative, and the locus an *hyperbola*.

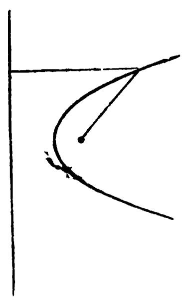
189. By the help of the preceding theorems, we can describe an ellipse or hyperbola mechanically.

If the extremities of a thread be fastened at two fixed points F and F′, it is plain that a pencil moved about so as to keep the thread always stretched will describe an ellipse whose foci are F and F′, and whose axis major is equal to the length of the thread.

In order to describe an hyperbola, let a ruler be fastened at one extremity (F), and capable of moving round it, then if a thread, fastened to a fixed point F′, and also to a fixed point on the ruler (R), be kept stretched by a ring at P, as the ruler is moved round, the point P will describe an hyperbola; for, since the sum of F′P and PR is constant, the difference of FP and F′P will be constant.



190. The polar of either focus is called the *directrix* of the conic section. The directrix must, therefore (Art. 173), be a line perpendicular to the axis major at a distance from the centre $= \pm \dfrac{a^2}{c}$.



Knowing the distance of the directrix from the centre, we can find its distance from any point on the curve. It must be equal to

$$\frac{a^2}{c} - x', \text{ or } = \frac{a}{c}(a - ex') = \frac{1}{e}(a - ex').$$

But the distance of any point on the curve from the focus $= a - ex'$.

Hence we obtain the important property, that *the distance of any point on the curve from the focus is in a constant ratio to its distance from the directrix*, viz., as e to 1.

Conversely, a conic section may be defined as the locus of a point whose distance from a fixed point (the focus) is in a constant ratio to its distance from a fixed line (the directrix). On this definition several writers have based the theory of conic sections. Taking the fixed line for the axis of x, the equation of the locus is at once written down

$$(x - x')^2 + (y - y')^2 = e^2y^2,$$

which it is easy to see will represent an ellipse, hyperbola, or parabola, according as e is less, greater than, or equal to 1.

Ex. If a curve be such that the distance of any point of it from a fixed point can be expressed as a rational function of the first degree of its co-ordinates, then the curve must be a conic section, and the fixed point its focus (see O'Brien's *Co-ordinate Geometry*, p. 85).

For, if the distance can be expressed

$$\rho = Ax + By + C,$$

since $Ax + By + C$ is proportional to the perpendicular let fall on the right line whose equation is $(Ax + By + C = 0)$, the equation signifies that the distance of any point of the curve from the fixed point is in a constant ratio to its distance from this line.

191. *To find the length of the perpendicular from the focus on the tangent.*

The length of the perpendicular from the focus $(+c, 0)$ on the line $\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1\right)$ is, by Art. 27,

$$= \frac{1 - \frac{cx'}{a^2}}{\sqrt{\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right)}};$$

but, Art. 179, $$\sqrt{\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right)} = \frac{b'}{ab}.$$

Hence $$FT = \frac{b}{b'}(a - ex) = \frac{b}{b'} FP.$$

Likewise, $$F'T' = \frac{b}{b'}(a + ex) = \frac{b}{b'} F'P.$$

Hence $$FT \cdot F'T' = b^2 \text{ (since } a^2 - e^2x^2 = b'^2),$$

or, *The rectangle under the focal perpendiculars on the tangent is constant, and equal to the square of the semiaxis minor.*

This property applies equally to the ellipse and the hyperbola.

192. Some important consequences may be drawn from the value of the perpendicular just found.

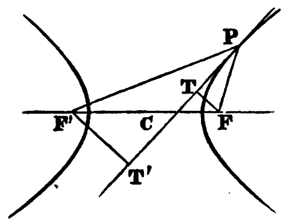
For we had $$FT = \frac{b}{b'} FP, \text{ or } \frac{FT}{FP} = \frac{b}{b'};$$

but $$\frac{FT}{FP} = \sin FPT.$$

Hence the sine of the angle which the focal radius vector makes with the tangent $= \frac{b}{b'}$.

We find, in like manner, the same value for sin F′PT′, the sine of the angle which the other focal radius vector makes with the tangent. Hence *the focal radii make equal angles with the tangent.*

This property is true both for the ellipse and hyperbola, and, on looking at the figures, it is evident that the tangent to the ellipse is the *external* bisector of the angle between the focal radii, and the tangent to the hyperbola the *internal* bisector.



Hence, *if an ellipse and hyperbola, having the same foci, pass through the same point, they will cut each other at right angles*, that is to say, the tangent to the ellipse at that point will be at right angles to the tangent to the hyperbola.

Ex. 1. Prove analytically that confocal conics cut at right angles.

The co-ordinates of the intersection of the conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1,$$

satisfy the relation obtained by subtracting the two equations

$$\frac{(a^2-a'^2)\,x'^2}{a^2a'^2}+\frac{(b^2-b'^2)\,y'^2}{b^2b'^2}=0.$$

But if the conics be confocal $a^2-a'^2=b^2-b'^2$, and this relation becomes

$$\frac{x'^2}{a^2a'^2}+\frac{y'^2}{b^2b'^2}=0.$$

But this is the condition (Art. 40) that the two tangents

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1, \qquad \frac{xx'}{a'^2}+\frac{yy'}{b'^2}=1,$$

should be perpendicular to each other.

Ex. 2. Find the length of a line drawn through the centre parallel to either focal radius vector, and terminated by the tangent.

This length is found by dividing the perpendicular from the centre on the tangent $\left(\frac{ab}{b'}\right)$ by the sine of the angle between the radius vector and tangent $\left(\frac{b}{b'}\right)$, and is therefore $=a$.

193. The normal, being perpendicular to the tangent, is the *internal* bisector of the angle between the focal radii in the case of the ellipse, and the *external* bisector in the case of the hyperbola.

We can give an independent proof of this, by showing that it cuts the distance between the foci into parts which are in the ratio of the focal radii (Euc. VI. 3), for the distance of the foot of the normal from the centre is (Art. 184) $= e^2x'$. Hence its distances from the foci are $c + e^2x'$ and $c - e^2x'$, quantities which are evidently e times $a + ex'$ and $a - ex'$.

Ex. To draw a normal to the ellipse from any point on the axis minor.

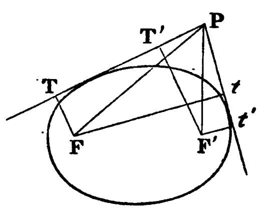
Ans. The circle through the given point, and the two foci, will meet the curve at the point whence the normal is to be drawn.

194. Another important consequence may be deduced from the theorem (of Art. 191), that the rectangle under the focal perpendiculars on the tangent is constant.

For, if we take any two tangents, we have

$$FT \cdot F'T' = Ft \cdot F't', \text{ or } \frac{FT}{Ft} = \frac{F't'}{F'T'};$$

but $\frac{FT}{Ft}$ is the ratio of the sines of the parts into which the line FP divides the angle at P, and $\frac{F't'}{F'T'}$ is the ratio of the sines of the parts into which F′P divides the same angle; we have, therefore, the angle $TPF = tPF'$.



If we conceive a conic section to pass through P, having F and F′ for foci, it was proved in Art. 191, that the tangent to it must be equally inclined to the lines FP, F′P; it follows, therefore, from the present Article, that it must be also equally inclined to PT, Pt; hence we derive a useful theorem, that *if through any point* (P) *of a conic section we draw tangents* (PT, Pt) *to a* confocal *conic section, these tangents will be equally inclined to the tangent at* P.

195. *To find the locus of the foot of the perpendicular let fall from either focus on the tangent.*

The perpendicular from the focus is expressed in terms of the angles it makes with the axis by putting $x' = c$, $y' = 0$ in the formula of Art. 182, viz.,

$$p = x' \cos a + y' \sin a - \sqrt{(a^2 \cos^2 a + b^2 \sin^2 a)}.$$

Hence the polar equation of the locus is

$$\rho = c\cos a - \sqrt{(a^2\cos^2 a + b^2\sin^2 a)},$$

or $$\rho^2 - 2c\rho\cos a + c^2\cos^2 a = a^2\cos^2 a + b^2\sin^2 a,$$

or $$\rho^2 - 2c\rho\cos a = b^2.$$

This (Art. 93) is the polar equation of a circle whose centre is on the axis of x, at a distance from the focus $= c$; the circle is, therefore, concentric with the curve. The radius of the circle is, by the same Article, $= a$.

Hence, *if we describe a circle having for diameter the transverse axis of an ellipse or hyperbola, the perpendicular from the focus will meet the tangent on the circumference of this circle.*

Or, conversely, *if from any point* F (see figure, p. 166) *we draw a radius vector* FT *to a given circle, and draw* TP *perpendicular to* FT, *the line* TP *will always touch a conic section having* F *for its focus, which will be an ellipse or hyperbola, according as* F *is within or without the circle.*

It may be inferred from Art. 192, Ex. 2, that the line CT, whose length $= a$, is parallel to the focal radius vector F′P.

196. *To find the angle subtended at the focus by the tangent drawn to a central conic from any point* (xy).

Let the point of contact be $(x'y')$, the centre being the origin, then, if the focal radii to the points (xy), $(x'y')$, be ρ, ρ', and make angles θ, θ', with the axis, it is evident that

$$\cos\theta = \frac{x+c}{\rho},\quad \sin\theta = \frac{y}{\rho};\quad \cos\theta' = \frac{x'+c}{\rho'},\quad \sin\theta' = \frac{y'}{\rho'}.$$

Hence $$\cos(\theta-\theta') = \frac{(x+c)(x'+c)+yy'}{\rho\rho'};$$

but from the equation of the tangent we must have

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

Substituting this value of yy', we get

$$\rho\rho'\cos(\theta-\theta') = xx' + cx + cx' + c^2 - \frac{b^2}{a^2}xx' + b^2,$$

or $$= e^2xx' + cx + cx' + a^2 = (a+ex)(a+ex');$$

or since $\rho' = a + ex'$ we have (see O'Brien's "Co-ordinate Geometry, p. 156)

$$\cos(\theta-\theta') = \frac{a+ex}{\rho}.$$

Since this value depends solely on the co-ordinates xy, and does not involve the co-ordinates of the point of contact, either tangent drawn from xy subtends the same angle at the focus. Hence, *The angle subtended at the focus by any chord is bisected by the line joining the focus to its pole.*

197. *The line joining the focus to the pole of any chord passing through it is perpendicular to that chord.*

This may be deduced as a particular case of the last Article, the angle subtended at the focus being in this case 180°; or directly as follows:—The equation of the perpendicular through any point $x'y'$ to the polar of that point $\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}=1\right)$ is, as in Art. 184,

$$\frac{a^2x}{x'}-\frac{b^2y}{y'}=c^2.$$

But if $x'y'$ be anywhere on the directrix, we have $x'=\frac{a^2}{c}$, and it will then be found that both the equation of the polar and that of the perpendicular are satisfied by the co-ordinates of the focus ($x=c$, $y=0$).

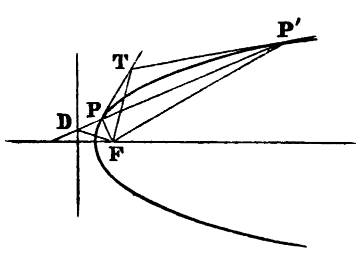
When in any curve we use polar co-ordinates, the portion intercepted by the tangent on a perpendicular to the radius vector drawn through the pole is called the *polar subtangent.* Hence the theorem of this Article may be stated thus: *The focus being the pole, the locus of the extremity of the polar subtangent is the directrix.*

It will be proved (Chap. XII.) that the theorems of this and the last Article are true also for the parabola.

Ex. 1. The angle is constant which is subtended at the focus, by the portion intercepted on a variable tangent between two fixed tangents.

By Art. 196, it is half the angle subtended by the chord of contact of the fixed tangents.

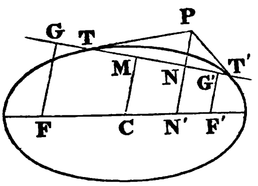
Ex. 2. If any chord PP′ cut the directrix in D, then FD is the external bisector of the angle PFP′. For FT is the internal bisector (Art. 196); but D is the polar of FT (since it is the intersection of PP′, the polar of T, with the directrix, the polar of F); therefore, DF is perpendicular to FT, and is therefore the external bisector.



[The following theorems (communicated to me by the Rev. W. D. Sadleir) are founded on the analogy between the equations of the polar and the tangent.]

Ex. 3. If a point be taken anywhere on a fixed perpendicular to the axis, the perpendicular from it on its polar will pass through a fixed point on the axis. For the intercept made by the perpendicular will (as in Art. 184) be $e^2 x'$, and will therefore be constant when x' is constant.

Ex. 4. Find the lengths of the perpendicular from the centre and from the foci on the polar of $x'y'$.



Ex. 5. Prove CM . PN' = b^2. This is analogous to the theorem that the rectangle under the normal and the central perpendicular on tangent is constant.

Ex. 6. Prove PN'. NN' $= \frac{b^2}{a^2}(a^2 - e^2x'^2)$. When P is on the curve this equation gives us the known expression for the normal $= \frac{bb'}{a}$ (Art. 185).

Ex. 7. Prove FG . F'G' = CM . NN'. When P is on the curve this theorem becomes FG . F'G' = b^2.

198. *To find the polar equation of the ellipse or hyperbola, the focus being the pole.*

The length of the focal radius vector (Art. 186) $= a - ex'$; but x' (being measured from the centre) $= \rho \cos\theta + c$.

Hence
$$\rho = a - e\rho\cos\theta - ec,$$
or
$$\rho = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{b^2}{a}\cdot\frac{1}{1+e\cos\theta}.$$

The double ordinate at the focus is called the *parameter;* its half is found by making $\theta = 90°$ in the equation just given, to be $= \frac{b^2}{a} = a(1-e^2)$. The parameter is commonly denoted by the letter p. Hence the equation is often written
$$\rho = \frac{p}{2}\cdot\frac{1}{1+e\cos\theta}.$$

The parameter is also called the *Latus Rectum.*

Ex. 1. The harmonic mean between the segments of a focal chord is constant, and equal to the semiparameter.

For, if the radius vector FP, when produced backwards through the focus, meet the curve again in P', then FP being $\frac{p}{2}\cdot\frac{1}{1+e\cos\theta}$, FP', which answers to $(\theta + 180°)$, will $= \frac{p}{2}\cdot\frac{1}{1-e\cos\theta}$.

Hence
$$\frac{1}{\text{FP}} + \frac{1}{\text{FP}'} = \frac{4}{p}.$$

Ex. 2. The rectangle under the segments of a focal chord is to the whole chord in a constant ratio.

This is merely another way of stating the result of the last Example; but it may be proved directly by calculating the quantities FP . FP', FP + FP', which are easily seen to be, respectively,

$$\frac{b^4}{a^2}\frac{1}{1-e^2\cos^2\theta}, \text{ and } \frac{2b^2}{a}\frac{1}{1-e^2\cos^2\theta}.$$

Ex. 3. Any focal chord is a third proportional to the transverse axis and the parallel diameter.

For it will be remembered that the length of a semidiameter making an angle θ with the transverse axis is (Art. 164)

$$R^2 = \frac{b^2}{1-e^2\cos^2\theta}.$$

Hence the length of the chord FP + FP' found in the last Example $= \frac{2R^2}{a}$.

Ex. 4. The sum of two focal chords drawn parallel to two conjugate diameters is constant.

For the sum of the squares of two conjugate diameters is constant (Art. 177).

Ex. 5. The sum of the reciprocals of two focal chords at right angles to each other is constant.

199. The equation of the ellipse, referred to the vertex, is

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

or

$$y^2 = \frac{2b^2}{a}x - \frac{b^2}{a^2}x^2 = px - \frac{b^2}{a^2}x^2.$$

Hence, in the ellipse, the square of the ordinate *is less than* the rectangle under the parameter and abscissa.

The equation of the hyperbola is found in like manner,

$$y^2 = px + \frac{b^2}{a^2}x^2.$$

Hence, in the hyperbola, the square of the ordinate *exceeds* the rectangle under the parameter and abscissa.

We shall show, in the next chapter, that in the parabola these quantities are *equal*.

It was from this property that the names *parabola*, *hyperbola*, and *ellipse*, were first given (see Pappus, *Math. Coll.*, Book vii.).

THE ASYMPTOTES.

200. We have hitherto discussed properties common to the ellipse and the hyperbola. There is, however, one class of pro-

perties of the hyperbola which have none corresponding to them in the ellipse, those, namely, depending on the asymptotes, which in the ellipse are imaginary.

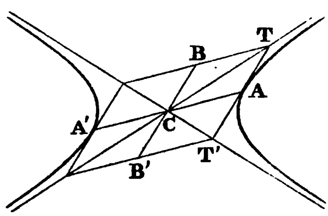
We saw that the equations of the asymptotes were always obtained by putting the highest powers of the variables = 0, the centre being the origin. Thus the equation of the curve, referred to any pair of conjugate diameters, being

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1,$$

that of the asymptotes is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0, \text{ or } \frac{x}{a'} - \frac{y}{b'} = 0, \text{ and } \frac{x}{a'} + \frac{y}{b'} = 0.$$

Hence the asymptotes are parallel to the diagonals of the parallelogram, whose adjacent sides are any pair of conjugate semi-diameters. For, the equation of CT is $\frac{y}{x} = \frac{b'}{a'}$, and must, therefore, coincide with one asymptote, while the equation of AB $\left(\frac{x}{a'} + \frac{y}{b'} = 1\right)$ is parallel to the other (see Art. 170).



Hence, given any two conjugate diameters, we can find the asymptotes; or, given the asymptotes, we can find the diameter conjugate to any given one; for if we draw AO parallel to one asymptote, to meet the other, and produce it equal to itself, we find B, the extremity of the conjugate diameter.

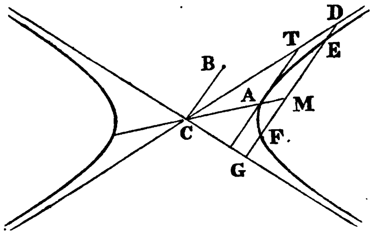
201. *The portion of any tangent intercepted by the asymptotes is bisected at the curve, and is equal to the conjugate diameter.*

This appears at once from the last Article, where we have proved $AT = b' = AT'$; or, directly, taking for axes the diameter through the point and its conjugate, the equation of the asymptotes is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0.$$

Hence, if we take $x = a'$, we have $y = \pm b'$; but the tangent at A being parallel to the conjugate diameter, this value of the ordinate is the intercept on the tangent.

202. *If any line cut an hyperbola, the portions* DE, FG, *intercepted between the curve and its asymptotes, are equal.*



For, if we take for axes a diameter parallel to DG and its conjugate, it appears from the last Article, that the portion DG is bisected by the diameter; so is also the portion EF; hence DE = FG.

The lengths of these lines can immediately be found, for, from the equation of the asymptotes $\left(\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0\right)$, we have

$$y\,(= \text{DM} = \text{MG}) = \pm \frac{b'}{a'}\,x.$$

Again, from the equation of the curve $\left(\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1\right)$,

we have
$$y\,(= \text{EM} = \text{FM}) = \pm\, b'\sqrt{\left(\frac{x^2}{a'^2} - 1\right)}.$$

Hence
$$\text{DE}\,(= \text{FG}) = b'\left\{\frac{x}{a'} - \sqrt{\left(\frac{x^2}{a'^2} - 1\right)}\right\},$$

and
$$\text{DF}\,(= \text{EG}) = b'\left\{\frac{x}{a'} + \sqrt{\left(\frac{x^2}{a'^2} - 1\right)}\right\}.$$

203. From these equations it at once follows, that *the rectangle* DE · DF *is constant, and* $= b'^2$. Hence, the greater DF is, the smaller will DE be. Now it is evident, that the further from the centre we draw DF the greater will it be, and that by taking x sufficiently large, we can make DF $\left[= b'\left\{\frac{x}{a'} + \sqrt{\left(\frac{x^2}{a'^2} - 1\right)}\right\}\right]$ greater than any assigned quantity. Hence, *the further from the centre we draw any line, the less will be the intercept between the curve and its asymptote, and by increasing the distance from the centre, we can make this intercept less than any assigned quantity.*

204. If the asymptotes be taken for axes, the coefficients D and E of the general equation vanish, since the origin is the centre; and the coefficients A and C vanish since the axes meet the curve at infinity (Art. 136, III.): hence the equation reduces to the form
$$xy = k^2.$$

The geometrical meaning of this equation evidently is, that *the area of the parallelogram formed by the co-ordinates is constant.*

205. *The equation being given in the form $xy = k^2$, to form the equation of any chord or of any tangent.*

We have
$$y' = \frac{k^2}{x'}, \text{ and } y'' = \frac{k^2}{x''};$$
therefore,
$$y' - y'' = -\frac{k^2(x' - x'')}{x'x''};$$
the equation of a chord is, therefore,
$$\frac{y - y'}{x - x'} = -\frac{k^2}{x'x''},$$
which may be written (since $k^2 = x'y' = x''y''$)
$$y'x + x''y = k^2 + y'x''.$$
Making $x' = x''$ and $y' = y''$, we find the equation of the tangent,
$$x'y + y'x = 2k^2,$$
or (writing $x'y'$ for k^2)
$$\frac{x}{x'} + \frac{y}{y'} = 2.$$

From this form it appears that the intercepts made on the asymptotes by any tangent $= 2x'$ and $2y'$; their rectangle is, therefore, $4k^2$. Hence, *the triangle which any tangent forms with the asymptotes has a constant area, and is equal to double the area of the parallelogram formed by the co-ordinates.*

Ex. 1. If two fixed points $(x'y', x''y'')$ on a hyperbola be joined to any variable point $(x'''y''')$, the portion which the joining lines intercept on either asymptote is constant.

The equation of one of the joining lines being
$$x'''y + y'x = y'x''' + k^2,$$
the intercept made by it from the origin on the axis of x is found by making $y = 0$ to be $x''' + x'$. Similarly the intercept from the origin made by the other joining line is $x''' + x''$, and the difference between these two $(x' - x'')$ is independent of the position of the point $x'''y'''$.

Ex. 2. Find the co-ordinates of the intersection of the tangents at $x'y'$, $x''y''$.

Solve for x and y from
$$x'y + y'x = 2k^2, \quad x''y + y''x = 2k^2,$$
and we find
$$x = \frac{2k^2(x' - x'')}{x'y'' - y'x''} = \frac{2x'x''}{x' + x''}; \quad y = \frac{2y'y''}{y' + y''}.$$

206. *To express the quantity k^2 in terms of the lengths of the axes of the curve.*

Since the axis bisects the angle between the asymptotes, the co-ordinates of its vertex are found, by putting $x = y$ in the equation $xy = k^2$, to be $x = y = k$.

Hence, if θ be the angle between the axis and the asymptote,

$$a = 2k\cos\theta$$

(since a is the base of an isosceles triangle whose sides $= k$ and base angle $= \theta$), but (Art. 170)

$$\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}};$$

hence

$$k = \frac{\sqrt{(a^2+b^2)}}{2}.$$

And the equation of the curve, referred to its asymptotes, is

$$xy = \frac{a^2+b^2}{4}.$$

207. *The perpendicular from the focus on the asymptote is equal to the conjugate axis b.*

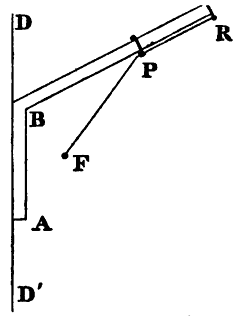
For it is CF $\sin\theta$, but CF $= \sqrt{(a^2+b^2)}$, and $\sin\theta = \dfrac{b}{\sqrt{(a^2+b^2)}}$.

This might also have been deduced as a particular case of the property, that the product of the perpendiculars from the focus on any tangent is constant, and $= b^2$. For the asymptote may be considered as a tangent, whose point of contact is at an infinite distance (Art. 157), and the perpendiculars from the foci on it are evidently equal to each other.

208. *The distance of the focus from any point on the curve is equal to the length of a line drawn through the point parallel to an asymptote to meet the directrix.*

For the distance from the focus is e times the distance from the directrix (Art. 190), and the distance from the directrix is to the length of the parallel line as $\cos\theta\left(=\frac{1}{e}, \text{ Art. } 170\right)$ is to 1.

Hence has been derived a method of describing the hyperbola by continued motion. A ruler ABR, bent at B, slides along the fixed line DD′; a thread of a length = RB is fastened at the two points R and F, while a ring at P keeps the thread always stretched; then as the ruler is moved along, the point P will describe an hyperbola, of which F is a focus, DD′ a directrix, and BR parallel to an asymptote, since PF must always = PB.



CHAPTER XII.

THE PARABOLA.

209. The equation of the second degree, we saw (Art. 136), will represent a parabola, when the first three terms form a perfect square, or when the equation is of the form

$$(ax + by)^2 + Dx + Ey + F = 0.$$

We saw that we could not transform this equation to any finite point so as to make the coefficients of x and y both vanish. The form of the equation, however, points at once to another method of simplifying it.

We know (Art. 27) that the quantity $Dx + Ey + F$ is proportional to the length of the perpendicular from the point (xy) on the right line whose equation is $Dx + Ey + F = 0$; and, in like manner, the quantity $ax + by$ is proportional to the perpendicular on the line $ax + by = 0$.

Hence if we construct the two lines whose equations are

$$ax + by = 0, \qquad Dx + Ey + F = 0,$$

the equation of the curve asserts that the square of the perpendicular from any point of the curve on the first line is in a constant ratio to the perpendicular on the second line.

Now if we transform our axes, and make the line $ax + by$ our new axis of x, and $Dx + Ey + F = 0$ our new axis of y, then our new y will, of course, be proportional to the perpendicular on the line

$ax + by$, and our new x to the perpendicular on $Dx + Ey + F = 0$, and the transformed equation must be of the form

$$y^2 = px.$$

It is evident that our new origin is a point on the curve, and since for every value of x we have two equal and opposite values of y, our new axis of x will be a diameter, and our new axis of y parallel to its ordinates. But the ordinate drawn at the extremity of any diameter is (Art. 145) a tangent to the curve, therefore, the new axis of y is the tangent at the origin. Hence, if we are given the equation of the parabola in the form

$$(ax + by)^2 + Dx + Ey + F = 0,$$

the equation $ax + by = 0$ represents the diameter passing through the origin, and the equation $Dx + Ey + F = 0$ represents the tangent at the point where this diameter meets the curve. And the equation of the curve, referred to a diameter and tangent at its extremity as axes, is of the form

$$y^2 = px.$$

210. Though we have transformed the equation of the parabola into a very simple form, yet our new axes have the inconvenience of not being in general rectangular. We shall prove, however, that it is possible to transform the equation into this form, still retaining the axes rectangular.

If we introduce an arbitrary constant k, the equation

$$(ax + by)^2 + Dx + Ey + F = 0$$

will be found to be equivalent to the equation

$$(ax + by + k)^2 + (D - 2ak)x + (E - 2bk)y + F - k^2 = 0.$$

Hence, as in the last Article,

$$ax + by + k = 0$$

is the equation of a diameter, and

$$(D - 2ak)x + (E - 2bk)y + F - k^2 = 0$$

of the tangent at its extremity. (This confirms our proof (Art. 139) that all the diameters of the parabola are parallel.)

Now, the condition that these two lines should be perpendicular is (Art. 40)

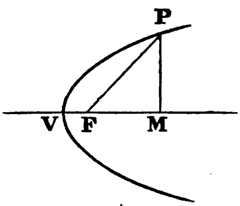
$$a(D - 2ak) + b(E - 2bk) = 0.$$

Hence $$k = \frac{aD + bE}{2(a^2 + b^2)}.$$

Since we get a simple equation to determine the particular value of k, which would make the new axes rectangular, there is *one* diameter whose ordinates cut it perpendicularly, and this diameter is called the *axis* of the curve. And we see, as in the last Article, that if we take for axes, this diameter $ax + by + k$, and the perpendicular tangent $(D - 2ak)x + (E - 2bk)y + F - k^2 = 0$, the transformed equation must be of the form

$$y^2 = px.$$

211. From the equation $y^2 = px$ we can at once perceive the figure of the curve. It must be symmetrical on both sides of the axis of x, since every value for x gives two equal and opposite for y. None of it can lie on the negative side of the origin, since if we make x negative y will be imaginary, and as we give increasing positive values to x, we obtain increasing values for y. Hence the figure of the curve is that here represented.



Although the parabola resembles the hyperbola in having infinite branches, yet there is an important difference between the nature of the infinite branches of the two curves. Those of the hyperbola, we saw, tend ultimately to coincide with two diverging right lines; but this is not true for the parabola, since, if we seek the points where any right line $(x = ky + l)$ meets the parabola $(y^2 = px)$, we obtain the quadratic

$$y^2 - pky - pl = 0,$$

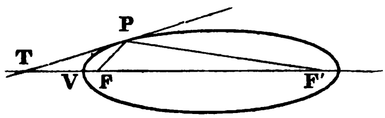
whose roots can never be infinite as long as k and l are finite.

There is no finite right line which meets the parabola in two coincident points at infinity; for any diameter $(y = m)$ which meets the curve once at infinity (Art. 140) meets it once also in the point $x = \frac{m^2}{p}$; and although this value increases as m increases, yet it will never become infinite as long as m is finite.

212. The figure of the parabola may be more clearly conceived from the following theorem:

If we suppose one vertex and focus of an ellipse given, while its axis major increases without limit, the curve will ultimately become a parabola.

The equation of the ellipse, referred to its vertex, is (Art. 199)



$$y^2 = \frac{2b^2}{a}x - \frac{b^2}{a^2}x^2.$$

We wish to express b in terms of the distance VF $(= m)$, which we suppose fixed. We have $m = a - \sqrt{(a^2 - b^2)}$ (Art. 186), whence $b^2 = 2am - m^2$, and the equation becomes

$$y^2 = \left(4m - \frac{2m^2}{a}\right)x - \left(\frac{2m}{a} - \frac{m^2}{a^2}\right)x^2.$$

Now, if we suppose a to become infinite, all but the first term of the right-hand side of the equation will vanish, and the equation becomes

$$y^2 = 4mx,$$

the equation of a parabola.

Hence we see that the focus and vertex of an ellipse being given, while the axis major is indefinitely increased, the parameter $(= \frac{2b^2}{a}$, Art. 198) will remain finite, and $= 4m$.

Hence if the equation of the parabola be given in the form $y^2 = px$, the quantity p is called the principal parameter.

A parabola may also be considered as an ellipse whose eccentricity is equal to 1. For $e^2 = 1 - \frac{b^2}{a^2}$. Now we saw that $\frac{b^2}{a^2}$, which is the coefficient of x^2 in the preceding equation, vanished as we supposed a increased according to the prescribed conditions; hence e^2 becomes finally $= 1$.

*213. *To find the parameter of the parabola*

$$(ax + by)^2 + Dx + Ey + F = 0.$$

We have seen (Art. 210) that the equation may be written in the form

$$(ax + by + k)^2 + (D - 2ak)x + (E - 2bk)y + F - k^2 = 0;$$

which, when k has the value found in that article, is

$$(ax+by+k)^2 + \frac{bD - aE}{a^2+b^2}(bx - ay + F') = 0,$$

where we have written for shortness,

$$F' = \frac{(a^2+b^2)(F-k^2)}{bD - aE}.$$

Now, let Y and X denote the perpendiculars from any point on the lines $ax + by + k = 0$, and $bx - ay + F' = 0$, and this equation becomes

$$(a^2+b^2)\,Y^2 = \frac{aE - bD}{\sqrt{(a^2+b^2)}}\,X,$$

and

$$p = \frac{aE - bD}{(a^2+b^2)^{\frac{3}{2}}}.$$

Ex. 1. Change to the form $y^2 = px$ the equation

$$9x^2 + 24xy + 16y^2 + 22x + 46y + 9 = 0.$$

Here $k = 5$ and the equation may be written

$$(3x + 4y + 5)^2 = 2(4x - 3y + 8);$$

or if the distances of any point from $3x + 4y + 5$ and $4x - 3y + 8$ be Y and X,

$$Y^2 = \tfrac{2}{5}X.$$

Ex. 2. Find the parameter of the parabola

$$\frac{x^2}{a^2} - \frac{2xy}{ab} + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0. \qquad \textit{Ans.}\ \frac{4a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}.$$

This value may also be deduced directly by the help of the following theorems, which will be proved afterwards:—"The focus of a parabola is the foot of a perpendicular let fall from the intersection of two tangents which cut at right angles on their chord of contact;" and "The parameter of a conic is found by dividing four times the rectangle under the segments of a focal chord, by the length of that chord" (Art. 198).

Ex. 3. If a and b be the lengths of two tangents to a parabola which intersect at right angles, and m one quarter of the parameter, prove

$$\frac{a^{\frac{2}{3}}}{b^{\frac{4}{3}}} + \frac{b^{\frac{2}{3}}}{a^{\frac{4}{3}}} = \frac{1}{m^{\frac{2}{3}}}.$$

*214. *To find the parameter of* $(ax+by)^2 + Dx + Ey + F = 0$, *the axes being oblique.*

We proceed as in Art. 210, but the axes being oblique, we must use the condition (Art. 41) that two lines should cut at right angles, and the equation which determines k becomes

$$a(D - 2ak) + b(E - 2bk) = \{a(E - 2bk) + b(D - 2ak)\}\cos\omega$$

$$k = \frac{aD + bE - (aE + bD)\cos\omega}{2(a^2 + b^2 - 2ab\cos\omega)}.$$

The transformed equation then is

$$(ax+by+k)^2+\frac{bD-aE}{a^2+b^2-2ab\cos\omega}\{(b-a\cos\omega)x-(a-b\cos\omega)y+F'\}=0,$$

where
$$F'=\frac{(F-k^2)(a^2+b^2-2ab\cos\omega)}{(bD-aE)}.$$

Now, let

$$Y=\frac{(ax+by+k)\sin\omega}{\sqrt{(a^2+b^2-2ab\cos\omega)}},\quad X=\frac{(b-a\cos\omega)x-(a-b\cos\omega)y+F'}{\sqrt{(a^2+b^2-2ab\cos\omega)}},$$

and we get
$$p=\frac{(aE-bD)\sin^2\omega}{(a^2+b^2-2ab\cos\omega)^{\frac{3}{2}}}.$$

Ex. Find the parameter of the parabola

$$\frac{x^2}{a^2}-\frac{2xy}{ab}+\frac{y^2}{b^2}-\frac{2x}{a}-\frac{2y}{b}+1=0.$$

Ans. $p=\dfrac{4a^2b^2\sin^2\omega}{(a^2+b^2+2ab\cos\omega)^{\frac{3}{2}}}.$

THE TANGENT.

215. The equation of any chord of the parabola can be easily obtained. For, since $y'^2=px'$ and $y''^2=px''$, we have

$$y'^2-y''^2=p(x'-x''),\text{ and }\frac{y'-y''}{x'-x''}=\frac{p}{y'+y''},$$

and the equation of the chord is

$$\frac{y-y'}{x-x'}=\frac{p}{y'+y''},$$

or
$$(y'+y'')y-px-y'y''=0.$$

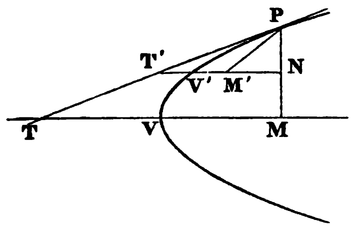
The equation of the tangent is found from this, by supposing $y'=y''$, or (remembering that $y'^2=px'$) is $2y'y=p(x+x')$.

If we seek the point where the tangent meets the axis, we obtain $x=-x'$, or TM (which is called the *subtangent*) is bisected at the vertex.

We saw that if the oblique axes were any diameter and a tangent through its vertex, the equation of the parabola was still

$$y^2=p'x.$$

The equations of the chord and tangent remain the same, and it will be equally true that the subtangent is = twice the abscissa.



This Article enables us, therefore, to draw a tangent at any point on the parabola, since we have only to take TV = VM and join PT; or again, having found this tangent, to draw an ordinate from P to any other diameter, since we have only to take V'M' = T'V', and join PM'.

216. It follows from Art. 144 (or may be proved as in Art. 172) that the equation of the polar of any point $x'y'$ is similar in form to that of the tangent, and is, therefore,

$$2y'y = p(x + x').$$

If we seek the point where this polar meets the axis of x, we get

$$x = -x'.$$

Hence we derive a theorem, which will be useful hereafter, *that the intercept which the polars of any two points cut off on the axis is equal to the intercept between perpendiculars from those points on that axis;* each of these quantities being equal to $(x' - x'')$.

DIAMETERS.

217. We have said, that if we take for axes any diameter and the tangent at its extremity, the equation will be of the form

$$y^2 = p'x.$$

We shall prove this again by actual transformation of the equation referred to rectangular axes ($y^2 = px$), because it is desirable to express the new p' in terms of the old p.

If we transform the equation $y^2 = px$ to parallel axes through any point $(x'y')$ on the curve, writing $x + x'$ and $y + y'$ for x and y, the equation becomes

$$y^2 + 2yy' = px.$$

Now if, preserving our axis of x, we take a new axis of y, inclined to x at an angle θ, then our old y = PN = PM' $\sin\theta$, and our old x = V'M' + PM' $\cos\theta$. (See figure, above.)

We, therefore, substitute $y \sin\theta$ for y, and $x + y\cos\theta$ for x, and our equation becomes

$$y^2 \sin^2\theta + 2y'y \sin\theta = px + py\cos\theta.$$

In order that this should reduce to the form $y^2 = px$, we must have

$$2y' \sin\theta = p\cos\theta, \text{ or } \tan\theta = \frac{p}{2y'}.$$

Now this is the very angle which the tangent makes with the angle of x, as we see from the equation

$$2y'y = p(x + x').$$

This equation, therefore, referred to a diameter and tangent, will take the form

$$y^2 = \frac{p}{\sin^2\theta}x, \text{ or } y^2 = p'x.$$

The quantity p' is called the parameter corresponding to the diameter V′M′, and we see that *the parameter of any diameter is to the principal parameter* (p), *inversely as the square of the sine of the angle which its ordinates make with the axis*, since $p' = \frac{p}{\sin^2\theta}$.

We can express the parameter of any diameter in terms of the co-ordinates of its vertex, from the equation $\tan\theta = \frac{p}{2y'}$; hence,

$$\sin\theta = \frac{p}{\sqrt{(p^2 + 4y'^2)}} = \sqrt{\left(\frac{p}{p + 4x'}\right)};$$

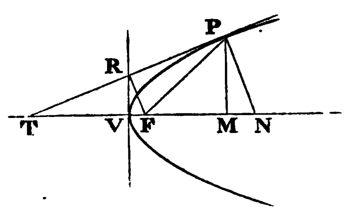
hence

$$p' = p + 4x.$$

THE NORMAL.

218. The equation of a line through ($x'y'$) perpendicular to the tangent $2yy' = p(x + x')$ is

$$p(y - y') + 2y'(x - x') = 0.$$



If we seek the intercept on the axis of x we have

$$x\,(= \text{VN}) = x' + \frac{p}{2};$$

and, since VM $= x$, we must have

$$\text{MN (the \textit{subnormal}, Art. 185)} = \frac{p}{2}.$$

Hence *in the parabola the subnormal is constant, and equal to the semiparameter.* The normal itself

$$= \sqrt{(\text{PM}^2 + \text{MN}^2)} = \sqrt{\left(y'^2 + \frac{p^2}{4}\right)} = \sqrt{\left\{p\left(x' + \frac{p}{4}\right)\right\}}.$$

THE FOCUS.

219. A point situated on the axis of a parabola, at a distance from the vertex equal to one-fourth of the principal parameter, is called the *focus* of the curve. This is the point which, Art. 214, has led us to expect to find analogous to the focus of an ellipse; and we shall show, in the present section, that a parabola may in every respect be considered as an ellipse, having one of its foci at this distance, and the other at infinity. To avoid fractions we shall often, in the following Articles, use the abbreviation $m = \frac{p}{4}$.

To find the distance of any point on the curve from the focus.

The co-ordinates of the focus being $(m, 0)$, the square of its distance from any point is

$$(x' - m)^2 + y'^2 = x'^2 - 2mx' + m^2 + 4mx' = (x' + m)^2.$$

Hence the distance of any point from the focus $= x' + m$.

This enables us to express more simply the result of Art. 217, and to say that *the parameter of any diameter is four times the distance of its extremity from the focus.*

220. The polar of the focus of a parabola is called the *directrix*, as in the ellipse and hyperbola.

Since the distance of the focus from the vertex $= m$, its polar is (Art. 216) a line perpendicular to the axis at the same distance on the other side of the vertex. The distance of any point from the directrix must, therefore, $= x' + m$.

Hence, by the last Article, *the distance of any point on the curve from the directrix is equal to its distance from the focus.*

We saw (Art. 190) that in the ellipse and hyperbola, the distance from the focus is to the distance from the directrix in the constant ratio e to 1. We see, now, that this is true for the parabola also, since in the parabola $e = 1$ (Art. 212).

The method given for mechanically describing an hyperbola, Art. 208, can be adapted to the mechanical description of the parabola, by simply making the angle ABR a right angle.

221. *The point where any tangent cuts the axis, and its point of contact, are equally distant from the focus.*

For, the distance from the vertex of the point where the tangent cuts the axis $= x'$ (Art. 215), its distance from the focus is, therefore, $x' + m$.

222. *Any tangent makes equal angles with the axis and with the focal radius vector.*

This is evident from inspection of the isosceles triangle, which, in the last Article, we proved was formed by the axis, the focal radius vector, and the tangent.

This is only an extension of the property of the ellipse (Art. 192), that the angle TPF = T′PF′; for, if we suppose the focus F′ to go off to infinity, the line PF′ will become parallel to the axis, and TPF = PTF. (See figure, p. 178.)

Hence the tangent at the extremity of the focal ordinate cuts the axis at an angle of 45°.

223. *To find the length of the perpendicular from the focus on the tangent.*

The perpendicular from the point $(m, 0)$ on the tangent $\{yy' = 2m(x + x')\}$ is

$$= \frac{2m(x' + m)}{\sqrt{(y'^2 + 4m^2)}} = \frac{2m(x' + m)}{\sqrt{(4mx' + 4m^2)}} = \sqrt{\{m(x' + m)\}}.$$

Hence (see fig. p. 182) FR is a mean proportional between FV and FP.

It appears, also, from this expression, and from Art. 218, that FR is half the normal, as we might have inferred geometrically from the fact that TF = FN.

224. *To express the perpendicular from the focus in terms of the angles which it makes with the axis.*

We have

$$\cos a = \sin \text{FTR} = (\text{Art. } 217) \sqrt{\left(\frac{m}{x' + m}\right)}.$$

Therefore (Art. 223),

$$\text{FR} = \sqrt{\{m(x' + m)\}} = \frac{m}{\cos a}.$$

The equation of the tangent, *the focus being the origin*, can, therefore, be expressed

$$x \cos a + y \sin a + \frac{m}{\cos a} = 0,$$

and hence we can express the perpendicular from any other point in terms of the angle it makes.

225. *The locus of the extremity of the perpendicular from the focus on the tangent is a right line.*

For, taking the focus for pole, we have at once the polar equation

$$\rho = \frac{m}{\cos a}, \quad \rho \cos a = m;$$

which obviously represents the tangent at the vertex.

Conversely, if from any point F we draw FR a radius vector to a right line VR, and draw PR perpendicular to it, the line PR will always touch a parabola having F for its focus.

We shall show hereafter how to solve generally questions of this class, where one condition less than is sufficient to determine a line is given, and it is required to find its *envelope*, that is to say, the curve which it always touches.

We leave, as a useful exercise to the reader, the investigation of the locus of the foot of the perpendicular by ordinary rectangular co-ordinates.

226. *To find the locus of the intersection of tangents which cut at right angles to each other.*

The equation of any tangent being (Art. 224)

$$x \cos^2 a + y \sin a \cos a + m = 0;$$

the equation of a tangent perpendicular to this (that is, whose perpendicular makes an angle = $90° + a$ with the axis) is found by substituting $\cos a$ for $\sin a$, and $-\sin a$ for $\cos a$, or

$$x \sin^2 a - y \sin a \cos a + m = 0.$$

a is eliminated by simply adding the equations, and we get

$$x + 2m = 0,$$

the equation of *the directrix*, since the distance of focus from directrix = $2m$.

227. *The angle between any two tangents is half the angle between the focal radii vectores to their points of contact.*

For, from the isosceles PFT, the angle PTF which the tangent makes with the axis is half the angle PFN, which the focal radius makes with it. Now, the angle between any two tangents is equal to the difference of the angles they make with the axis, and the angle between two focal radii is equal to the difference of the angles which *they* make with the axis.

The theorem of the last Article follows as a particular case of the present theorem; for if two tangents make with each other an angle of 90°, the focal radii must make with each other an angle of 180°, therefore, the two tangents must be drawn at the extremities of a chord through the focus, and, therefore, from the definition of the directrix, must meet on the directrix.

228. *The line joining the focus to the intersection of two tangents bisects the angle which their points of contact subtend at the focus.*

The equations of two tangents being

$$x\cos^2\alpha + y\sin\alpha\cos\alpha + m = 0, \quad x\cos^2\beta + y\sin\beta\cos\beta + m = 0;$$

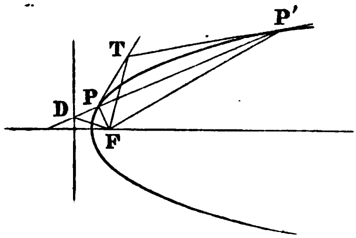
subtracting them, we find for the line joining their intersection to the focus, $x\sin(\alpha+\beta) - y\cos(\alpha+\beta) = 0.$

This is the equation of a line making the angle $\alpha + \beta$ with the axis of x. But since α and β are the angles made with the axis by the perpendiculars on the tangent, we have VFP $= 2\alpha$ and VFP′ $= 2\beta$; therefore the line making an angle with the axis $= \alpha + \beta$ must bisect the angle PFP′. This theorem may also be proved by calculating, as in Art. 196, the angle $(\theta - \theta')$ subtended at the focus by the tangent to a parabola from the point xy; when it will be found that $\cos(\theta - \theta') = \frac{x+m}{\rho}$, a value which, being independent of the co-ordinates of the point of contact, will be the same for each of the two tangents which can be drawn through xy. (See O'Brien's *Co-ordinate Geometry*, p. 156.)

Cor. 1. If we take the case where the angle PFP′ = 180°, then PP′ passes through the focus; the tangents TP, TP′ will intersect on the directrix, and the angle TFP = 90°. (See Art. 197.) This may also be proved directly by forming the equations of the polar of any point $(-m, y')$ on the directrix, and also the equation of the line joining that point to the focus. These two equations are

$$y'y = 2m(x-m), \quad 2m(y-y') + y'(x-m) = 0,$$

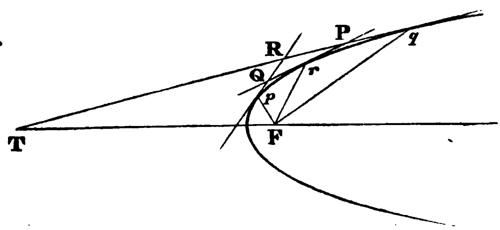
which obviously represent two right lines at right angles to each other.



Cor. 2. If any chord PP' cut the directrix in D, then FD is the external bisector of the angle PFP'. This is proved as at p. 168.

Cor. 3. If any variable tangent to the parabola meet two fixed tangents, the angle subtended at the focus by the portion of the variable tangent intercepted between the fixed tangents, is the supplement of the angle between the fixed tangents.

For the angle QRT is half pFq (Art. 227), and, by the present Article, PFQ is obviously also half pFq, therefore, PFQ is = QRT, or is the supplement of PRQ.

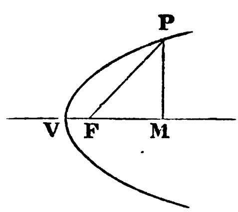


Cor. 4. *The circle circumscribing the triangle formed by any three tangents to a parabola will pass through the focus.* For the circle described through PRQ must pass through F, since the angle contained in the segment PFQ will be the supplement of that contained in PRQ.

229. *To find the polar equation of the parabola, the focus being the pole.*

We proved (Art. 219) that the focal radius

$$= x' + m = VM + m = FM + 2m = \rho \cos\theta + 2m.$$



Hence
$$\rho = \frac{2m}{1 - \cos\theta}.$$

This is exactly what the equation of Art. 198 becomes, if we suppose $e = 1$ (Art. 212). The properties proved in the Examples to Art. 198 are equally true of the parabola.

In this equation θ is supposed to be measured from the side

FM; if we suppose it measured from the side FV, the equation becomes

$$\rho = \frac{2m}{1 + \cos\theta}.$$

This equation may be written

$$\rho \cos^2 \tfrac{1}{2}\theta = m,$$

or $$\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta = (m)^{\frac{1}{2}},$$

and is, therefore, one of a class of equations,

$$\rho^n \cos n\theta = a^n,$$

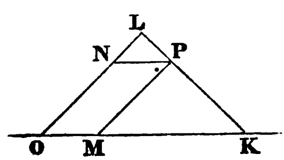
some of whose properties we shall mention hereafter.

CHAPTER XIII.

EXAMPLES AND MISCELLANEOUS PROPERTIES OF THE CONIC SECTIONS.

230. THE method of applying algebra to problems relating to conic sections is essentially the same as that employed in the case of the right line and circle, and will present no difficulty to any reader who has carefully worked out the Examples given in Chapters III. and VII. We, therefore, only think it necessary to select a few out of the great multitude of examples which lead to loci of the second order, and we shall then add some properties of conic sections, which it was not found convenient to insert in the preceding chapters.

Ex. 1. A line of constant length moves about in the legs of a given angle: to find the locus described by a fixed point on it.



Denoting PL by n, PK by m, and LK by l, we have, by similar triangles,

$$OL = \frac{ly}{m}, \text{ and } OK = \frac{lx}{n}.$$

But since $$LK^2 = OL^2 + OK^2 - 2OK \cdot OL \cos\omega,$$

we have $$l^2 = \frac{l^2y^2}{m^2} + \frac{l^2x^2}{n^2} - \frac{2l^2xy\cos\omega}{mn},$$

or $$\frac{x^2}{n^2} + \frac{y^2}{m^2} - \frac{2xy\cos\omega}{mn} = 1;$$

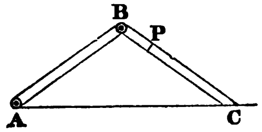
the equation of an *ellipse* having the point O for its centre, since $B^2 - 4AC$ is here negative, being $= -\frac{4}{m^2n^2}\sin^2\theta$.

Ex. 2. If P be a fixed point, and LK any right line drawn through it, to find the locus of intersection of the parallels to OK, OL, through the points L and K.

Ex. 3. Or of perpendiculars erected to OK, OL, through the same points.

Ex. 4. If a point Q be taken on LK so that QK = PL, to find its locus.

Ex. 5. Two equal rulers, AB, BC, are connected by a pivot at B; the extremity A is fixed, while the extremity C is made to traverse the right line AC; find the locus described by any fixed point P on BC.



Ex. 6. Given base and difference of base angles of a triangle: to find the locus of vertex.

We may proceed exactly as at page 85, where the *sum* of the base angles is given. The locus will be found to be an equilateral hyperbola, of which the base is a diameter. The difference of base angles being given, it is easy to see that the internal and external bisectors of the vertical angle must be parallel to fixed lines, and these lines will be parallel to the asymptotes of the locus. Conversely, if we consider the triangle whose base is any diameter of an equilateral hyperbola, and whose vertex is on the curve, the sides are parallel to conjugate diameters (Art. 183); but conjugate diameters of an equilateral hyperbola make equal angles with the asymptotes (Art. 178).

Ex. 7. Given base and the product of the tangents of the base angles of a triangle: find the locus of vertex.

It will be a conic section, of which the extremities of the base are vertices. This is the converse of Art. 174.

Ex. 8. Given base and the product of the tangents of the halves of the base angles: find the locus of vertex.

Expressing the tangents of the half angles in terms of the sides, it will be found that the sum of sides is given: and, therefore, that the locus is an ellipse, of which the extremities of the base are the foci.

Ex. 9. Given base and sum of sides of a triangle: find the locus of the centre of the inscribed circle.

It may be immediately inferred, from the last two examples, that the locus is an ellipse, whose vertices are the extremities of the given base.

Ex. 10. Given the vertical angle of a triangle in magnitude and position, and also the area: find the locus of a point dividing the base in a given ratio.

Ex. 11. Given base of a triangle, and that one base angle is double the other; find locus of vertex.

Ex. 12. Trisect a given arch of a circle.

Ans. The point of trisection is determined as the intersection of the given arch with a given hyperbola.

Ex. 13. Given base and area of a triangle; find the locus of the intersection of perpendiculars.

Ex. 14. Find the locus of the centre of a circle which touches two others; or which touches a given circle and a given right line.

Ex. 15. Given the base of a triangle, and the length of the intercept made by the sides on a given line; find the locus of vertex.

Ex. 16. Two vertices of a *given* triangle move along fixed right lines; find the locus of the third.

Ex. 17. Two vertices of a triangle move along fixed right lines, and the sides pass through fixed points; find the locus of the third vertex.

Ex. 18. Find the locus of the centre of a circle which makes given intercepts on two given lines.

Ex. 19. A triangle ABC circumscribes a given circle; the angle at C is given, and B moves along a fixed line; find the locus of A.

Let us use polar co-ordinates, the centre O being the pole, and the angles being measured from the perpendicular on the fixed line; let the co-ordinates of A, B, be ρ, θ; ρ', θ'. Then we have $\rho' \cos\theta' = p$. But it is easy to see that the angle AOB is given ($=\alpha$). And since the perpendicular of the triangle AOB is given, we have

$$r = \frac{\rho\rho' \sin\alpha}{\sqrt{(\rho^2 + \rho'^2 - 2\rho\rho' \cos\alpha)}}.$$

But $\theta + \theta' = \alpha$; therefore the polar equation of the locus is

$$r^2 = \frac{p^2\rho^2 \sin^2\alpha}{\rho^2 \cos^2(\alpha - \theta) + p^2 - 2p\rho \cos\alpha \cos(\alpha - \theta)},$$

which represents a conic.

Ex. 20. Given two conic sections, to find the locus of the pole, with respect to one, of any tangent to the other.

Let their equations be $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

the polar of any point, with regard to the second, is (Art. 144)

$$(2Ax' + By' + D)\,x + (2Cy' + Bx' + E)\,y + Dx' + Ey' + 2F = 0.$$

But the condition that this should touch the first is (p. 150)

$$a^2(2Ax' + By' + D)^2 + b^2(2Cy' + Bx' + E)^2 = (Dx' + Ey' + 2F)^2.$$

This condition, which must be satisfied by the point $(x'y')$, is the equation of its locus, and is plainly of the second degree.

231. We give in this Article some examples on the focal properties of conics.

Ex. 1. The distance of any point on a conic from the focus is equal to the whole length of the ordinate at that point, produced to meet the tangent at the extremity of the focal ordinate.

Ex. 2. If from the focus a line be drawn making a given angle with any tangent, find the locus of the point where it meets it.

Ex. 3. To find the locus of the pole of a fixed line with regard to a series of concentric and confocal conic sections.

We know that the pole of any line $\left(\frac{x}{m} + \frac{y}{n} = 1\right)$, with regard to the conic $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$, is found from the equations $mx = a^2$ and $ny = b^2$ (Art. 172).

Now, if the foci of the conic are given, $a^2 - b^2 = c^2$ is given; hence, the locus of the pole of the fixed line is

$$mx - ny = c^2,$$

the equation of a right line perpendicular to the given line.

If the given line touch one of the conics, its pole will be the point of contact (Art. 144). Hence, given two confocal conics, if we draw any tangent to one and tangents to the second where this line meets it, these tangents will intersect on the normal to the first conic.

Ex. 4. The focus being the pole, prove that the polar equation of the tangent, at the point whose angular co-ordinate is α, is $\frac{p}{2\rho} = e\cos\theta + \cos(\theta - \alpha)$.

This expression is due to Mr. Davies (*Philosophical Magazine* for 1842, p. 192, cited by Walton, Examples, p. 368).

Ex. 5. Prove that the polar equation of the chord through points whose angular co-ordinates are $\alpha + \beta$, $\alpha - \beta$, is

$$\frac{p}{2\rho} = e\cos\theta + \sec\beta\cos(\theta - \alpha).$$

This expression is due to Mr. Frost (*Cambridge and Dublin Math. Journal*, i. 68, cited by Walton, Examples, p. 375).

These equations may be conveniently used in investigating theorems concerning angles subtended at the focus. Still simpler methods, however, of obtaining these will be given in Chapter XIV.

Ex. 6. If a chord PP′ of a conic pass through a fixed point O, then $\tan\frac{1}{2}\text{PFO}\,.\,\tan\frac{1}{2}\text{P}'\text{FO}$ is constant.

The reader will find an investigation of this theorem by the help of the equation of the last Example (Walton's Examples, p. 377). I insert here the geometrical proof given by Mr. MacCullagh, to whom, I believe, the theorem is due. Imagine a point O taken anywhere on PP′ (see figure, p. 187), and let the distance FO be e' times the distance of O from the directrix; then since the distances of P and O from the directrix are proportional to PD and OD, we have

$$\frac{\text{FP}}{\text{PD}} \div \frac{\text{FO}}{\text{OD}} = \frac{e}{e'}, \quad \text{or} \quad \frac{\sin \text{PDF}}{\sin \text{PFD}} \div \frac{\sin \text{ODF}}{\sin \text{OFD}} = \frac{e}{e'}.$$

Hence (Art. 197),

$$\frac{\cos \text{OFT}}{\cos \text{PFT}} = \frac{e}{e'};$$

or, since (Art. 196) PFT is half the sum, and OFT half the difference, of PFO and P′FO

$$\tan\tfrac{1}{2}\text{PFO}\,.\,\tan\tfrac{1}{2}\text{P}'\text{FO} = \frac{e - e'}{e + e'}.$$

It is obvious that the product of these tangents remains constant if O be not fixed, but be anywhere on a conic having the same focus and directrix as the given conic.

Ex. 7. If normals be drawn at the extremities of any focal chord, a line drawn through their intersection parallel to the axis will bisect the chord.

Take any point on the directrix whose co-ordinates are $x = \frac{a^2}{c}$, $y = \beta$, then the equation of the polar of that point, which passes through the focus, will be $\frac{x}{c} + \frac{\beta y}{b^2} = 1$.

Substituting for x from this equation in the equation of the curve, the ordinates of the points where this line meets the curve are given by the equation

$$(b^2 + e^2\beta^2)y^2 - 2b^2e^2\beta y - \frac{b^6}{a^2} = 0.$$

Hence, if y', y'', be the ordinates of the point of intersection, we have

$$y' + y'' = \frac{2b^2e^2\beta}{b^2 + e^2\beta^2}, \quad y'y'' = \frac{-b^6}{a^2(b^2 + e^2\beta^2)},$$

but (Art. 185, Ex. 4)

$$y = \frac{b^2 - a^2}{b^4} y'y''\beta = \frac{b^2e^2\beta}{b^2 + e^2\beta^2} = \frac{y' + y''}{2}.$$

It may be found, in like manner, that the abscissæ of the intersection of the chord with the curve are determined by the equation

$$(b^2 + e^2\beta^2)x^2 - 2b^2cx + c^2(b^2 - \beta^2) = 0,$$

whence

$$x' + x'' = \frac{2b^2c}{b^2 + e^2\beta^2}, \quad x'x'' = \frac{c^2(b^2 - \beta^2)}{b^2 + e^2\beta^2},$$

and the abscissa of the intersection of normals is

$$x = \frac{c^3(b^2 - \beta^2)}{a^2(b^2 + e^2\beta^2)}.$$

Ex. 8. If a chord pass through a focus, the line joining the intersection of tangents at its extremities to the intersection of the corresponding normals will pass through the other focus.

The equation of the joining line is $c\beta(x + c) = (a^2 + c^2)y$.

Ex. 9. Find the locus of the intersection of normals at the extremities of a focal chord.

Solving for β^2 from $x = \dfrac{c^3(b^2 - \beta^2)}{a^2(b^2 + e^2\beta^2)}$, we have

$$\beta^2 = \frac{b^2(c^3 - a^2x)}{c^2(c + x)}, \quad b^2 + e^2\beta^2 = \frac{b^2c(a^2 + c^2)}{a^2(c + x)}.$$

But since $y = \dfrac{b^2e^2\beta}{b^2 + e^2\beta^2}$ we have $\beta = \dfrac{(a^2 + c^2)y}{c(c + x)}$.

Hence

$$\frac{(a^2 + c^2)^2y^2}{c^2(c + x)^2} = \frac{b^2(c^3 - a^2x)}{c^2(c + x)},$$

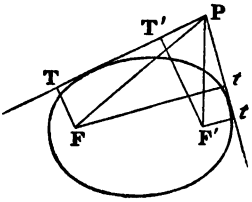
and the locus is the ellipse

$$(a^2 + c^2)^2y^2 = b^2(c + x)(c^3 - a^2x),$$

or

$$(a^2 + c^2)^2y^2 + a^2b^2x^2 + b^4cx = b^2c^4.$$

Ex. 10. If θ be the angle between the tangents to an ellipse from any point P; and if ρ, ρ' be the distances of that point from the focus, prove that $\cos\theta = \dfrac{\rho^2 + \rho'^2 - 4a^2}{2\rho\rho'}$.



For (Art. 189)

$$\sin TPF . \sin tPF = \frac{FT . FT'}{PF . PF'} = \frac{b^2}{\rho\rho'}.$$

But $\cos FPF' - \cos TPt = 2 \sin TPF . \sin tPF$;

And $2\rho\rho' \cos FPF' = \rho^2 + \rho'^2 - 4c^2$.

232. We give in this Article some examples on the parabola. The reader will have no difficulty in distinguishing those of the examples of the last Article, the proofs of which apply equally to the parabola.

Ex. 1. Find the co-ordinates of the intersection of the two tangents at the points $x'y'$, $x''y''$, to the parabola $y^2 = px$.

Ans. $y = \dfrac{y' + y''}{2}$, $x = \dfrac{y'y''}{p}$.

Ex. 2. The three perpendiculars of the triangle formed by three tangents intersect on the directrix (Steiner, Gergonne, *Annales*, xix. 59, Walton, p. 119).

The equation of one of those perpendiculars is (Art. 42)

$$\frac{y'y''' - y'y''}{p}\left(x - \frac{y''y'''}{p}\right) + \frac{y''' - y''}{2}\left(y - \frac{y'' + y'''}{2}\right) = 0;$$

which, after dividing by $y''' - y''$, may be written

$$y'\left(x + \frac{p}{4}\right) - \frac{y'y''y'''}{p} + \frac{py}{2} - \frac{p(y' + y'' + y''')}{4} = 0.$$

The symmetry of the equation shows that the three perpendiculars intersect on the directrix at a height

$$y = \frac{2y'y''y'''}{p^2} + \frac{y' + y'' + y'''}{2}.$$

Ex. 3. The area of the triangle formed by three tangents is half that of the triangle formed by joining their points of contact (Gregory, *Cambridge Journal*, ii. 16, Walton, p. 137).

Substituting the co-ordinates of the vertices of the triangles in the expression of Art. 31, we find for the latter area, $\dfrac{1}{2p}(y' - y'')(y'' - y''')(y''' - y')$; and for the former area half this quantity.

Ex. 4. Find an expression for the radius of the circle circumscribing a triangle inscribed in a parabola.

The radius of the circle circumscribing a triangle, the lengths of whose sides are d, e, f, and whose area $= \Sigma$ is easily proved to be $\dfrac{def}{4\Sigma}$. But if d be the length of the chord joining the points $x''y''$, $x'''y'''$, and θ' the angle which this chord makes with the axis, it is obvious that $d\sin\theta' = y'' - y'''$. Using, then, the expression for the area found in the last Example, we have $R = \dfrac{p}{2\sin\theta'\sin\theta''\sin\theta'''}$. We might express the radius, also, in terms of the focal chords parallel to the sides of the triangle. For (Art. 198, Ex. 2) the length of a chord making an angle θ with the axis is $c = \dfrac{p}{\sin^2\theta}$. Hence $R^2 = \dfrac{c'c''c'''}{4p}$.

It follows, from Art. 217, that c', c'', c''' are the parameters of the diameters which bisect the sides of the triangle.

Ex. 5. Express the radius of the circle circumscribing the triangle formed by three tangents to a parabola in terms of the angles which they make with the axis.

Ans. $R = \dfrac{p}{2\sin\theta'\sin\theta''\sin\theta'''}$; or $R^2 = \dfrac{p'p''p'''}{4p}$, where p', p'', p''' are the parameters of the diameters through the points of contact of the tangents (see Art. 217).

Ex. 6. Find the angle contained by the two tangents through the point $x'y'$ to the parabola $y^2 = 4mx$.

The equation of the pair of tangents is (as in Art. 150) found to be

$$(y'^2 - 4mx')(y^2 - 4mx) = \{yy' - 2m(x + x')\}^2.$$

A parallel pair of lines through the origin is

$$x'y^2 - y'xy + mx^2 = 0.$$

The angle contained by which is (Art. 70)

$$\tan\phi = \frac{\sqrt{(y'^2 - 4mx')}}{x' + m}.$$

Ex. 7. Find the locus of tangents to a parabola which cut at a given angle.

Ans. The hyperbola $y^2 - 4mx = (x+m)^2\tan^2\phi$, or $y^2 + (x-m)^2 = (x+m)^2\sec^2\phi$. From the latter form of the equation it is evident (see Art. 190) that the hyperbola has the same focus and directrix as the parabola, and that its eccentricity $= \sec\phi$.

Ex. 8. Find the locus of the foot of the perpendicular from the focus of a parabola on the normal.

The length of the perpendicular from $(m, 0)$ on $2m(y - y') + y'(x - x') = 0$ is

$$\frac{y'(x' + m)}{\sqrt{(y'^2 + 4m^2)}} = \sqrt{\{x'(x' + m)\}}.$$

But if θ be the angle made with the axis by the perpendicular (Art. 217)

$$\sin\theta = \sqrt{\left(\frac{m}{x' + m}\right)}, \quad \cos\theta = \sqrt{\left(\frac{x'}{x' + m}\right)}.$$

Hence the polar equation of the locus is

$$\rho = \frac{m\cos\theta}{\sin^2\theta}, \text{ or } y^2 = mx.$$

Ex. 9. Find the co-ordinates of the intersection of the normals at the points $x'y'$, $x''y''$.

$$\textit{Ans.} \quad x = 2m + \frac{y'^2 + y'y'' + y''^2}{4m}, \quad y = -\frac{y'y''(y' + y'')}{8m^2}.$$

Or if α, β, be the co-ordinates of the corresponding intersection of tangents, then (Ex. 1)

$$x = 2m + \frac{\beta^2}{m} - \alpha, \quad y = -\frac{\alpha\beta}{m}.$$

Ex. 10. Find the locus of the intersection of normals at the extremities of chords which pass through a given point $x'y'$.

We have then the relation $\beta y' = 2m(x' + \alpha)$; and on substituting in the results of the last Example the value of α derived from this relation, we have

$$2mx + \beta y' = 4m^2 + 2\beta^2 + 2mx'; \quad 2m^2y = 2\beta mx' - \beta^2y';$$

whence, eliminating β, we find

$$2\{2m(y - y') + y'(x - x')\}^2 = (4mx' - y'^2)(y'y + 2x'x - 4mx' - 2x'^2),$$

the equation of a parabola whose axis is parallel to the perpendicular from the given point on its polar.

Ex. 11. Find the locus of the intersection of normals at right angles to each other.

In this case $\alpha = -m$, $x = 3m + \dfrac{\beta^2}{m}$, $y = \beta$, $y^2 = m(x - 3m)$.

Ex. 12. If the lengths of two tangents be a, b, and the angle between them ω; find the parameter.

Draw the diameter bisecting the chord of contact; then the parameter of that diameter is $p' = \frac{y^2}{x}$, and the principal parameter is $p = \frac{y^2 \sin^2\theta}{x} = \frac{\varpi^2 y^2}{4x^3}$ (where ϖ is the length of the perpendicular on the chord from the intersection of the tangents). But $2\varpi y = ab \sin\omega$, and

$$16x^2 = a^2 + b^2 + 2ab\cos\omega \text{ ; hence } p = \frac{4ab\sin\omega}{(a^2 + b^2 + 2ab\cos\omega)^{\frac{3}{2}}} \text{ (see p. 180).}$$

Ex. 13. Show, from the equation of the circle circumscribing three tangents to a parabola, that it passes through the focus.

The equation of the circle circumscribing a triangle being (Art. 105) $\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0$, the absolute term in this equation is found (by writing at full length for α, $x\cos\alpha + y\sin\alpha - p$, &c.) to be $p'p''\sin(\beta - \gamma) + p''p\sin(\gamma - \alpha) + pp'\sin(\alpha - \beta)$. But if the line α be a tangent to a parabola, and the origin the focus, we have (Art. 224) $p = \frac{m}{\cos\alpha}$, and the absolute term $= \frac{m^2}{\cos\alpha\cos\beta\cos\gamma}\{\sin(\beta - \gamma)\cos\alpha + \sin(\gamma - \alpha)\cos\beta + \sin(\alpha - \beta)\cos\gamma\}$, which vanishes identically.

Ex. 14. Find the locus of the intersection of tangents to a parabola, being given either (1) the product of sines, (2) the product of tangents, (3) the sum or (4) difference of cotangents of the angles they make with the axis.

Ans. (1) a circle, (2) a right line, (3) a right line, (4) a parabola.

233. We add a few miscellaneous examples.

Ex. 1. If an equilateral hyperbola circumscribe a triangle, it will also pass through the intersection of its perpendiculars (Brianchon & Poncelet: Gergonne, *Annales*, xi. 205; Walton, p. 283).

The equation of a conic meeting the axes in given points is (Ex. 1, p. 137)

$$bb'x^2 + Bxy + aa'y^2 - bb'(a + a')x - aa'(b + b')y + aa'bb' = 0.$$

And if the axes be rectangular, this will represent an equilateral hyperbola (Art. 178) if $aa' = -bb'$. If, therefore, the axes be any side of the given triangle, and the perpendicular on it from the opposite vertex, the portions a, a', b, are given, therefore b' is also given; or the curve meets the perpendicular in the fixed point $y = -\frac{aa'}{b}$, which is (Ex. 7, p. 35) the intersection of the perpendiculars of the triangle.

Ex. 2. Given a triangle, such that any vertex is the pole of the opposite side with respect to an equilateral hyperbola; the circle circumscribing the triangle passes through the centre of the curve. [This is a particular case of a theorem to be proved in the next Chapter (Brianchon & Poncelet, Gergonne, xi. 210; Walton, p. 304).]

Take two sides of the triangle for axes; now the pole of the axis of x, with regard to a conic given by the general equation, lies on the diameter bisecting chords parallel to that axis ($2Ax + By + D = 0$), and also on the polar of the origin ($Dx + Ey + 2F = 0$). If, then, we have $DE = 2BF$, both these lines will meet the axis of y in the same point, and the pole of the axis of x will be the point $y = -\frac{D}{B}$ on the axis of y. In the same case the pole of the axis of y will be the point on the axis of x, $x = -\frac{E}{B}$.

The equation of the circle through the origin and through these two points is

$$B(x^2 + 2xy\cos\omega + y^2) + Ex + Dy = 0,$$

or $$x(2Cy + Bx + E) + y(2Ax + By + D) - 2(A + C - B\cos\omega)\,xy = 0,$$

an equation which will evidently be satisfied by the co-ordinates of the centre, provided we have $A + C = B\cos\omega$, that is to say, provided the curve be an equilateral hyperbola (Arts. 70, 178). If DE be not $= 2BF$, we have still proved that the circle passes through the centre, which is described through the origin and through the points $\left(0, -\frac{D}{B}\right)$, $\left(-\frac{E}{B}, 0\right)$, that is to say, through the points where each axis is met by the diameter bisecting chords parallel to the other. Hence, *a circle described through the centre of an equilateral hyperbola, and through any two points, will also pass through the intersection of lines drawn through each of these points parallel to the polar of the other.*

Ex. 3. If on any tangent to a conic there be taken points A, B, such that AB may be constant; find the locus of the intersection of tangents from A and B (see the section on the Anharmonic Properties of Conics).

The points where a pair of tangents to a conic, given by the general equation, meet the axis of x are found (Art. 150) from the equation.

$$\{(4AC - B^2)y'^2 + (4AE - 2BD)y' + 4AF - D^2\}x^2 + 2\{(BD - 2AE)x'y' + (2CD - BE)y'^2 + (D^2 - 4AF)x' + (DE - 2BF)y'\}x + \{(4AF - D^2)x'^2 + (4BF - 2DE)x'y' + (4CF - E^2)y'^2\} = 0.$$

Forming the difference of the roots of this equation, and putting it equal to a constant, we obtain the equation of the locus which will be in general of the fourth degree; but if $D^2 = 4AF$, the axis of x will touch the given conic, and the equation of the locus will become divisible by y^2, and will reduce to the second degree. We could, by the help of the same equation, find the locus of the intersection of tangents; if the sum, product, &c., of the intercepts on the axis be given.

THE ECCENTRIC ANGLE.*

234. It is always advantageous to express the position of a point on a curve, if possible, by *a single independent* variable, rather than by the *two* co-ordinates $x'y'$. We shall, therefore, find it useful, in discussing properties of the ellipse, to make a substitution similar to that employed (Art. 100) in the case of the circle; and shall write

$$x' = a\cos\phi, \quad y' = b\sin\phi,$$

* The use of this angle occurred to me some years ago, as a particular case of the methods given in Chapter XIV. It has, however, been already recommended by Mr. O'Brien in the *Cambridge Mathematical Journal*, vol. iv. p. 99, and has since been introduced by him, under the name here adopted, into his treatise on *Plane Co-ordinate Geometry*, p. 111.

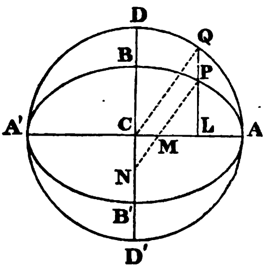
a substitution, evidently, consistent with the equation of the ellipse

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1.$$

The geometric meaning of the angle ϕ is easily explained.

If we describe a circle on the axis major as diameter, and produce the ordinate at P to meet the circle at Q, then the angle QCL $= \phi$, for CL $=$ CQ cos QCL, or $x' = a\cos\phi$; and PL $= \frac{b}{a}$ QL (Art. 166); or, since QL $= a\sin\phi$, we have $y' = b\sin\phi$.

235. Some important consequences may be drawn from this construction.



If we draw through P a parallel PN to the radius CQ, then

PM : CQ :: PL : QL :: $b : a$,

but CQ $= a$, therefore PM $= b$.

PN parallel to CQ is, of course, $= a$.

Hence, if from any point of an ellipse a line $= a$ be inflected to the minor axis, its intercept to the axis major $= b$. If the ordinate PQ were produced to meet the circle again in the point Q′, it could be proved, in like manner, that a parallel through P to the radius CQ′ is cut into parts of a constant length. Hence, conversely, if a line MN, of a constant length, move about in the legs of a right angle, and a point P be taken so that MP may be constant, the locus of P is an ellipse, whose axes are equal to MP and NP. (See Art. 230, Ex. 1.)

On this principle has been constructed an instrument for describing an ellipse by continued motion, called the *Elliptic Compasses.* CA, CD′, are two fixed rulers, MN a third ruler of a constant length, capable of sliding up and down between them, then a pencil fixed at any point of MN will describe an ellipse.

If the pencil be fixed at the middle point of MN it will describe a circle. (O'Brien's *Co-ordinate Geometry*, p. 112.)

236. The consideration of the angle ϕ affords a simple method of constructing geometrically the diameter conjugate to a given one, for

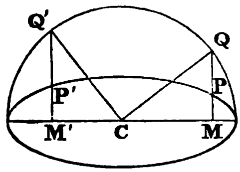
$$\tan\theta = \frac{y'}{x'} = \frac{b}{a}\tan\phi.$$

Hence the relation

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}\ \text{(Art. 174)}$$

becomes
$$\tan\phi\tan\phi' = -1,$$
or
$$\phi - \phi' = 90^\circ.$$

Hence we obtain the following construction for drawing the diameter conjugate to any given one. Let the ordinate at the given point P, when produced, meet the semicircle on the axis major at Q, join CQ, and erect CQ′ perpendicular to it; then the perpendicular let fall on the axis from Q′ will pass through P′, a point on the conjugate diameter.



Hence, too, can easily be found the co-ordinates of P′ given in Art. 176, for, since

$$\cos\phi' = \sin\phi,\ \text{we have}\ \frac{x''}{a} = \frac{y'}{b},$$

and since
$$\sin\phi' = -\cos\phi,\ \text{we have}\ \frac{y''}{b} = -\frac{x'}{a}.$$

From these values it appears that the areas of the triangles PCM, P′CM′, are equal.

Ex. 1. To express the lengths of two conjugate semidiameters in terms of the angle ϕ. *Ans.* $a'^2 = a^2\cos^2\phi + b^2\sin^2\phi$; $b'^2 = a^2\sin^2\phi + b^2\cos^2\phi$.

Ex. 2. To express the equation of any chord of the ellipse in terms of ϕ and ϕ' (see p. 93),

Ans. $\frac{x}{a}\cos\frac{1}{2}(\phi+\phi') + \frac{y}{b}\sin\frac{1}{2}(\phi+\phi') = \cos\frac{1}{2}(\phi-\phi')$.

Ex. 3. To express similarly the equation of the tangent.

Ans. $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$.

Ex. 4. To express the length of the chord joining two points α, β,

$$D^2 = a^2(\cos\alpha - \cos\beta)^2 + b^2(\sin\alpha - \sin\beta)^2$$
$$D = 2\sin\tfrac{1}{2}(\alpha-\beta)\left\{a^2\sin^2\tfrac{1}{2}(\alpha+\beta) + b^2\cos^2\tfrac{1}{2}(\alpha+\beta)\right\}^{\frac{1}{2}}.$$

But (Ex. 1) the quantity between the parentheses is the semidiameter conjugate to that to the point $\frac{1}{2}(\alpha+\beta)$; and (Ex. 2, 3) the tangent at the point $\frac{1}{2}(\alpha+\beta)$ is parallel to the chord joining the points α, β; hence, if b' denote the length of the semidiameter parallel to the given chords, $D = 2b'\sin\frac{1}{2}(\alpha-\beta)$.

Ex. 5. To find the area of the triangle formed by three given points α, β, γ.

By Art. 31 we have

$$2\Sigma = ab\{\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)\}$$

$$= ab\{2\sin\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\alpha-\beta) - 2\sin\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\alpha+\beta-2\gamma)\}$$

$$= 4ab\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)$$

$$\Sigma = 2ab\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha).$$

Ex. 6. To find the radius of the circle circumscribing the triangle formed by three given points α, β, γ.

If d, e, f be the sides of the triangle formed by the three points, $R = \dfrac{def}{4\Sigma} = \dfrac{b'b''b'''}{ab}$ where b', b'', b''' are the semidiameters parallel to the sides of the triangle. If c', c'', c''' be the parallel focal chords, then (see p. 193) $R^2 = \dfrac{c'c''c'''}{4p}$. (These expressions are due to Mr. MacCullagh, *Dublin Exam. Papers*, 1836, p. 22; see also Crelle, vol. XL. p. 31.)

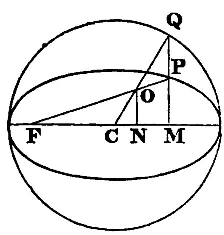
Ex. 7. To find the equation of the circle circumscribing this triangle.

Ans. $$x^2+y^2-\frac{2(a^2-b^2)x}{a}\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha)-\frac{2(b^2-a^2)y}{b}$$
$$\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\gamma+\alpha)=\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\{\cos(\alpha+\beta)$$
$$+\cos(\beta+\gamma)+\cos(\gamma+\alpha)\}.$$

From this equation the co-ordinates of the centre of this circle are at once obtained.

Ex. 8. To find the locus of the intersection of the focal radius vector FP with the radius of the circle CQ.

Let the central co-ordinates of P be $x'y'$, of O xy, then we have, from the similar triangles, FON, FPM,



$$\frac{y}{x+c} = \frac{y'}{x'+c} = \frac{b\sin\phi}{a(e+\cos\phi)}.$$

Now, since ϕ is the angle made with the axis by the radius vector to the point O, we at once obtain the polar equation of the locus by writing $\rho\cos\phi$ for x, $\rho\sin\phi$ for y, and we find

$$\frac{\rho}{c+\rho\cos\phi} = \frac{b}{a(e+\cos\phi)},$$

or
$$\rho = \frac{bc}{c+(a-b)\cos\phi}.$$

Hence (Art. 199) the locus is an ellipse, of which C is one focus, and it can easily be proved that F is the other.

Ex. 9. The normal at P is produced to meet CQ; find the locus of their intersection.

The equation of the normal is (Art. 184)

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = c^2,$$

or (Art. 234)
$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = c^2;$$

but we may, as in the last example, write $\rho\cos\phi$ and $\rho\sin\phi$ for x and y, and the equation becomes

$$(a-b)\rho = c^2,$$

or
$$\rho = a+b.$$

The locus is, therefore, a circle concentric with the ellipse.

Ex. 10. It is useful in astronomy to express the angle PFC in terms of the angle ϕ. It will be found that

$$\tan \tfrac{1}{2}\text{PFC} = \sqrt{\left(\frac{1-e}{1+e}\right)} \tan \tfrac{1}{2}\phi.$$

Ex. 11. If from the vertex of an ellipse a radius vector be drawn to any point on the curve, find the locus of the point where a parallel radius through the centre meets the tangent at the point.

The tangent of the angle made with the axis by the radius vector to the vertex $= \frac{y'}{x'+a}$; therefore, the equation of the parallel radius through the centre is

$$\frac{y}{x} = \frac{y'}{x'+a} = \frac{b \sin\phi}{a(1+\cos\phi)} = \frac{b}{a}\frac{1-\cos\phi}{\sin\phi};$$

or

$$\frac{y}{b}\sin\phi + \frac{x}{a}\cos\phi = \frac{x}{a},$$

and the locus of the intersection of this line with the tangent

$$\frac{y}{b}\sin\phi + \frac{x}{a}\cos\phi = 1$$

is, obviously, $\frac{x}{a} = 1$, the tangent at the other extremity of the axis.

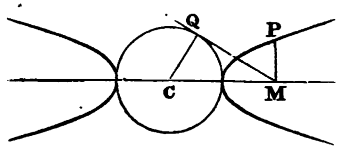
The same investigation will apply, if the first radius vector be drawn through any point of the curve, by substituting a' and b' for a and b; the locus will then be the tangent at the diametrically opposite point.

237. The methods of the preceding Articles do not apply to the hyperbola. For the hyperbola, however, we may substitute

$$x' = a \sec\phi, \qquad y' = b\tan\phi,$$

since

$$\left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{b}\right)^2 = 1.$$



This angle may be represented geometrically by drawing a tangent MQ from the foot of the ordinate M to the circle described on the transverse axis, then the angle QCM = ϕ, since

$$\text{CM} = \text{CQ} \sec \text{QCM}.$$

We have also QM = $a \tan\phi$, but PM = $b \tan\phi$. Hence, if from the foot of any ordinate of a hyperbola we draw a tangent to the circle described on the transverse axis, this tangent is in a constant ratio to the ordinate.

238.* Since the equation of the conjugate hyperbola is

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* This Article is taken from a paper by Mr. Turner in the *Cambridge and Dublin Math. Jour.*, vol. i. p. 122.

$$\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1,$$

any point on the conjugate hyperbola may be expressed by

$$y'' = b \sec\phi', \text{ and } x'' = a \tan\phi'.$$

Now if θ be the angle made by any diameter with the axis of x, we have

$$\tan\theta = \frac{y'}{x'} = \frac{b}{a}\sin\phi.$$

In like manner

$$\tan\theta' = \frac{y''}{x''} = \frac{b}{a}\frac{1}{\sin\phi'};$$

hence the relation connecting two conjugate diameters

$$\tan\theta \tan\theta' = \frac{b^2}{a^2}$$

becomes

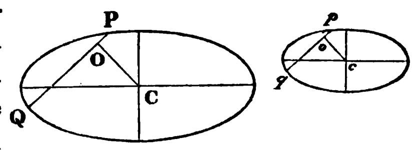
$$\sin\phi = \sin\phi';$$

or, simply,

$$\phi = \phi'.$$

SIMILAR CONIC SECTIONS.

239. Any two figures are said to be *similar and similarly placed*, if radii vectores drawn to the first from a certain point O are in a constant ratio to parallel radii drawn to the second from another point o. If it be possible to find any two such points O and o, we can find an infinity of others; for, take any point C, draw oc parallel to OC, and in the constant ratio $\frac{op}{OP}$, then from the similar triangles OCP, ocp, cp is parallel to CP and in the given ratio. In like manner, any other radius vector through c can be proved to be proportional to the parallel radius through C.



If two *central conic sections* be similar, all diameters of the one are constantly proportional to the parallel diameters of the other, since the rectangles OP · OQ, $op \cdot oq$, are proportional to the squares of the parallel diameters (Art. 152).

240. We now propose to investigate the condition that two conic sections, whose equations are given,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$
$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

should be similar, and similarly placed.

The equation of the first, referred to its centre as origin, must (Art. 155) be of the form

$$Ax^2 + Bxy + Cy^2 = F',$$

and the square of any semidiameter

$$R^2 = \frac{F'}{A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta};$$

the square of a parallel semidiameter of the second is

$$r^2 = \frac{f'}{a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta}.$$

The ratio $\frac{R^2}{r^2}$ cannot be independent of θ, unless we have

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}.$$

Hence, *two conic sections will be similar, and similarly placed, if the coefficients of the highest powers of the variables are the same in both, or only differ by a constant multiplier.*

241. It is evident that the directions of the axes of similar conics must be the same, since the greatest and least diameters of one must be parallel to the greatest and least diameters of the other.

If the diameter of one become infinite, so must also the parallel diameter of the other, that is to say, *the asymptotes of similar and similarly placed hyperbolæ are parallel.* The same thing follows from the result of the last Article, since (Art. 157) the directions of the asymptotes are wholly determined by the highest terms of the equation.

Similar conics have the same eccentricity; for $\frac{a^2 - b^2}{a^2}$ must be $= \frac{m^2a^2 - m^2b^2}{m^2a^2}$. Similar and similarly placed conic sections have hence sometimes been defined as those whose axes are parallel, and which have the same eccentricity.

If two hyperbolæ have parallel asymptotes they are similar, for their axes must be parallel, since they bisect the angles between the asymptotes (Art. 157), and the eccentricity wholly depends on the angle between the asymptotes (Art. 170).

242. Since the eccentricity of all parabolæ is constantly = 1, we should be led to infer that all parabolæ are similar and similarly placed, the direction of whose axes is the same. In fact, the equation of one parabola, referred to its vertex, being $y^2 = px$, or

$$\rho = \frac{p \cos \theta}{\sin^2\theta},$$

it is plain that a parallel radius vector through the vertex of the other will be to this radius in the constant ratio $\frac{p}{p'}$.

Ex. 1. If on any radius vector to a conic section through a fixed point O, OQ be taken in a constant ratio to OP, find the locus of Q.

We have only to substitute $m\rho$ for ρ in the polar equation, and the locus is found to be a conic similar to the given conic, and similarly placed.

The point O may be called *the centre of similitude* of the two conics; and it is obviously (see also Art. 120) the point where common tangents to the two conics intersect, since when the radii vectores OP, OP′ to the first conic become equal, so must also OQ, OQ′ the radii vectores to the other.

Ex. 2. If a pair of radii be drawn through a centre of similitude of two similar conics, the chords joining their extremities will be either parallel, or will meet on the chord of intersection of the conics.

This is proved precisely as in Art. 121.

Ex. 3. Given three similar conics, their six centres of similitude will lie three by three on right lines (see figure, page 113).

Ex. 4. If any line cut two similar and concentric conics, its parts intercepted between the conics will be equal.

Any chord of the outer conic which touches the interior will be bisected at the point of contact.

These are proved in the same manner as the theorems at pages 171, 172, which are but particular cases of them; for the asymptotes of any hyperbola may be considered as a conic section similar to it, since the highest terms in the equation of the asymptotes are the same as in the equation of the curve.

Ex. 5. If a tangent drawn at V, the vertex of the inner of two concentric and similar ellipses, meet the outer in the points T and T′, then any chord of the inner drawn through V is half the algebraic sum of the parallel chords of the outer through T and T′.

243. Two figures will be similar, although not similarly placed, if the proportional radii make a constant angle with each other, instead of being parallel; so that, if we could imagine one

of the figures turned round through the given angle, they would be then both similar and similarly placed.

To find the condition that the two conic sections,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$
$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

should be similar, even though not similarly placed. (Mr. Jellett: *Dublin Examination Papers*, 1847.)

We have only to transform the first equation to axes making any angle θ with the given axes, and examine whether any value can be assigned to θ which will make the new A, B, C proportional to a, b, c.

Let $A' = ma$, $B' = mb$, $C' = mc$. But the axes being supposed rectangular, we have seen (Art. 160) that the quantities $B^2 - 4AC$, $A + C$, are unaltered by transformation of co-ordinates; hence we have

$$A + C = A' + C' = m(a + c),$$
$$B^2 - 4AC = B'^2 - 4A'C' = m^2(b^2 - 4ac),$$

and the required condition is

$$\frac{B^2 - 4AC}{(A + C)^2} = \frac{b^2 - 4ac}{(a + c)^2}.$$

If the axes be oblique it is seen in like manner (Art. 161) that the condition for similarity is

$$\frac{B^2 - 4AC}{(A + C - B\cos\omega)^2} = \frac{b^2 - 4ac}{(a + c - b\cos\omega)^2}.$$

It will be seen (Arts. 70, 157) that the condition found expresses that the angle between the (real or imaginary) asymptotes of the one curve is equal to that between those of the other.

THE CONTACT OF CONIC SECTIONS.

244. We proved (Art. 15) that we obtain an equation of the mn^{th} degree to determine the co-ordinates of the points of intersection of two curves of the m^{th} and n^{th} degrees. Hence, *two conic sections will in general intersect in four points.*

If two of these points of intersection coincide, the conic sections are said to touch each other, and the line joining the coincident points will be the common tangent.

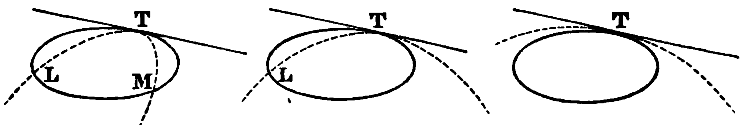
Let the equations of the conics, referred to the tangent and normal, be (see p. 160)

$$Ax^2 + Bxy + Cy^2 + Ey = 0,$$
$$A'x^2 + B'xy + C'y^2 + E'y = 0,$$

then the equation of the line (LM) joining the other two points of intersection will be, as in Ex. 2, p. 160,

$$(BA' - AB')x + (CA' - AC')y + (EA' - AE') = 0.$$

This is called *a contact of the first order.*



Now the contact of the conics will evidently be more close if *three* of their points of intersection coincide. In this case one of the points L, M must coincide with T, the line LM must pass through the origin, and we must have the condition

$$EA' - AE' = 0.$$

This is called *a contact of the second order.* Curves which have a contact higher than the first order are said to *osculate*, and it appears that conics which osculate, in general, meet each other in *one* other point.

The contact of two conics will be the closest possible when they have *four* consecutive points in common. In this case the line LM must coincide with the tangent at T ($y = 0$), and we must have the two conditions

$$EA' - AE' = 0, \qquad BA' - AB' = 0.$$

This is called *a contact of the third order;* and since two conic sections cannot have more than four points common, it is the highest order of contact which two different conics can have.

Hence, if the equation of one curve be

$$x^2 + Bxy + Cy^2 + Ey = 0,$$

that of the other will be

$$x^2 + Bxy + C'y^2 + Ey = 0.$$

245. Hence an infinity of conic sections can be drawn having a contact of the third order with a given conic at a given point,

and any *one* condition will enable us to determine the touching conic. Thus, for example, the *parabola* having a contact of the third order with the conic

$$x^2 + Bxy + Cy^2 + Ey = 0$$

will be

$$x^2 + Bxy + \frac{B^2}{4} y^2 + Ey = 0.$$

We cannot describe a *circle* to have a contact of the third order with a given conic, because *two* conditions must be fulfilled in order that this equation should represent a circle; or, in other words, we cannot describe a circle through four consecutive points on a conic, since three points are sufficient to determine a circle. We can, however, easily find the equation of the circle passing through *three* consecutive points on the curve. This circle is called the *osculating circle*, or the *circle of curvature*.

The equation of the conic being

$$Ax^2 + Bxy + Cy^2 + Ey = 0,$$

that of any circle touching it is (Art. 77, Cor. 2)

$$x^2 + y^2 + 2ry = 0,$$

and the condition that the circle should osculate is (Art. 244)

$$E = 2Ar, \text{ or } r = \frac{E}{2A}.$$

The quantity r is called the *radius of curvature* of the conic at the point T.

246. *To find an expression for the radius of curvature at any point of an ellipse.*

It is plain, from the last Article, that this can be found by transforming the equation to the tangent and normal at the point.

The equation referred to a diameter through the point and its conjugate $\left(\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1\right)$, is transferred to parallel axes through the given point, by substituting $x + a'$ for x, and becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{2x}{a'} = 0.$$

The axes are now a tangent and diameter through the point, and we wish, allowing the axis of y to remain unaltered, to make the normal the axis of x.

Now, if X and Y be rectangular co-ordinates, x and y oblique co-ordinates, inclined at an angle θ, the axis of y remaining unaltered, we see (as in Art. 217) that

$$x \sin\theta = X; \quad y + x\cos\theta = Y;$$

and, therefore, $$x = \frac{X}{\sin\theta}; \quad y = Y - \frac{X}{\tan\theta}.$$

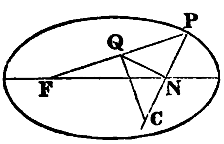
On making these substitutions, the coefficient of X will be $\frac{2}{a'\sin\theta}$ and that of Y^2 will be $\frac{1}{b'^2}$; hence the radius of curvature will be $\frac{b'^2}{a'\sin\theta}$. Now, $a'\sin\theta$ is the perpendicular from the centre on the tangent; therefore the radius of curvature

$$= \frac{b'^2}{p} = (\text{Art. } 179)\ \frac{b'^3}{ab}.$$

247. This value enables us to construct simply for the radius of curvature at any point. We proved (Art. 185) that the length of the normal $= \frac{bb'}{a}$, and that $\cos\psi = \frac{b}{b'}$ (ψ being the angle between the focal radius and the normal); hence

$$R = \frac{N}{\cos^2\psi}.$$

If, therefore, we erect a perpendicular to the normal at the point where it meets the axis, and again at the point Q, where this perpendicular meets the focal radius, draw CQ perpendicular to it, then C will be the centre of curvature, and CP the radius of curvature.



Another useful construction is founded on the principle that *if a circle intersect a conic, its chords of intersection will make equal angles with the axis.* For, the rectangles under the segments of the chords are equal (Euc. III. 35), and therefore the parallel diameters of the conic are equal (Art. 152), and, therefore, make equal angles with the axis (Art. 165).

Now in the case of the circle of curvature, the tangent at T (see figure, p. 205) is one chord of intersection, and the line TL the other; we have, therefore, only to draw TL, making the same angle with the axis as the tangent, and we have the point

L; then the circle described through the points T, L, and touching the conic at T, is the circle of curvature.

This construction shows that the osculating circle at either vertex has a contact of the third degree.

Ex. 1. Using the notation of the eccentric angle, find the condition that four points $\alpha, \beta, \gamma, \delta$ should lie on the same circle (Joachimstal, *Crelle*, xxxvi. 95).

The chord joining two of them must make the same angle with one side of the axis as the chord joining the other two does with the other; and the chords being

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta)+\frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta)=\cos\tfrac{1}{2}(\alpha-\beta);\ \frac{x}{a}\cos\tfrac{1}{2}(\gamma+\delta)+\frac{y}{b}\sin\tfrac{1}{2}(\gamma+\delta)=\cos\tfrac{1}{2}(\gamma-\delta),$$

we have $\tan\frac{1}{2}(\alpha+\beta)+\tan\frac{1}{2}(\gamma+\delta)=0$; $\alpha+\beta+\gamma+\delta=0$; or $=2m\pi$.

Ex. 2. Find the co-ordinates of the point where the osculating circle meets the conic again.

We have $\alpha=\beta=\gamma$; hence $\delta=-3\alpha$; or $X=\dfrac{4x'^3}{a^2}-3x'$; $Y=\dfrac{4y'^3}{b^2}-3y'$.

Ex. 3. There are three points on a conic whose osculating circles pass through a given point on the curve; these lie on a circle passing through the point, and form a triangle of which the centre of the curve is the intersection of bisectors of sides (Steiner, *Crelle*, xxxii. 300; Joachimstal, *Crelle*, xxxvi. 95).

Here we are given δ, the point where the circle meets the curve again, and from the last Example the point of contact is $\alpha=-\dfrac{\delta}{3}$. But since the sine and cosine of δ would not alter if δ were increased by 360°, we might also have $\alpha=-\dfrac{\delta}{3}+120°$, or $=-\dfrac{\delta}{3}+240°$, and from Ex. 1, these three points lie on a circle passing through δ. If in the last Example we suppose XY given, since the cubics which determine x' and y' want the second terms, the sums of the three values of x and of y are respectively equal to cipher: and therefore (Ex. 4, p. 5) the origin is the intersection of the bisectors of sides of the triangle formed by the three points. It is easy to see that the normals at these points are the three perpendiculars of this triangle, and therefore that they meet in a point.

248. *To find the radius of curvature of the parabola.*

The equation, referred to any diameter and tangent ($y^2=p'x$), is transferred to the tangent and normal by the same substitution as in Art. 246, and we find

$$R=\frac{p'}{2\sin\theta}=\frac{p'^{\frac{3}{2}}}{2p^{\frac{1}{2}}}\ \text{(Art. 217)};$$

or since (Arts. 217, 218)

$$N=\frac{p'}{2}\sin\theta,\qquad R=\frac{N}{\sin^2\theta}=\frac{N}{\cos^2\psi}.$$

The construction, therefore, used in the last Article, applies also to the case of the parabola.

Ex. 1. In all the conic sections the radius of curvature is equal to the cube of the normal divided by the square of the semi-parameter.

Ex. 2. Express the radius of curvature of an ellipse in terms of the angle which the normal makes with the axis.

Ex. 3. Find the lengths of the chords of the circle of curvature which pass through the centre or the focus of a central conic section.

Ans. $\frac{2b'^2}{a'}$, and $\frac{2b'^2}{a}$.

Ex. 4. The focal chord of curvature of any conic is equal to a focal chord of the conic drawn parallel to the tangent at the point.

Ex. 5. In the parabola the focal chord of curvature is equal to the parameter of the diameter passing through the point.

249. *To find the co-ordinates of the centre of curvature of a central conic.*

These are evidently found by subtracting from the co-ordinates of the point on the conic the projections of the radius of curvature upon each axis. Now it is plain that this radius is to its projection on y as the normal to the ordinate y. We find the projection, therefore, of the radius of curvature on the axis of y by multiplying the radius $\frac{b'^2}{p}$ by $\frac{y'}{N} = \frac{b'^2y'}{b^2}$. The y of the centre of curvature then is $\frac{b^2 - b'^2}{b^2}y'$. But $b'^2 = b^2 + \frac{c^2}{b^2}y'^2$, therefore the y of the centre of curvature is $\frac{b^2 - a^2}{b^4}y'^3$. In like manner its x is $\frac{a^2 - b^2}{a^4}x'^3$.

We should have got the same values by making $\alpha = \beta = \gamma$ in Ex. 7, Art. 236.

Or again, the centre of the circle circumscribing a triangle is the intersection of perpendiculars to the sides at their middle points; and when the triangle is formed by three consecutive points on a curve, two sides are consecutive tangents to the curve, and the perpendiculars to them are the corresponding normals, and *the centre of curvature of any curve is the intersection of two consecutive normals.* Now if we make $x' = x'' = X$, $y' = y'' = Y$, in Ex. 4, p. 161, we obtain again the same values as those just determined.

250. *To find the co-ordinates of the centre of curvature of a parabola.*

The projection of the radius on the axis of y is found in like manner by multiplying the radius of curvature

$$\frac{N}{\sin^2\theta} \text{ by } \frac{y'}{N} = \frac{y'}{\sin^2\theta};$$

and subtracting this quantity from y' we have

$$Y = -\frac{y'}{\tan^2\theta} = -\frac{4y'^3}{p^2} \text{ (Art. 217).}$$

In like manner its x is $x' + \dfrac{p}{2\sin^2\theta} = x' + \dfrac{p + 4x'}{2}$.

The same values may be found from Ex. 9, p. 194.

251. The *evolute* of a curve is the locus of the centres of curvature of its different points. If it were required to find the evolute of a central conic, we should solve for $x'y'$ in terms of the x and y of the centre of curvature, and, substituting in the equation of the curve, should have (writing $\frac{c^2}{a} = A$, $\frac{c^2}{b} = B$),

$$\frac{x^{\frac{2}{3}}}{A^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{B^{\frac{2}{3}}} = 1.$$

In like manner the equation of the evolute of a parabola is found to be

$$27py^2 = 16(x - \tfrac{1}{2}p)^3,$$

which represents a curve called the *semicubical parabola*.

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*CHAPTER XIV.

METHODS OF ABRIDGED NOTATION.

252. We have proved (Art. 15) that we obtain an equation of the mn^{th} degree to determine the co-ordinates of the points of intersection of two curves of the m^{th} and n^{th} degrees; and since an equation of the mn^{th} degree has always mn roots, real or imaginary, we infer (as in Art. 69) that a curve of the m^{th} degree will *always* intersect a curve of the n^{th} degree in mn points, real or imaginary. Two conic sections, therefore, $S = 0$, $S' = 0$, always intersect each other in four points, real or imaginary;

and (Art. 36) $S + kS' = 0$ is the equation of another conic through these four points of intersection.

253. This will, of course, still be true if either or both the quantities S, S′ be resolvable into factors. Thus, let S′ be resolvable into factors, and represent the pair of right lines α, β; then $S + k\alpha\beta = 0$, which is evidently satisfied by the co-ordinates of the points where either α or β meets S, will represent a conic passing through the four points where S is met by this pair of right lines. It is, therefore, *the equation of a conic having α and β for its chords of intersection with* S. If either α or β do not meet S in real points, it must still be considered as a *chord of imaginary intersection*, and will preserve many important properties in relation to the two curves, as we have already seen in the case of the circle (Art. 111).

If both S and S′ break up into factors, the equation $\alpha\gamma + k\beta\delta = 0$ represents the conic circumscribing the quadrilateral $(\alpha\beta\gamma\delta)$, as we have already seen, p. 97.

It is obvious that in what precedes α need not denote a line whose equation has been reduced to the form $x \cos \alpha + y \sin \alpha = p$, but that $S + LM = 0$ (see convention, Art. 52) will in like manner represent a conic passing through the points where L and M meet S, &c.

Ex. 1. What is the equation of a conic passing through the points where a given conic S meets the axes?

Here the axes $x = 0$, $y = 0$ are the chords of intersection, and the equation must be of the form $S + kxy = 0$, where k is indeterminate. Compare Ex. 1, p. 137.

Ex. 2. Find the equation of the conic passing through five given points.

Having formed the equations of α, β, γ, δ, the sides of the quadrilateral formed by four of the given points, we know that the equation must be of the form $\alpha\gamma = k\beta\delta$; and, substituting in this equation the co-ordinates of the fifth point, we are able to determine k.

Ex. 3. Form the equation of the conic which passes through the points (1, 2), (3, 5) (− 1, 4), (− 3, − 1), (− 4, 3).

Considering the quadrilateral formed by the first four points, we see that the equation must be of the form

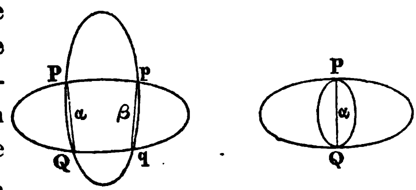
$$(3x - 2y + 1)(5x - 2y + 13) = k(x - 4y + 17)(3x - 4y + 5).$$

Substituting in this the co-ordinates − 4, 3, which must satisfy it, we obtain $k = -\frac{221}{19}$.

Substituting this value, and reducing the equation, it becomes

$$79x^2 - 320xy + 301y^2 + 1101x - 1665y + 1586 = 0.$$

254. We have seen that the equation $S + k\alpha\beta = 0$ represents a conic passing through the four points P, Q; p, q; where α, β meet S: and it is evident that the closer to each other the lines $\alpha\beta$ are, the nearer the point P is to p, and Q to q.



Suppose then that the lines α and β coincide, then the points P, p; Q, q coincide, and the second conic will touch the first at the points P, Q. We learn then that the equation $S + k\alpha^2 = 0$ *represents a conic having double contact with* S, *and whose chord of contact is* α. In like manner $\alpha\gamma + k\beta^2 = 0$ represents a conic, to which α and γ are tangents, while β is their chord of contact, as we have already seen (Art. 104). Similarly $S + L^2 = 0$ represents a conic having double contact with S, L being the chord of contact; and $LN = M^2$ denotes a conic to which L and N are tangents, while M is their chord of contact.

If the line α were a tangent to S, the two points P and Q would coincide, and the conic $S + k\alpha^2$ would have four consecutive points common with S, and would therefore have with it a contact of the third degree. Thus, for instance, we have seen (Art. 244) that the equations of two conics which have contact of the third order at a point on the axis of x are of the form

$$S = 0 \text{ and } S + ky^2 = 0.$$

255. The forms given in the preceding articles receive important modifications, if any of the lines which they involve be at an infinite distance. It was proved (Art. 64) that when a line is removed to an infinite distance, its equation is reduced to the constant term. If, then, in any of the preceding equations, we substitute a constant for any of the quantities α, β, &c., we shall have the form which that equation will assume when the line α, β, &c., is at an infinite distance.

Thus we know that the lines L, N touch the conic $LN = M^2$ at the points where they meet M; if, then, we substitute for M a constant m, we see that the conic $LN = m^2$ is touched by the lines L, N at the points at infinity on those lines: in other words, that the lines L, N are asymptotes to this conic. If we suppose the lines L, N to be the axes, we obtain the known form of the equation of a conic referred to its asymptotes $xy = m^2$ (Art. 204).

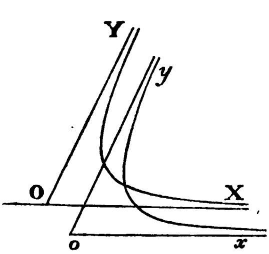
In like manner, the equation $lN = M^2$ (where l is a constant) denotes a conic to which N is one tangent, and l, the line at infinity, is another. In this equation the highest terms form the perfect square M^2, and therefore the curve is a parabola. Conversely, *every parabola has one tangent altogether at an infinite distance.* In fact, the equation which determines the direction of the points at infinity on a parabola is a perfect square (Art. 136); the two points of the curve at infinity therefore coincide; and therefore the line at infinity is to be regarded as a tangent (Art. 81). And the form of the equation of the parabola $px = y^2$ denotes that the line at infinity p is one tangent, the line x another, and that the diameter y is the line joining their points of contact.

So, in general, the equation

$$(ax + by)^2 + Dx + Ey + F = 0$$

denotes a parabola to which $Dx + Ey + F = 0$ is a tangent, and $ax + by = 0$ the diameter through the point of contact.

256. In like manner, it may be inferred from Art. 253 that the equations $S = 0$, $S + lM = 0$ (where l is a constant), denote two conics intersecting each other in the two finite points where M meets either, and also in the two infinitely distant points where the line at infinity l meets either. Now, it is plain that the coefficients of x^2, xy, and y^2 are the same in the two equations $S = 0$, $S + lM = 0$; and therefore (Art. 240) that these equations denote two conics similar and similarly placed. We learn, therefore, that *two conics similar and similarly placed can cut each other only in two finite points;* and that this is because *they also cut each other in two real, coincident, or imaginary points at infinity.*



257. We may arrive geometrically at the same conclusion.

First. If the curves be hyperbolæ. The asymptotes of similar hyperbolæ are parallel (Art. 241), that is, they intersect each other at infinity; but each asymptote intersects its own curve at infinity; hence we infer that similar and similarly placed hyperbolæ intersect each other in the two points at infinity, where each is intersected by its own asymptotes (see the figure, where the two hyperbolæ evidently tend to intersect at the two points at infinity, where OX meets ox, and OY meets oy).

Secondly. If the curves be ellipses. Ellipses only differ from hyperbolæ in having imaginary instead of real asymptotes. The directions of the points at infinity on either of two similar ellipses are determined from the same equation ($Ax^2 + Bxy + Cy^2 = 0$) (Arts. 134 and 240). Now, although the roots of this equation are in both cases imaginary, yet they are in both cases *the same* imaginary roots; we infer, therefore, that two similar ellipses pass through the same two imaginary points at infinity.

Thirdly. If the curves be parabolæ. They are both touched by the line at infinity (Art. 255). The direction of the point of contact at infinity is the same as that of the diameters (Art. 140), and is therefore the same for two similarly placed parabolæ (Art. 242). Hence *two similarly placed parabolæ touch each other at infinity.*

258. It may be inferred in precisely the same way, from Art. 254, that the equation $S + l^2 = 0$, where l is constant, denotes a conic touching the conic S in two points at infinity. Now if the equations of two conics only differ in the constant terms, since the co-ordinates of the centre do not contain F (Art. 138), the conics must have the same centre; and since the first three terms are the same in both, the conics are similar; hence the conics S and $S + l^2$ are similar and concentric. We learn then that *similar and concentric conics are to be regarded as touching each other at two points at an infinite distance.* This is otherwise evident, since we have proved in the last Article that the curves pass through the same points at infinity; and since they have the same, real or imaginary, asymptotes, they have also the same tangents at those points.

If the curves be parabolæ, then since the line at infinity touches both, by Art. 254, the conics S and $S + l^2$ have with each other a contact of the third order at infinity. Two parabolæ whose equations only differ in the constant term will be equal to each other; for the parabolæ $y^2 = px$, and $y^2 = p(x + n)$, are obviously equal, and if the origin be transferred to any other point the equations will continue to differ only in the constant term. We have seen too (Art. 213) that the expression for the parameter of a parabola does not involve the absolute term. The parabolæ, then, S and $S + l^2$, are equal to each other, and we learn that *two equal and similarly placed parabolæ may be considered as having with each other a contact of the third order at infinity.*

259. Since all circles are similar curves, it follows, as a particular case of the last Articles, that *all circles pass through the same two imaginary points at infinity*, and that *concentric circles touch each other in two imaginary points at infinity.* Thus we see the reason why two circles cannot cut each other in more than two finite points, and why two concentric circles do not meet in any finite point, although two curves of the second degree in general intersect in four points. We shall also show that the theorems established (p. 103, &c.), concerning circles which pass through the same two points, are only particular cases of more general theorems concerning conic sections which pass through the same four points.

260. We proceed to notice some inferences which follow immediately on interpreting the preceding equations by the help of Art. 27. Thus the equation $\alpha\gamma = k\beta^2$ implies that *the product of the perpendiculars from any point of a conic on two fixed tangents is in a constant ratio to the square of the perpendicular on their chord of contact.*

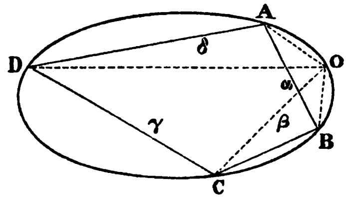
The equation $\alpha\gamma = k\beta\delta$, similarly interpreted, leads to the important theorem: *The product of the perpendiculars let fall from any point of a conic on two opposite sides of an inscribed quadrilateral is in a constant ratio to the product of the perpendiculars let fall on the other two sides.*

From this property we at once infer, that *the anharmonic ratio of a pencil, whose sides pass through four fixed points of a conic, and whose vertex is any variable point of it, is constant.*

For the perpendicular

$$\alpha = \frac{OA \cdot OB \cdot \sin AOB}{AB}, \quad \gamma = \frac{OC \cdot OD \cdot \sin COD}{CD}, \text{ \&c.}$$

Now if we substitute these values in the equation $\alpha\gamma = k\beta\delta$, the continued product OA·OB·OC·OD will appear on both sides of the equation, and may therefore be suppressed, and there will remain



$$\frac{\sin AOB \cdot \sin COD}{\sin BOC \cdot \sin AOD} = k \cdot \frac{AB \cdot CD}{BC \cdot AD};$$

but the right-hand member of this equation is constant, while the left-hand member is the anharmonic ratio of the pencil OA, OB, OC, OD.

The consequences of this theorem are so numerous and important, that we shall devote a section of the next chapter to develop them more fully.

261. If $S = 0$ be the equation to a circle, then (Art. 88) S is the square of the tangent from any point xy to the circle; hence $S - k\alpha\beta = 0$ (the equation of a conic whose chords of intersection with the circle are α and β) expresses that *the locus of a point, such that the square of the tangent from it to a fixed circle is in a constant ratio to the product of its distances from two fixed lines, is a conic passing through the four points in which the fixed lines intersect the circle.*

This theorem is equally true whatever be the magnitude of the circle, and whether the right lines meet the circle in real or imaginary points; thus, for example, if the circle be infinitely small, *the locus of a point, the square of whose distance from a fixed point is in a constant ratio to the product of its distances from two fixed lines, is a conic section;* and the fixed lines may be considered as chords of imaginary intersection of the conic with an infinitely small circle whose centre is the fixed point.

262. Similar inferences can be drawn from the equation $S - k\alpha^2 = 0$, where S is a circle. We learn that *the locus of a point, such that the tangent from it to a fixed circle is in a constant ratio to its distance from a fixed line, is a conic touching the circle*

at the two points where the fixed line meets it; or, conversely, that *if a circle have double contact with a conic, the tangent drawn to the circle from any point on the conic is in a constant ratio to the perpendicular from the point on the chord of contact.*

In the particular case where the circle is infinitely small, we obtain the fundamental property of the focus and directrix, and we infer that *the focus of any conic may be considered as an infinitely small circle, touching the conic in two imaginary points situated on the directrix.*

263. In general, *if in the equation of any conic the co-ordinates of any point be substituted, the result will be proportional to the rectangle under the segments of a chord drawn through the point parallel to a given line.**

For (Art. 151) this rectangle

$$= \frac{F'}{A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta},$$

where, by Art. 129, F' is the result of substituting in the equation the co-ordinates of the point; if, therefore, the angle θ be constant, this rectangle will be proportional to F'. Hence, we may extend the last-proved theorems to the case where S is any conic. For example: "If two conics have double contact, the square of the perpendicular from any point of one upon the chord of contact, is in a constant ratio to the rectangle under the segments of that perpendicular made by the other;" or, in general, "If a line parallel to a given one meets two conics in the points P, Q, p, q, and we take on it a point O, such that the rectangle $OP \cdot OQ$ may be to $Op \cdot Oq$ in a constant ratio, the locus of O is a conic through the points of intersection of the given conics."

264. *If two conics have each double contact with a third, their chords of contact with the third conic, and a pair of their chords of intersection with each other, will all pass through the same point, and will form an harmonic pencil.*

Let the equation of the third conic be $S = 0$, and those of the other two conics,

$$S + L^2 = 0, \qquad S + M^2 = 0.$$

* This is equally true for curves of any degree.

Now, on subtracting these equations, we find for the equation of the chords of intersection,

$$L^2 - M^2 = 0.$$

The chords of intersection, therefore ($L - M = 0$, $L + M = 0$), pass through the intersection of the chords of contact (L and M), and form an harmonic pencil with them (Art. 55).

It is important that the student should acquire the habit of taking notice of the number of particular theorems often included under one general enunciation; thus, for example, the present theorem holds good, and is proved, in like manner, if the conic S reduce to two right lines; hence, *the chords of contact of two conics with their common tangents pass through the intersection of their common chords.*

Again, if S be any conic, while $S + L^2$ and $S + M^2$ both reduce to pairs of right lines, these right lines will then form a circumscribing quadrilateral, and the chords of intersection ($L^2 - M^2$) will be the diagonals of that quadrilateral, while the chords of contact (L and M) obviously are the diagonals of the inscribed quadrilateral formed by joining the points of contact. Hence, *the diagonals of any inscribed, and of the corresponding circumscribed quadrilateral, pass through the same point, and form an harmonic pencil.*

The theorem of this Article may also be stated thus: *If a conic section pass through two given points, and have double contact with a given conic, the chord of contact passes through a fixed point.* For, suppose any conic ($S + L^2 = 0$) through the two given points to be fixed, then the intersection of its chord of contact (L), with the line joining the given points, determines a point through which, by the present Article, any other chord of contact must pass.

In like manner: *Given two tangents and two points on a conic section; the chord of contact will pass through a fixed point on the line joining the two given points.*

265. *If three conics have each double contact with a fourth, their six chords of intersection will pass three by three through the same points*, thus forming the sides and diagonals of a quadrilateral.

Let the conics be

$$S + L^2 = 0, \qquad S + M^2 = 0, \qquad S + N^2 = 0.$$

By the last Article the chords will be

$$\begin{array}{lll} L - M = 0, & M - N = 0, & N - L = 0; \\ L + M = 0, & M + N = 0, & N - L = 0; \\ L + M = 0, & M - N = 0, & N + L = 0; \\ L - M = 0, & M + N = 0, & N + L = 0. \end{array}$$

As in the last Article, we may deduce hence many particular theorems, by supposing one or more of the conics to break up into right lines.

Thus, for example, if S break up into right lines, it represents two common tangents to $S + M^2$, $S + N^2$; and if L denote any right line through the intersection of those common tangents, then $S + L^2$ also breaks up into right lines, and represents any two right lines passing through the intersection of the common tangents. Hence, *if through the intersection of the common tangents of two conics we draw any pair of right lines, the chords of each conic joining the extremities of those lines will meet on one of the common chords of the conics.* This is the extension of Art. 121. Or, again, *tangents at the extremities of either of these right lines will meet on one of the common chords.*

266. If $S + L^2$, $S + M^2$, $S + N^2$, all break up into pairs of right lines, they will form a hexagon circumscribing S, the chords of intersection will be diagonals of that hexagon, and the proposition of this Article becomes Brianchon's theorem: "*The three opposite diagonals of every hexagon circumscribing a conic intersect in a point.*"

By the *opposite* diagonals we mean (if the sides of the hexagon be numbered 1, 2, 3, 4, 5, 6) the lines joining (1, 2) to (4, 5), (2, 3) to (5, 6), and (3, 4) to (6, 1); and by changing the order in which we take the sides, we may consider the same lines as forming a number (sixty) of different hexagons, for each of which the present theorem is true.

By supposing two sides of the hexagon to be indefinitely near, we obtain from this theorem a very simple construction for the solution of the problem,—"Given five tangents, to find the point of contact of any of them,"—since any tangent is intersected by a consecutive tangent at its point of contact (p. 130).

267. *If three conic sections have one chord common to all, their three other common chords will pass through the same point.*

Let the equation of one be $S = 0$, and of the common chord $L = 0$, then the equations of the other two are of the form

$$S + LM = 0, \qquad S + LN = 0,$$

which must have, for their intersection with each other,

$$L(M - N) = 0;$$

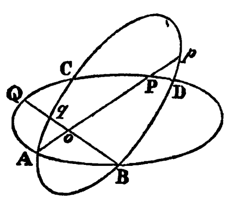
but $M - N$ is a line passing through the point (MN).

According to the remark in Art. 259, this is only an extension of the theorem (Art. 113), that the radical axes of three circles meet in a point. For three circles have one chord (the line at infinity) common to all, and the radical axes are their other common chords.

The theorem of Art. 265 may be considered as a still further extension of the same theorem, and three conics which have each double contact with a fourth may be considered as having four radical centres, through each of which pass three of their common chords.

The theorem of this Article may, as in Art. 113, be otherwise enunciated: *Given four points on a conic section, its chord of intersection with a fixed conic passing through two of these points will pass through a fixed point.*

A number of particular inferences may also be drawn from the theorem of the present Article, by supposing one or more of the conics to break up into two right lines. Thus, for example, if one of the conics break up into the pair of lines OA, OB, we obtain the theorem: "If through one of the points of intersection of two conics we draw any line meeting the conics in the points P, p, and through any other point of intersection B a line meeting the conics in the points Q, q, then the lines PQ, pq, will meet on CD, the other chord of intersection." Next let the points A, B coincide, then the two conics will touch at A, and we learn



that "if two right lines, drawn through the point of contact of two conics, meet the curves in points P, p, Q, q, then the chords PQ, pq, will meet on the chord of intersection of the conics."

This is a particular case of a theorem given in Art. 265, since one intersection of common tangents to two conics which touch, reduces to the point of contact (Art. 123).

268. The equation of a conic circumscribing a quadrilateral ($\alpha\gamma = k\beta\delta$) furnishes us with a proof of "Pascal's theorem," that *the three intersections of the opposite sides of any hexagon inscribed in a conic section are in one right line.*

Let the vertices be $abcdef$, and let $ab = 0$ denote the equation of the line joining the points a, b, then, since the conic circumscribes the quadrilateral $abcd$, its equation must be capable of being put into the form
$$ab \cdot cd - bc \cdot ad = 0.$$
But since it also circumscribes the quadrilateral $defa$, the same equation must be capable of being expressed in the form
$$de \cdot fa - ef \cdot ad = 0.$$
From the identity of these expressions we have
$$ab \cdot cd - de \cdot fa = (bc - ef)\, ad.$$
Hence we learn that the left-hand side of this equation (which from its form represents a figure circumscribing the quadrilateral formed by the lines ab, de, cd, af) is resolvable into two factors, which must therefore represent the diagonals of that quadrilateral. But ad is evidently the diagonal which joins the vertices a and d, therefore $bc - ef$ must be the other, and must join the points (ab, de), (cd, af); and since from its form it denotes a line through the point (bc, ef), it follows that these three points are in one right line. We shall in the next chapter give another demonstration of this important theorem.

By supposing two vertices of the hexagon to be indefinitely near, we may, "given five points on a conic, draw a tangent at any of these points."

269. We may, as in the case of Brianchon's theorem, obtain a number of different theorems concerning the same six points, according to the different orders in which we take them. Thus

since the conic circumscribes the quadrilateral *bcef*, its equation can be expressed in the form

$$be \,.\, cf - bc \,.\, ef = 0.$$

Now, from identifying this with the first form given in the last Article, we have

$$ab \,.\, cd - be \,.\, cf = (ad - ef)\, bc;$$

whence, as before, we learn that the three points (ab, cf), (cd, be), (ad, ef) lie in one right line, viz. $ad - ef = 0$.

In like manner, from identifying the second and third forms of the equation of the conic, we learn that the three points (de, cf), (fa, be), (ad, bc) lie in one right line, viz. $bc - ad = 0$. But the three right lines

$$bc - ef = 0, \quad ef - ad = 0, \quad ad - bc = 0,$$

meet in a point (Art. 37). Hence we have Steiner's theorem, that "the three Pascal's lines which are obtained by taking the vertices in the orders respectively, *abcdef*, *adcfeb*, *afcbed*, meet in a point." For some further developments on this subject we refer the reader to the note at the end of the volume.

TRILINEAR CO-ORDINATES.

270. We proved (Art. 61) that being given three lines (α, β, γ), we can express the equation of any other right line in the form

$$A\alpha + B\beta + C\gamma = 0.$$

In the same manner we can show that there is no conic section whose equation may not be written in the form

$$A\alpha^2 + B\alpha\beta + C\beta^2 + D\alpha\gamma + E\beta\gamma + F\gamma^2 = 0.$$

For this equation is obviously of the second degree; and since it contains five independent constants, we may (as in Art. 128) determine these constants so that the curve which it represents may pass through five given points, and therefore may coincide with any given conic. In short, since the equation just written contains the same number of constants as the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

it must be equally capable of representing any particular conic.

In like manner, in general, there is no curve of any degree whose equation may not be expressed as a homogeneous function of the quantities α, β, γ. For it can readily be proved that the number of terms in the *complete* equation of the n^{th} order between *two* variables is the same as the number of terms in the *homogeneous* equation of the n^{th} order between *three* variables.

271. If (as in Art. 66) we render the Cartesian equation homogeneous by the introduction of the linear unit z, we at once perceive the identity of the two forms

$$A\alpha^2 + B\alpha\beta + C\beta^2 + D\alpha\gamma + E\beta\gamma + F\gamma^2 = 0,$$
$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0;$$

the latter being the form assumed by the former, when two of the lines of reference ($\alpha\beta$) are the axes (xy), and the third (γ) is the line at infinity z. It is important to keep constantly in view the analogy which subsists between these two forms of equations. If, for instance, we make $\gamma = 0$ in the first equation, the result $A\alpha^2 + B\alpha\beta + C\beta^2 = 0$ is plainly the equation of the lines joining the point ($\alpha\beta$) to the points where γ cuts the curve. In like manner, if we make $z = 0$ in the second equation, the result $Ax^2 + Bxy + Cy^2 = 0$ must be the equation of the pair of lines joining the origin (xy) to the points where the line at infinity cuts the curve (Art. 134).

Precisely the same argument which proves (Art. 36) that the curve represented by

$$(A\alpha^2 + B\alpha\beta + C\beta^2) + \gamma(D\alpha + E\beta + F\gamma) = 0$$

passes through the intersections of the line γ with the pair of lines ($A\alpha^2 + B\alpha\beta + C\beta^2$), proves likewise that the curve passes through the intersections of the same pair of lines with the line $D\alpha + E\beta + F\gamma = 0$. This latter equation then denotes the fourth side of a quadrilateral inscribed in the conic, of which the other three sides are the line γ, and the lines joining to $\alpha\beta$ the points where γ meets the curve. In like manner $Dx + Ey + F = 0$ is the equation of a line joining the two finite points where the curve is met by two lines drawn through the origin to meet the curve at infinity.

In general let the equation of a curve of any degree be written

$$u_n + u_{n-1}z + u_{n-2}z^2 + u_{n-3}z^3 + \&c. = 0,$$

(where we use the abbreviations u_n, u_{n-1}, &c. to denote terms of the n^{th}, $n-1^{st}$, &c. degrees). Now, if we seek the points where the line at infinity meets the curve, we have only to make $z=0$, when we obtain the equation $u_n=0$; hence we infer that the directions of the points at infinity on any curve are found by putting the highest terms of the equation $=0$.

Again, we saw (Art. 136), that, if $A=0$ in the equation of the second degree, the axis of x will meet the curve in one infinitely distant point. The same thing appears, by making $y=0$ in the equation, which will then reduce to

$$Dxz + Fz^2 = 0.$$

The axis, therefore, meets the curve, not only in the finite point where it meets the line $(Dx+F)$, but also in the point at infinity where it meets the line z.

In like manner, if both A and $D=0$, the points where the axis meets the curve are given by the equation $Fz^2=0$; hence, the axis meets the curve in two coincident points at infinity, and is, therefore, an asymptote.

272. We shall commence our examples of the use of trilinear co-ordinates with the equation (Art. 254) of a conic section, referred to two tangents and their chord of contact,

$$LM = R^2,$$

and shall first show how to express the equation of any line connected with the conic in terms of L, M, R.

We can express the position of any point on the curve by a *single* variable (Art. 234); for if $\mu L = R$ be the equation of the line joining any point on the curve to (LR), then, substituting in the equation of the curve, we get

$$M = \mu R \text{ and } \mu^2 L = M$$

for the equations of the lines joining this point to (MR) and (LM): any two of these three equations, therefore, will determine a point on the curve. We shall call this point the point μ.

We can form, by Art. 59, the equation of the line joining two points on the curve μ and μ', and we get

$$\mu\mu' L - (\mu + \mu') R + M = 0,$$

an equation evidently satisfied by either of the suppositions

$$(\mu L = R,\ \mu R = M), \text{ or } (\mu' L = R,\ \mu' R = M).$$

If μ and μ' coincide, we find *the equation of the tangent*, viz.,

$$\mu^2L - 2\mu R + M = 0.$$

Hence, conversely, if the equation of a right line $(\mu^2L - 2\mu R + M = 0)$ contain an indeterminate quantity μ *in the second degree*, the right line will always touch a conic section ($LM = R^2$).

273. *Given four points of a conic, the anharmonic ratio of the pencil joining them to any fifth point is constant.*

The lines joining four points μ', μ'', μ''', μ'''' to any fifth point μ, are

$$\mu'(\mu L - R) + (M - \mu R) = 0, \quad \mu''(\mu L - R) + (M - \mu R) = 0,$$
$$\mu'''(\mu L - R) + (M - \mu R) = 0, \quad \mu''''(\mu L - R) + (M - \mu R) = 0,$$

and their anharmonic ratio is (Art. 55)

$$\frac{(\mu' - \mu'')(\mu''' - \mu'''')}{(\mu' - \mu''')(\mu'' - \mu'''')},$$

and is, therefore, independent of the position of the point μ.

We shall, for brevity, use the expression, "the anharmonic ratio of four points of a conic," when we mean the anharmonic ratio of a pencil joining those points to any fifth point on the curve.

274. *Four fixed tangents cut any fifth in points whose anharmonic ratio is constant.*

Let the fixed tangents be those at the points μ', μ'', μ''', μ''''; and the variable tangent that at the point μ; then the anharmonic ratio in question is the same as that of the pencil joining the four points of intersection to the point LM. Now if we eliminate R from the equations of any two tangents,

$$\mu^2L - 2\mu R + M = 0,$$
$$\mu'^2L - 2\mu' R + M = 0,$$

we obtain

$$\mu\mu' L - M = 0,$$

the equation of the line joining LM to the intersection of these two tangents. The anharmonic ratio in question is therefore that of the four lines,

$$\mu\mu' L - M = 0, \;\; \mu\mu'' L - M = 0, \;\; \mu\mu''' L - M = 0, \;\; \mu\mu'''' L - M = 0,$$

which by Art. 55 is

$$\frac{(\mu' - \mu'')(\mu''' - \mu'''')}{(\mu' - \mu''')(\mu'' - \mu'''')},$$

a result independent of μ. Hence too we see that the anharmonic ratio of four tangents is the same as that of their points of contact.

275. Since the equation of the line joining any point to (LM) is $\mu^2 L - M$, we see that the two points $+\mu$ and $-\mu$ lie on a right line passing through LM.

The expression given in the last Article for the anharmonic ratio of four points on a conic, μ', μ'', μ''', μ'''', remains unchanged, if we alter the sign of each of these quantities; hence we derive an important theorem, that *if we draw four lines through any point* LM, *the anharmonic ratio of four of the points* (μ', μ'', μ''', μ'''') *where these lines meet the conic, is equal to the anharmonic ratio of the other four points* ($-\mu'$, $-\mu''$, $-\mu'''$, $-\mu''''$) *where these lines meet the conic.*

The equation in this form enables us easily to investigate properties of two conic sections relating to the point of intersection of their common tangents. For, let L and M be common tangents to two conics, and their equations will be

$$LM - R^2 = 0, \qquad LM - R'^2 = 0.$$

A point of one conic may be said to *correspond* to a point of the other if the line joining them passes through (LM) the intersection of common tangents. This will be the case if they have the same μ, since the equation $\mu^2 L - M = 0$ does not involve R or R′. Points are said to correspond inversely if they have the same μ with opposite signs. The chord joining any two points of one conic is said to correspond to the chord joining the corresponding points of the other.

Corresponding lines must meet on one or other of the common chords of the curves (Art. 265).

The chords of intersection of $LM - R^2$ and $LM - R'^2$ are

$$R^2 - R'^2 = 0,$$

but

$$\mu\mu' L - (\mu + \mu') R + M = 0,$$
$$\mu\mu' L - (\mu + \mu') R' + M = 0,$$

evidently intersect on the common chord $R - R'$. If the lines correspond inversely they meet on the common chord $R + R'$, as will be seen by changing the signs of μ and μ' in the latter equation.

The anharmonic ratio of four points of one conic is equal to the anharmonic ratio of the four corresponding points of the other.

This useful theorem follows immediately from the expression for the anharmonic ratio of four points given in the last Article, and from the fact that corresponding points have the same μ.

276. *To find the equation of the polar of any point.*

Let the co-ordinates of the point substituted in the equation of either tangent through it give the result

$$\mu^2 L' - 2\mu R' + M' = 0.$$

Now, at the point of contact, $\mu^2 = \frac{M}{L}$, and $\mu = \frac{R}{L}$ (Art. 272). Therefore, the co-ordinates of the point of contact satisfy the equation

$$ML' - 2RR' + M'L = 0,$$

which is, therefore, that of the polar required.

We may sometimes express a point by the equations

$$aL - R = 0, \qquad bR - M = 0;$$

in this case, by exactly the same method, the equation of the polar is found to be

$$abL - 2aR + M = 0.$$

277. It is evident that if we were given any relation between the μ's of two points, we could find the envelope of the chord joining them, or the locus of the intersection of their tangents. One or two simple cases of this are worth mentioning. For example, if we were given the *product* of two μ's, $\mu\mu' = a$, then (Art. 274) the intersection of their tangents will lie on the right line $aL - M = 0$; and by substituting a for $\mu\mu'$ in the equation of the chord joining the points, we see that this chord must pass through the fixed point $(aL + M, R)$.

In general the chord joining two points,

$$\mu\mu' L - (\mu + \mu') R + M = 0,$$

will pass through a fixed point (Art. 50) if

$$a\mu\mu' - b(\mu + \mu') + c = 0,$$

where a, b, c are any constants; that is, if

$$\mu' = \frac{b\mu - c}{a\mu - b}.$$

If the *ratio* of two μ's be given, $\mu' = k\mu$, the equation of the chord becomes

$$k\mu^2L - (1 + k)\mu R + M = 0;$$

the chord must, therefore (Art. 272), always touch the conic

$$4kLM = (1 + k)^2R^2.$$

This property may be expressed in a more symmetrical form, as follows: "The chord joining the points $\mu \tan\phi$, $\mu \cot\phi$, will always touch the conic $LM \sin^2 2\phi = R^2$ at the point μ on that conic." It can be proved, in like manner, that "the locus of the intersection of tangents at the points $\mu \tan\phi$ and $\mu \cot\phi$, will be the conic $LM = R^2 \sin^2 2\phi$."

278. Since the expression for the anharmonic ratio of four points on a conic (Art. 273) remains unaltered, if we multiply each μ either by $\tan\phi$ or by $\cot\phi$, we obtain an important theorem: "*If two conics have double contact, the anharmonic ratio of four of the points in which any four tangents to the one meet the other, is the same as that of the other four points in which the four tangents meet the curve, and also the same as that of the four points of contact.*"*

Or, again: "If from four points of one of the conics pairs of tangents be drawn to the other, the anharmonic ratio of one set of points of contact is equal to the anharmonic ratio of the other set."

If, in the expression for the anharmonic ratio of four points (Art. 273), we substitute for each μ, $\dfrac{a + b\mu}{c + d\mu}$ (a, b, c, d being constants), the anharmonic ratio will remain unaltered. It will be found that this is the most general substitution we can make for μ, which will leave the anharmonic ratio unchanged. The chord joining μ, $\dfrac{a + b\mu}{c + d\mu}$, will envelope a conic having double contact with the given one. For its equation is

$$\mu(a + b\mu)L - \{(a + b\mu) + \mu(c + d\mu)\}R + (c + d\mu)M = 0,$$

or $\quad (bL - dR)\mu^2 + (aL - bR - cR + dM)\mu + cM - aR = 0,$

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* This extension of the theorem in page 226 was communicated to me by Mr. Townsend, who had obtained it geometrically.

a line always touching a conic whose equation can be written in the form

$$4(bc-ad)(LM-R^2)+\{aL+(b-c)R-dM\}^2=0,$$

and which, therefore, has double contact with the given conic. We may see, from Art. 277, that the touched conic will reduce to a point if $b=-c$.

Hence, "Given three chords of a conic, AA′, BB′, CC′; the envelope of a fourth chord DD′, such that the anharmonic ratio of ABCD is equal to that of A′B′C′D′, will be a conic having double contact with the given one."

279. We give now some examples of the application of the preceding formulæ to the investigation of questions relating to the *position* of lines (Art. 1). We suppress some formulæ relating to the magnitude of lines and angles, as, where these are concerned, it is in general more advantageous to use ordinary rectangular co-ordinates.

Ex. 1. A triangle is circumscribed to a given conic; two of its vertices move on fixed right lines: to find the locus of the third.

Let us take for lines of reference the two tangents through the intersection of the fixed lines, and their chord of contact. Let the equations of the fixed lines be

$$aL-M=0,\quad bL-M=0,$$

while that of the conic is $\qquad LM-R^2=0.$

Now we proved (Art. 277) that two tangents which meet on $aL-M$ must have the product of their μ's $=a$; hence, if one side of the triangle touch at the point μ, the others will touch at the points $\frac{a}{\mu}$, $\frac{b}{\mu}$, and their equations will be

$$\frac{a^2}{\mu^2}L-2\frac{a}{\mu}R+M=0,\quad \frac{b^2}{\mu^2}L-2\frac{b}{\mu}R+M=0,$$

μ can easily be eliminated from the last two equations, and the locus of the vertex is found to be

$$LM=\frac{4ab}{(a+b)^2}R^2,$$

the equation of a conic having double contact with the given one along the line R.

Ex. 2. To find the envelope of the base of a triangle, inscribed in a conic, and whose two sides pass through fixed points.

Take the line joining the fixed points for R, let the equation of the conic be $LM=R^2$, and those of the lines joining the fixed points to LM be

$$aL+M=0,\quad bL+M=0.$$

Now, it was proved (Art. 277) that the extremities of any chord passing through $(aL+M, R)$, must have the product of their μ's $=a$.

Hence, if the vertex be μ, the base angles must be $\frac{a}{\mu}$ and $\frac{b}{\mu}$, and the equation of the base must be

$$abL - (a + b)\mu R + \mu^2 M = 0.$$

The base must, therefore (Art. 272), always touch the conic

$$LM = \frac{(a + b)^2}{4ab} R^2,$$

a conic having double contact with the given one along the line joining the given points.

Ex. 3. To inscribe in a conic section a triangle whose sides pass through three given points.

Two of the points being assumed, as in the last Example, we saw that the equation of the base must be $$abL - (a + b)\mu R + \mu^2 M = 0.$$

Now, if this line pass through the point $cL - R = 0$, $dR - M = 0$, we must have

$$ab - (a + b)\mu c + \mu^2 cd = 0,$$

an equation sufficient to determine μ.

Now, at the point μ we have $\mu L = R$, $\mu^2 L = M$; hence the co-ordinates of this point must satisfy the equation $$abL - (a + b) cR + cdM = 0.$$

The question, therefore, admits of two solutions, for either of the points in which this line meets the curve may be taken for the vertex of the required triangle.

The solution here given, although algebraically complete, has the disadvantage of not pointing out how to construct geometrically the line whose equation has just been given; it will be a useful exercise, however, on the preceding formulæ, if the student verify by this method the following construction, which we shall prove otherwise in the next chapter:—"Form the triangle whose sides are the polars of the three given points, join each point to the opposite vertex of this triangle, and the line joining the points in which two of these lines meet the opposite sides of the polar triangle will be the required line."

The three given points are

$$(aL + M, R), \quad (bL + M, R), \quad (cL - R, dR - M),$$

and the three polars, $aL - M$, $bL - M$, $cdL - 2cR + M$;

the three joining lines are

$$b(a + cd)L - 2c(a + b)R + (a + cd)M = 0,$$
$$a(b + cd)L - 2c(a + b)R + (b + cd)M = 0,$$
$$cdL - M = 0.$$

Now, the line whose equation we want to construct passes through the intersection of the first of these lines with $bL - M$, and of the second with $aL - M$.

Ex. 4. Mac Laurin's method of generating conic sections. The three sides of a triangle pass through three fixed points, and two vertices move on fixed lines, the third vertex will describe a conic section.

Let the triangle formed by the given points be L, M, N.

Let the given lines be $$L + aM + bN = 0, \quad (1)$$
$$L + a'M + b'N = 0. \quad (2)$$

Let the base of the triangle be $L = \mu M$. (3)

Substituting this value of L in (1) we find, for the equation of the line joining (1, 3) to (M, N),

$$(\mu + a)\,M + bN = 0.$$

In like manner, the line joining (2, 3) to (L, N) is

$$(\mu + a')\,L + \mu b'N = 0.$$

Eliminating μ from the last two equations, the equation of the locus is

$$a'LM = (aM + bN)\,(L + b'N).$$

The locus is, therefore, a conic passing through the points (L, N), (M, N), (L, 1), (M, 2).

Ex. 5. The base of a triangle touches a given conic, its extremities move on two fixed tangents to the conic, and the other two sides of the triangle pass through fixed points: find the locus of the vertex.

Let the fixed tangents be L, M, and the equation of the conic $LM = R^2$. Then the point of intersection of the line L with any tangent $(\mu^2L - 2\mu R + M)$ will have its coordinates L, R, M respectively proportional to 0, 1, 2μ. And (by Art. 59) the equation of the line joining this point to any fixed point $L'R'M'$ will be

$$LM' - L'M = 2\mu\,(LR' - L'R).$$

Similarly, the equation of the line joining the fixed point $L''R''M''$ to the point $(2, \mu, 0)$, which is the intersection of the line M with the same tangent, is

$$2\,(RM'' - R''M) = \mu\,(LM'' - L''M).$$

Eliminating μ, the locus of the vertex is found to be

$$(LM' - L'M)\,(LM'' - L''M) = 4\,(LR' - L'R)\,(RM'' - R''M),$$

the equation of a conic through the two given points.

Ex. 6. If in the last example the extremities of the base lie on any conic having double contact with the given conic, and passing through the given points, to find the locus of the vertex.

Let the conics be

$$LM - R^2 = 0, \quad LM - \frac{R^2}{\sin^2 2\phi} = 0,$$

then, if any line touch the latter at the point μ, it will, by Art. 277, meet the former in the points $\mu\tan\phi$ and $\mu\cot\phi$, and if the fixed points are μ', μ'', the equations of the sides are

$$\mu\mu'\tan\phi L - (\mu' + \mu\tan\phi)\,R + M = 0,$$
$$\mu\mu''\cot\phi L - (\mu'' + \mu\cot\phi)\,R + M.$$

Eliminating μ, the locus is found to be

$$(M - \mu'R)\,(\mu''L - R) = \tan^2\phi\,(M - \mu''R)\,(\mu'L - R).$$

FOCAL PROPERTIES.

280. We shall next discuss the equation $L^2 + M^2 - R^2 = 0$, which is one of great importance, and, as well as the equation $LM = R^2$, admits of our expressing the position of any point on the curve by a single indeterminate. We may suppose

$$L = R\cos\phi, \qquad M = R\sin\phi;$$

then (as at pp. 93, 198) the chord joining any two points is

$$L \cos \tfrac{1}{2}(\phi + \phi') + M \sin \tfrac{1}{2}(\phi + \phi') = R \cos \tfrac{1}{2}(\phi - \phi'),$$

and the tangent at any point is

$$L \cos \phi + M \sin \phi = R.$$

281. *The equation* $L^2 + M^2 - R^2 = 0$ *represents a conic such that any of the lines* L, M, R *is the polar with regard to it of the intersection of the other two.* For it may be written in any of the forms

$$L^2 = R^2 - M^2; \quad M^2 = R^2 - L^2; \quad R^2 = M^2 + L^2.$$

The first form shows that the lines $R + M$, $R - M$ (which intersect in RM) are tangents, and L their chord of contact; consequently RM is the pole of L. Similarly, the second form shows that RL is the pole of M. The third form shows that the imaginary lines $L + M\sqrt{-1}$, $L - M\sqrt{-1}$ (which intersect in the real point LM), are tangents, and R their chord of contact; consequently the point LM is in like manner the pole of R, but it lies inside the conic, since the tangents through it are imaginary.

It is evident in like manner that the equation

$$A\alpha^2 + B\alpha\beta + C\beta^2 = \gamma^2$$

denotes a conic such that the point $\alpha\beta$ is the pole with regard to it of the line γ; for the left-hand side of the equation can be resolved into the product of factors representing two lines which pass through $\alpha\beta$.

282. The most important application of the equation $L^2 + M^2 = R^2$ is in obtaining the properties of the foci. For if $x = 0$, $y = 0$, be any lines at right angles to each other through a focus, and $\gamma = 0$ the equation of the directrix, the equation of the curve is

$$x^2 + y^2 = e^2\gamma^2,$$

a particular form of the equation we are examining.

The form of the equation shows that the focus (xy) is the pole of the directrix γ, and that the polar of any point on the directrix is perpendicular to the line joining it to the focus (Art. 197), for y, the polar of $(x\gamma)$, is perpendicular to x, but x may be any line drawn through the focus.

The form of the equation shows that the two imaginary lines represented by the equation $(x^2 + y^2 = 0)$ are tangents drawn through the focus. Now, since these lines are the same whatever γ be, it appears that *all conics which have the same focus have two imaginary common tangents passing through this focus.* All conics, therefore, which have *both* foci common, have *four* imaginary common tangents, and may be considered as conics inscribed in the same quadrilateral. The imaginary tangents through the focus ($x^2 + y^2 = 0$) are the same as the lines drawn to the two imaginary points at infinity on any circle (see Art. 259). Hence we obtain the following general conception of foci, which we shall find useful afterwards: "Through each of the two imaginary points at infinity on any circle draw two tangents to the conic; these tangents will form a quadrilateral, two of whose vertices will be real and the foci of the curve, the other two may be considered as imaginary foci of the curve."

283. The tangents through (γ, x) to the curve are evidently $e\gamma + x$ and $e\gamma - x$. If, therefore, the curve be a parabola, $e = 1$; and the tangents are the internal and external bisectors of the angle (γx). Hence, "tangents to a parabola from any point on the directrix are at right angles to each other."

In general, since $x = e\gamma \cos\phi$, $y = e\gamma \sin\phi$, we have

$$\frac{y}{x} = \tan\phi;$$

or ϕ expresses the angle which any radius vector makes with x.

Hence we can find the envelope of a chord which subtends a constant angle at the focus, for the chord

$$x \cos\tfrac{1}{2}(\phi + \phi') + y \sin\tfrac{1}{2}(\phi + \phi') = e\gamma \cos\tfrac{1}{2}(\phi - \phi'),$$

if $\phi - \phi'$ be constant, must, by the present section, always touch

$$x^2 + y^2 = e^2\gamma^2 \cos^2\tfrac{1}{2}(\phi - \phi'),$$

a conic having the same focus and directrix as the given one.

284. The line joining the focus to the intersection of two tangents is found by subtracting

$$x\cos\phi + y\sin\phi - e\gamma = 0,$$
$$x\cos\phi' + y\sin\phi' - e\gamma = 0,$$

to be
$$x \sin\tfrac{1}{2}(\phi + \phi') - y \cos\tfrac{1}{2}(\phi + \phi') = 0,$$

the equation of a line making an angle $\tfrac{1}{2}(\phi + \phi')$ with the axis of x, and therefore *bisecting the angle between the focal radii.*

The line joining to the focus the point where the chord of contact meets the directrix is

$$x \cos \tfrac{1}{2}(\phi + \phi') + y \sin \tfrac{1}{2}(\phi + \phi') = 0,$$

a line evidently *at right angles to the last.*

To find the locus of the intersection of tangents at points which subtend a given angle 2δ at the focus.

By an elimination precisely the same as that in Ex. 1 and 2, p. 93, the equation of the locus is found to be $(x^2 + y^2)\cos^2\delta = e^2\gamma^2$, which represents a conic having the same focus and directrix as the given one, and whose eccentricity $= \dfrac{e}{\cos\delta}$.

If the curve be a parabola, the angle between the tangents is in this case given. For the tangent $(x\cos\phi + y\sin\phi - \gamma)$ bisects the angle between $x\cos\phi + y\sin\phi$ and γ. The angle between the tangents is, therefore, half the angle between $x\cos\phi + y\sin\phi$ and $x\cos\phi' + y\sin\phi'$, or $= \frac{1}{2}(\phi - \phi')$. Hence, *the angle between two tangents to a parabola is half the angle which the points of contact subtend at the focus;* and again, *the locus of the intersection of tangents to a parabola, which contain a given angle, is a hyperbola with the same focus and directrix, and whose eccentricity is the secant of the given angle,* or whose asymptotes contain double the given angle (Art. 170).

ENVELOPES.

285. We have seen that the line represented by the equation

$$\mu^2 L - 2\mu R + M = 0,$$

always touches the curve $LM = R^2$.

We wish the reader to take notice that this will be the case whether L, M, R represent *right* lines or not. For the equation

$$\mu\mu' L - (\mu + \mu') R + M = 0$$

must be satisfied for any points which satisfy the equations

$$(\mu L - R = 0,\ \mu R - M = 0), \quad (\mu' L - R = 0,\ \mu' R - M = 0),$$

and is therefore the equation of a curve passing through the points in which $\mu L - R$ and $\mu' L - R$ meet $LM - R^2$. Now let $\mu = \mu'$, and we see that $\mu^2 L - 2\mu R + M$ *touches* $LM - R^2$ in the points where $\mu L - R$ meets it.

Similar remarks apply to the equation

$$L\cos\phi + M\sin\phi = R,$$

which indeed may be reduced to the preceding form by assuming $\tan\frac{1}{2}\phi = \mu$, as we have then

$$\cos\phi = \frac{1-\mu^2}{1+\mu^2}, \quad \sin\phi = \frac{2\mu}{1+\mu^2},$$

and substituting these values, and clearing of fractions, we have an equation in which μ only enters in the second degree.

If, therefore, we are required to find the curve always touched by a variable line, we have only to form its equation so as to contain only a single indeterminate, and, if this indeterminate *be only in the second degree*, the envelope can be found as above. We can in like manner find the envelope of a line whose equation contains *two* indeterminates, provided these be connected by some given relation, for we have only to eliminate one of the indeterminates by the help of the given relation.

Ex. 1. To find the envelope of a line such that the product of the perpendiculars on it from two fixed points may be constant.

Take for axes the line joining the fixed points and a perpendicular through its middle point, so that the co-ordinates of the fixed points may be $y = 0$, $x = \pm c$; then if the variable line be $y - mx + n = 0$, we have by the conditions of the question

$$(n + mc)(n - mc) = b^2(1 + m^2),$$

or $$n^2 = b^2 + b^2m^2 + c^2m^2,$$

but $$n^2 = y^2 - 2mxy + m^2x^2,$$

therefore $$m^2(x^2 - b^2 - c^2) - 2mxy + y^2 - b^2 = 0;$$

and the envelope is $$x^2y^2 = (x^2 - b^2 - c^2)(y^2 - b^2),$$

or $$\frac{x^2}{b^2+c^2} + \frac{y^2}{b^2} = 1.$$

Ex. 2. Find the envelope of a line such that the sum of the squares of the perpendiculars on it from two fixed points may be constant. *Ans.* $\frac{x^2}{b^2-c^2} + \frac{y^2}{b^2} = 1.$

Ex. 3. Find the envelope if the difference of squares of perpendiculars be given. *Ans.* A parabola.

Ex. 4. Through a fixed point O any line OP is drawn to meet a fixed line; to find the envelope of PQ drawn so as to make the angle OPQ constant.

Let OP make the angle θ with the perpendicular on the fixed line, and its length is $p\sec\theta$; but the perpendicular from O on PQ makes a fixed angle β with OP, therefore its length is $= p\sec\theta\cos\beta$; and since this perpendicular makes an angle $= \theta + \beta$ with the perpendicular on the fixed line, if we assume the latter for the axis of x, the equation of PQ is

$$x\cos(\theta+\beta) + y\sin(\theta+\beta) = p\sec\theta\cos\beta,$$

or $$x\cos(2\theta+\beta)+y\sin(2\theta+\beta)=2p\cos\beta-x\cos\beta-y\sin\beta,$$

an equation of the form $$L\cos\phi+M\sin\phi=R,$$

whose envelope, therefore, is

$$x^2+y^2=(x\cos\beta+y\sin\beta-2p\cos\beta)^2,$$

the equation of a parabola having the point O for its focus.

Ex. 5. To find the envelope of the line $\frac{A}{\mu}+\frac{B}{\mu'}=1$, where the indeterminates are connected by the relation $\mu+\mu'=C$.

We may substitute for μ', $C-\mu$, and clear of fractions; the envelope is thus found to be

$$A^2+B^2+C^2-2AB-2AC-2BC=0,$$

an equation to which the following form will be found to be equivalent,

$$\pm\sqrt{A}\pm\sqrt{B}\pm\sqrt{C}=0.$$

Thus, for example,—Given vertical angle and sum of sides of a triangle, to find the envelope of base.

The equation of the base is $$\frac{x}{a}+\frac{y}{b}=1,$$

where $a+b=c$.

The envelope is, therefore,

$$x^2+y^2-2xy-2cx-2cy+c^2=0,$$

a parabola touching the sides x and y.

In like manner,—Given in position two conjugate diameters of an ellipse, and the sum of their squares, to find its envelope.

If in the equation $$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$$

we have $a'^2+b'^2=c^2$, the envelope is

$$x\pm y\pm c=0.$$

The ellipse, therefore, must always touch four fixed right lines.

Ex. 6. Again,* given the two equations

$$\frac{\mu}{A}+\frac{\mu'}{B}+\frac{\mu''}{C}=0,\quad \sqrt{(\mu a)}+\sqrt{(\mu' b)}+\sqrt{(\mu'' c)}=0,$$

if we eliminate μ'', the equation in $\frac{\mu}{\mu'}$ will be only of the second order, and the envelope will be found to be $$Aa+Bb+Cc=0.\dagger$$

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* This example, and its applications, are taken from Mr. Hearn's *Researches on Conic Sections*.

† In general, given the two equations

$$(\mu A)^m+(\mu' B)^m+(\mu'' C)^m=0,\quad (\mu a)^n+(\mu' b)^n+(\mu'' c)^n=0,$$

it can be proved that the envelope is

$$\left(\frac{a}{A}\right)^{\frac{mn}{m-n}}+\left(\frac{b}{B}\right)^{\frac{mn}{m-n}}+\left(\frac{c}{C}\right)^{\frac{mn}{m-n}}=0.$$

Thus, for example, in the equation of a conic circumscribing a triangle,

$$\frac{\mu}{\alpha} + \frac{\mu'}{\beta} + \frac{\mu''}{\gamma} = 0$$

(Art. 105), if the constants be connected by the relation

$$\sqrt{(\mu a)} + \sqrt{(\mu' b)} + \sqrt{(\mu'' c)} = 0,$$

the conic will touch the right line

$$a\alpha + b\beta + c\gamma = 0.$$

Or, again, in the equation of a conic inscribed in a triangle,

$$\sqrt{(\mu\alpha)} + \sqrt{(\mu'\beta)} + \sqrt{(\mu''\gamma)} = 0$$

(Art. 108), if the constants be connected by the relation

$$\frac{\mu}{A} + \frac{\mu'}{B} + \frac{\mu''}{C} = 0,$$

the conic will touch the right line

$$A\alpha + B\beta + C\gamma = 0.$$

286. These principles enable us to write the equation of a conic having double contact with two given conics, S and S′. Let E and F be their chords of intersection, so that $S - S' = EF$, then the equation of any conic touching the two will be

$$\mu^2E^2 - 2\mu(S + S') + F^2 = 0.$$

For, if we seek the envelope of this conic, we find

$$E^2F^2 - (S + S')^2 = 0, \text{ or } 4SS' = 0;$$

hence this conic touches both the given ones.

Since μ is of the second degree, we see that through any point can be drawn *two* conics, each of which will have double contact with the given ones; and it can be proved that one of the chords of intersection of these conics is the line joining the given point to (EF), and the other the fourth harmonic to this line, E and F.

287. The equation of a conic having double contact with two *circles* assumes a simpler form, viz.

$$\mu^2 - 2\mu(C + C') + (C - C')^2 = 0.$$

The chords of contact of the conic with the circles are found to be

$$C - C' + \mu = 0, \text{ and } C - C' - \mu = 0,$$

which are, therefore, parallel to each other, and equidistant from the radical axis of the circles. This equation may also be written in the form

$$\sqrt{C} \pm \sqrt{C'} = \sqrt{\mu}.$$

Hence, *the locus of a point, the sum or difference of whose tangents to two given circles is constant, is a conic having double contact with the two circles.* If we suppose both circles infinitely small, we obtain the fundamental property of the foci of the conic.

If μ be taken equal to the intercept between the circles on one of their common tangents, the equation denotes a pair of common tangents to the circles.

Ex. 1. Solve by this method the Examples (p. 110) of finding common tangents to circles. *Ans.* Ex. 1. $\sqrt{C} + \sqrt{C'} = 4$ or $= 2$. *Ans.* Ex. 2. $\sqrt{C} + \sqrt{C'} = 1$ or $= \sqrt{-80}$.

Ex. 2. Given three circles; let L, L' be the common tangents to C', C''; M, M' to C'', C; N, N' to C, C'; then if L, M, N meet in a point, so will L', M', N'.

Let the equations of the pairs of common tangents be

$$\sqrt{C'} + \sqrt{C''} = t, \quad \sqrt{C''} + \sqrt{C} = t', \quad \sqrt{C} + \sqrt{C'} = t''.$$

Then the condition that L, M, N should meet in a point is $t' \pm t = t''$; and it is obvious that when this condition is fulfilled, L', M', N' also meet in a point.

288. The equation of a conic inscribed in a *quadrilateral* is found as a particular case of Art. 286, and is

$$\mu^2 E^2 - 2\mu (AC + BD) + F^2 = 0,$$

where ABCD are the sides, EF the diagonals, and $AC - BD = EF$. This equation, however, will assume a simple form if expressed in terms of the three diagonals of the quadrilateral. Let L, M, N represent the diagonals, then (1) $L + M + N$, (2) $M + N - L$, (3) $L - M + N$, (4) $L + M - N$, represent the four sides; for L passes through the intersections of (12), (34); M through those of (13), (24); N through those of (14), (23); and the equation of the conic touching the four sides may be written

$$\mu^2 L^2 - \mu (L^2 + M^2 - N^2) + M^2 = 0.$$

For this always touches $(L^2 + M^2 - N^2)^2 - 4L^2M^2 =$

$$(L + M + N)(M + N - L)(L - M + N)(L + M - N).$$

The equation of the touching conic may be written

$$L^2 = \frac{M^2}{\mu} + \frac{N^2}{1 - \mu}.$$

Ex. 1. Find the equation of the conic touching the four sides of the quadrilateral whose equations are given (Ex. 8, p. 27).

It will be seen that we have here

$$L = \left(\frac{1}{a} + \frac{1}{a'}\right)x, \quad M = \left(\frac{1}{b} + \frac{1}{b'}\right)y, \quad N = \left(\frac{1}{a} - \frac{1}{a'}\right)x + \left(\frac{1}{b} - \frac{1}{b'}\right)y - 2.$$

And the conic is

$$\left\{\left(\frac{1}{a}-\frac{1}{a'}\right)x+\left(\frac{1}{b}-\frac{1}{b'}\right)y-2\right\}^2=\left(\frac{1}{a}+\frac{1}{a'}\right)^2\frac{x^2}{\mu}+\left(\frac{1}{b}+\frac{1}{b'}\right)^2\frac{y^2}{1-\mu}.$$

Ex. 2. Find the locus of the centre of the conic touching four right lines.

The centre of the conic whose equation is given in the last example is determined by the equations,

$$\left(\frac{1}{a}-\frac{1}{a'}\right)\left\{\left(\frac{1}{a}-\frac{1}{a'}\right)x+\left(\frac{1}{b}-\frac{1}{b'}\right)y-2\right\}=\left(\frac{1}{a}+\frac{1}{a'}\right)^2\frac{x}{\mu},$$

$$\left(\frac{1}{b}-\frac{1}{b'}\right)\left\{\left(\frac{1}{a}-\frac{1}{a'}\right)x+\left(\frac{1}{b}-\frac{1}{b'}\right)y-2\right\}=\left(\frac{1}{b}+\frac{1}{b'}\right)^2\frac{y}{1-\mu}.$$

Eliminating μ, we have

$$\left(\frac{1}{a}-\frac{1}{a'}\right)x+\left(\frac{1}{b}-\frac{1}{b'}\right)y-2=\frac{(a+a')^2}{aa'(a'-a)}x+\frac{(b+b')^2}{bb'(b'-b)}y,$$

or

$$\frac{2x}{a-a'}+\frac{2y}{b-b'}=1,$$

the equation of the line joining the middle points of the diagonals.

GENERAL EQUATION OF THE SECOND DEGREE.

289. We have already seen that the general trilinear equation of the second degree is

$$A\alpha^2+B\alpha\beta+C\beta^2+D\alpha\gamma+E\beta\gamma+F\gamma^2=0,$$

which for the sake of symmetry we shall write in the form

$$a\alpha^2+a'\beta^2+a''\gamma^2+2b\beta\gamma+2b'\gamma\alpha+2b''\alpha\beta=0.$$

This equation is evidently equivalent to the equation

$$(a\alpha+b'\gamma+b''\beta)^2+(aa'-b''^2)\beta^2+2(ab-b'b'')\beta\gamma+(aa''-b'^2)\gamma^2=0,$$

but the last three terms are the equation of two right lines drawn through $(\beta\gamma)$; hence (Art. 281) $a\alpha+b'\gamma+b''\beta$ is the chord of contact of two tangents drawn through $(\beta\gamma)$, that is to say, the *polar* of the point $(\beta\gamma)$.

In like manner, the polars of $(\gamma\alpha)$ and $(\alpha\beta)$ are

$$a'\beta+b\gamma+b''\alpha=0,\qquad a''\gamma+b\beta+b'\alpha=0.$$

290. The form of the equation of the tangents through $(\beta\gamma)$ leads to an important property of the sides of a circumscribing hexagon, and affords a useful test for determining whether six lines touch a conic.

The tangents are

$$(aa'-b''^2)\beta^2+2(ab-b'b'')\beta\gamma+(aa''-b'^2)\gamma^2=0,$$
$$(a'a''-b^2)\gamma^2+2(a'b'-b''b)\gamma\alpha+(aa'-b''^2)\alpha^2=0,$$
$$(a''a-b'^2)\alpha^2+2(a''b''-bb')\alpha\beta+(a'a''-b^2)\beta^2=0.$$

Now, if the roots of the first equation be $\beta = k\gamma$, $\beta = k'\gamma$, we have

$$kk' = \frac{aa'' - b'^2}{aa' - b''^2}.$$

The corresponding quantities for the other equations are $\frac{aa' - b''^2}{a'a'' - b^2}$ and $\frac{a'a'' - b^2}{a''a - b'^2}$, and these three multiplied together are = 1. Now, recollecting the meaning of k (Art. 53) we learn, that if A, F, B, D, C, E, be the vertices of a circumscribing hexagon,

$$\frac{\sin EAB \,.\, \sin FAB \,.\, \sin FBC \,.\, \sin DBC \,.\, \sin DCA \,.\, \sin ECA}{\sin EAC \,.\, \sin FAC \,.\, \sin FBA \,.\, \sin DBA \,.\, \sin DCB \,.\, \sin ECB} = 1.$$

Hence, also, if the equations of three pairs of lines can be put into the form

$$L^2 + M^2 - 2n'LM = 0,$$
$$M^2 + N^2 - 2l'MN = 0,$$
$$N^2 + L^2 - 2m'NL = 0,$$

they will touch the same conic section, for the equations last given can be reduced to this form by writing $\sqrt{(a'a'' - b^2)}\,L$ for α, &c.

291. It appears from Art. 289 that the equation of the polar of any point $(\beta\gamma)$, with regard to the conic, $S = 0$, is the *first derived equation* of $S = 0$, *considered as a function of* α. We shall anticipate the notation of the calculus, and denote this derived equation by $\frac{dS}{d\alpha}$.

In like manner, the polar of $(\alpha\gamma)$, with regard to S, is the first derived equation of S, considered as a function of β, $= \frac{dS}{d\beta}$, and the polar of $\alpha\beta$ is $\frac{dS}{d\gamma}$. Hence, *if the equation of a conic be expressed in terms of the equations of three right lines, the equation of the polar of the intersection of any two of them is the first derived of the equation of the conic, considered as a function of the third line.* The equations of polars given already are particulars cases of this. For example, the polar of the origin (xy), with regard to

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0,$$

is

$$Dx + Ey + 2Fz = 0;$$

that is, its first derived equation with regard to z.

Again, the equation of the diameter which bisects chords parallel to the axis of x is

$$\frac{dS}{dx} = 0, \quad \text{or} \quad 2Ax + By + Dz = 0,$$

and we shall show hereafter that this diameter may be considered as the polar of the point (yx) at infinity on the axis of x.

Ex. 1. Given four points on a conic, the polar of any other given point will pass through a fixed point (Ex. 3, p. 137).

The equation of the conic must be of the form $S + kS' = 0$, where S and S' are any two conics through the four points: now the polar of any point $\beta\gamma$ with regard to this is $\frac{d(S + kS')}{d\alpha} = 0$, which, it will be seen, is equivalent to

$$\frac{dS}{d\alpha} + k\frac{dS'}{d\alpha} = 0;\text{*}$$

and since this equation only involves k in the first degree, it will pass through a fixed point.

Ex. 2. To find the locus of the pole of a given line (γ), with regard to a conic of which four points are given.

We have to eliminate k from the equations

$$\frac{dS}{d\alpha} + k\frac{dS'}{d\alpha} = 0, \quad \frac{dS}{d\beta} + k\frac{dS'}{d\beta} = 0,$$

and we find

$$\frac{dS}{d\alpha}\frac{dS'}{d\beta} - \frac{dS}{d\beta}\frac{dS'}{d\alpha} = 0,$$

the equation of a conic section.

If we suppose the given line at an infinite distance, we obtain the locus of the centres (Ex. 4, p. 137).

Ex. 3. Given two points and two tangents to a conic, the polar of a fixed point touches a conic section.

Let LM be the two tangents, R the line joining the given points, and $LM - N^2$ one conic touching the two lines, and passing through the given points; then the equation of any other must be of the form

$$LM - (N + kR)^2 = 0;$$

the polar is, therefore,

$$2k^2R\frac{dR}{d\alpha} + 2k.\frac{d(NR)}{d\alpha} + \frac{d(N^2 - LM)}{d\alpha} = 0,$$

which must always touch a conic section, since k enters in the second degree. In the

* We may mention here, that if the axes of S be parallel to the axes of S', so will the axes of $S + kS'$; for if we take the axes of S for axes of co-ordinates, neither S nor S' will contain the term xy. If S' be a circle, the axes of $S + kS'$ must be always parallel to the axes of S. If $S + kS'$ reduce to a pair of right lines, its axes will become the internal and external bisectors of the angles between these right lines: thus we obtain the theorem of p. 208.

same manner it may be proved that the locus of the pole of a given line is a conic section.

In general, if the equation of a conic section involve an indeterminate in the second degree, the polar of any fixed point will touch a conic section. Thus, for example, the locus of centres of conic sections which have double contact with two given conics (Art. 286) is a conic section.

292. *To find the equation of the polar of any point* $(\alpha'\beta'\gamma')$, *with regard to a conic section.*

This may be done by a method similar to that used Art. 150. It is proved, as in Art. 7, that if α', α'' be the lengths of the perpendiculars from two points upon a given line, $\frac{l\alpha'' + m\alpha'}{l+m}$ will be the length of the perpendicular on that line from the point which divides in the ratio $l:m$ the line joining the given points. But since equations in trilinear co-ordinates are always homogeneous, they are not affected if the co-ordinates of any point be all multiplied or divided by the same quantity. Hence $l\alpha'' + m\alpha'$, $l\beta'' + m\beta'$, $l\gamma'' + n\gamma'$, may be taken as the trilinear co-ordinates of the point dividing in the ratio $l:m$ the line joining $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$. If then we substitute these values in the general equation $S = 0$, we have, to determine the points where this conic is met by the line joining $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$, the quadratic

$$l^2\{a\alpha''^2 + a'\beta''^2 + a''\gamma''^2 + 2b\beta''\gamma'' + 2b'\gamma''\alpha'' + 2b''\alpha''\beta''\}$$
$$+ 2lm\{(a\alpha''+b'\gamma''+b''\beta'')\alpha'+(a'\beta''+b\gamma''+b''\alpha'')\beta'+(a''\gamma''+b\beta''+b'\alpha'')\gamma'\}$$
$$+ m^2\{a\alpha'^2 + a'\beta'^2 + a''\gamma'^2 + 2b\beta'\gamma' + 2b'\gamma'\alpha' + 2b''\alpha'\beta'\} = 0.$$

Now, as in Art. 150, when $\alpha''\beta''\gamma''$ is on the polar of $\alpha'\beta'\gamma'$, the coefficient of lm must vanish, since we know that the line joining the points must in this case be cut harmonically; the equation of the polar of $\alpha'\beta'\gamma'$ is, therefore,

$$(a\alpha + b'\gamma + b''\beta)\alpha' + (a'\beta + b\gamma + b''\alpha)\beta' + (a''\gamma + b\beta + b'\alpha)\gamma' = 0,$$

which we may write for shortness

$$\alpha'\frac{dS}{d\alpha} + \beta'\frac{dS}{d\beta} + \gamma'\frac{dS}{d\gamma} = 0.$$

When $\alpha'\beta'\gamma'$ is on the curve, this equation, of course, represents the tangent at that point.

Ex. 1. To find the equation of the pair of tangents at the points where the conic S is cut by the line γ.

The γ' of either point of contact will in this case $= 0$, and the equation of the tangent at it will become

$$\alpha'\frac{dS}{d\alpha} + \beta'\frac{dS}{d\beta} = 0.$$

But making $\gamma = 0$ in the general equation, the points of contact are determined by the equation

$$a\alpha'^2 + 2b''\alpha'\beta' + a'\beta'^2 = 0.$$

Eliminating $\alpha'\beta'$ between these equations, we find for the equation of the pair of tangents

$$a\left(\frac{dS}{d\beta}\right)^2 - 2b''\left(\frac{dS}{d\beta}\right)\left(\frac{dS}{d\alpha}\right) + a'\left(\frac{dS}{d\alpha}\right)^2 = 0.$$

As a particular case of this, we find for the equation of the asymptotes of a conic given by its Cartesian equation (since the asymptotes are the pair of tangents at the points where the curve is met by z the line at infinity),

$$A\left(\frac{dS}{dy}\right)^2 - B\left(\frac{dS}{dy}\right)\left(\frac{dS}{dx}\right) + C\left(\frac{dS}{dx}\right)^2 = 0.$$

Ex. 2. The lines joining corresponding vertices of any triangle, and of its conjugate triangle with respect to a conic, meet in a point. By the conjugate triangle is understood the triangle whose sides are the polars of the vertices of the first triangle (see Ex. 3, p. 230).

It is obvious that the result of substituting the co-ordinates of any point (1) in the equation of the polar of (2) is the same as the result of substituting the co-ordinates of (2) in the polar of (1). Let us denote this result by t'''; and, in like manner, let t' denote the result of substituting the co-ordinates of (2) in the equation of the polar of (3), and t'' the result of substituting the co-ordinates of (3) in the polar of (1). Let the equations of the three polars of the vertices of the first triangle be $P' = 0$, $P'' = 0$, $P''' = 0$. Then the equation of any line through the intersection of the last two lines will be $P'' = kP'''$, and if this line pass through the point (1), the co-ordinates of this point substituted in the last equation give $t''' = kt''$. Hence the equations of the three lines joining corresponding vertices are

$$t'P' = t''P'',\quad t''P'' = t'''P''',\quad t'''P''' = t'P',$$

which obviously meet in a point.

Ex. 3. The intersections of corresponding sides of two conjugate triangles lie in one right line.

We can (by Art. 59) write in the form $lP' + mP'' + nP''' = 0$, the equation of the line joining $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$. Remembering that the co-ordinates of the first point substituted in P', P'', P''' give results, s', t''', t''; while those of the second point give results, t''', s'', t'; the equation of the joining line is found to be

$$(t't''' - s''t'')\,P' + (t''t''' - s't')\,P'' + (s's'' - t'''^2)\,P''' = 0.$$

Similarly the equations of the other sides of the first triangle are

$$(t't'' - s'''t'')\,P' + (s's''' - t''^2)\,P'' + (t''t''' - s't')\,P''' = 0,$$
$$(s''s''' - t'^2)\,P' + (t't'' - s'''t'')\,P'' + (t't''' - s''t')\,P''' = 0.$$

And the intersections of corresponding sides of the two triangles lie on the right line

$$\frac{P'}{t''t''' - s't'} + \frac{P''}{t't''' - s''t''} + \frac{P'''}{t't'' - s'''t'''} = 0.$$

Ex. 4. The anharmonic ratio of four points on a right line is the same as that of their four polars.

For the anharmonic ratio of the four points

$$l\alpha' + m\alpha'', \; l'\alpha' + m'\alpha'', \; l''\alpha' + m''\alpha'', \; l'''\alpha' + m'''\alpha'',$$

is evidently the same as that of the four lines

$$lP' + mP'', \; l'P' + m'P'', \; l''P' + m''P'', \; l'''P' + m'''P''.$$

Ex. 5. To express the equation of the conic S in terms of P′, P″, P‴.

From the general principles of trilinear co-ordinates, it follows that the equation of the conic can be expressed in the form

$$AP'^2 + A'P''^2 + A''P'''^2 + 2BP''P''' + 2B'P'''P' + 2B''P'P'' = 0.$$

Now the equation of the polar of any point being formed according to the rules of Art. 292, the equation of the polar of $\alpha'\beta'\gamma'$, whose co-ordinates in this system are s', t''', t'', is

$$(As' + B't'' + B''t''')\,P' + (A't''' + Bt'' + B''s')\,P'' + (A''t'' + Bt''' + B's')\,P''' = 0\,;$$

and since the polar of this point is P′, we must have

$$A't''' + Bt'' + B''s' = 0, \quad A''t'' + Bt''' + B's' = 0.$$

In like manner, we have

$$At''' + B't' + B''s'' = 0, \quad A''t' + Bs'' + B't''' = 0,$$
$$At'' + B's''' + B''t' = 0, \quad A't' + Bs''' + B''t'' = 0.$$

These equations are sufficient to determine the six unknown quantities A, A′, &c., and we find for the equation of the conic,

$$(s''s''' - t'^2)\,P'^2 + (s's''' - t''^2)\,P''^2 + (s's'' - t'''^2)\,P'''^2 + 2\,(t''t''' - t's')\,P''P'''$$
$$+ 2\,(t'''t' - t''s'')\,P'''P' + 2\,(t't'' - t'''s''')\,P'P'' = 0.$$

Ex. 6. To inscribe in a conic a triangle whose sides pass through three given points.

Let $\alpha\beta\gamma$ be the co-ordinates of the vertex of the triangle; we find, as in Art. 150, the co-ordinates of the point where the line joining $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ meets the conic again, by substituting in the equation of the curve $l\alpha + m\alpha'$, $l\beta + m\beta'$, $l\gamma + m\gamma'$, for α, β, γ. And since $\alpha\beta\gamma$ is on the curve, this gives us

$$2lmP' + m^2 s' = 0\,; \quad \frac{m}{l} = -\frac{2P'}{s'}\,;$$

and the co-ordinates of the point required are $s'\alpha - 2P'\alpha'$, $s'\beta - 2P'\beta'$, $s'\gamma - 2P'\gamma'$. If these values be substituted in P′, P″, P‴, they give results,

$$-P's', \; s'P'' - 2t'''P', \; s'P''' - 2t''P'.$$

Similarly, the co-ordinates in the same system of the point where the line joining $\alpha\beta\gamma$, $\alpha''\beta''\gamma''$, meets the conic again, are $s''P' - 2t'''P''$, $-s''P''$, $s''P''' - 2t'P''$. The condition that these points should lie in a right line with t'', t', s''', is

$$P'^2 s''\,(s'''t'' - t't') + P''^2 s'\,(s'''t' - t't'') + P'P''\,(4t't''t''' - 2s'''t'''^2 - s''t''^2 - s't'^2)$$
$$+ P'P'''s'\,(s''t'' - t't''') + P''P'''s''\,(s't' - t''t''') = 0.$$

The vertex of the required triangle is thus determined as the intersection of the given conic with another conic, but the solution assumes a simple form if we subtract from the equation just found the equation of the conic given in the last Example multiplied by t''', when we get

$$(P't' + P''t'' - P'''t''')\,\{P'(t't''' - s''t') + P''(t''t''' - s't') + P'''(s's'' - t'''^2)\} = 0.$$

It is obvious, from Ex. 2, that the first factor in this product represents the same right line as that described in the solution of the same problem given at p. 230. The second factor is irrelevant to the geometrical solution: for it represents (see Ex. 3) the line joining the points $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$; and though either of the points in which this line meets the curve fulfils the condition which we have expressed analytically, namely, that if it be joined to $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$, the points in which the joining lines meet the curve lie on a right line which passes through $\alpha'''\beta'''\gamma'''$; yet as the joining lines coincide, they cannot be sides of a triangle.

Ex. 7. If two conics have double contact, any tangent to the one is cut harmonically at its point of contact, the points where it meets the other, and where it meets the chord of contact.

If we substitute in the equation $S + R^2 = 0$, $l\alpha' + m\alpha''$, $l\beta' + m\beta''$, $l\gamma' + m\gamma''$, for $\alpha\beta\gamma$ (where the points $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ satisfy the equation $S = 0$), we get

$$(lR' + mR'')^2 + 2lmt'' = 0.$$

Now, if the line joining $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$, touch $S + R^2$, this equation must be a perfect square: and it is evident that the only way this can happen is if $t'' = -2R'R''$, when the equation becomes $(lR' - mR'')^2 = 0$; whence the truth of the theorem is manifest.

INSCRIBED AND CIRCUMSCRIBED TRIANGLES.

293. We gave (p. 99) the equation of a conic circumscribed about a triangle,*

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

we may prove, precisely as at p. 100, that the tangents at the three vertices are

$$l\beta + m\alpha = 0, \quad m\gamma + n\beta = 0, \quad n\alpha + l\gamma = 0;$$

that the three points in which each tangent meets the opposite side are in one right line,

$$\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} = 0;$$

and that the lines joining each vertex to the opposite vertex of the circumscribed triangle are

$$\frac{\alpha}{l} - \frac{\beta}{m} = 0, \quad \frac{\beta}{m} - \frac{\gamma}{n} = 0, \quad \frac{\gamma}{n} - \frac{\alpha}{l} = 0,$$

which evidently meet in a point.

To find the equation of a conic circumscribing $\alpha\beta\gamma$, and having its centre at a given point $(\alpha'\beta'\gamma')$.

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* This equation was, I believe, first discussed by M. Bobillier (*Annales de Mathématiques*, vol. xviii. p. 320).

The polar of any point is (Art. 292)

$$\alpha'(m\gamma + n\beta) + \beta'(n\alpha + l\gamma) + \gamma'(l\beta + m\alpha) = 0.$$

Now it is required to determine lmn, so that this equation should represent a line at an infinite distance (Art. 157).

Comparing this equation, therefore, with the equation of a line at infinity (Art. 64),

$$a\alpha + b\beta + c\gamma = 0,$$

where abc are the lengths of the sides of the triangle $\alpha\beta\gamma$, it will be found that we may take

$$l = \alpha'(b\beta' + c\gamma' - a\alpha');\ m = \beta'(a\alpha' + c\gamma' - b\beta');\ n = \gamma'(a\alpha' + b\beta' - c\gamma').$$

In like manner we could determine l, m, n, so that the polar of $(\alpha'\beta'\gamma')$ should be any right line, $A\alpha + B\beta + C\gamma$, by writing A, B, C, for a, b, c.

If we were given three points on a conic and any fourth condition, this fourth condition will give a relation between l, m, n; then, by writing in this relation the values of l, m, n, just found, we can find the locus of centres of the conic, or the locus of the poles of a given line.* Thus, for example, if we are given a fourth point on the conic, we must have

$$\frac{l}{\alpha''} + \frac{m}{\beta''} + \frac{n}{\gamma''} = 0,$$

and therefore the locus of the centre of the conic circumscribing a quadrilateral is

$$\frac{\alpha(b\beta + c\gamma - a\alpha)}{\alpha''} + \frac{\beta(a\alpha + c\gamma - b\beta)}{\beta''} + \frac{\gamma(a\alpha + b\beta - c\gamma)}{\gamma''} = 0,$$

a conic through the middle points of the given quadrilateral; for $a\alpha + b\beta - c\gamma$ represents the line joining the middle points of $\alpha\beta$, &c.

If we are given a tangent to the conic we must have

$$\sqrt{(lA)} + \sqrt{(mB)} + \sqrt{(nC)} = 0,$$

in order that the conic should touch

$$A\alpha + B\beta + C\gamma = 0 \text{ (p. 237)},$$

* The method given in this and the following Article, of finding the locus of the centre of a conic section described under certain conditions, is taken from Mr. Hearn's *Researches on Conic Sections.*

therefore the locus of centre, three points and a tangent being given, is

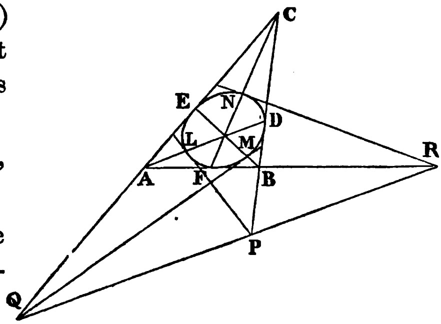
$$\sqrt{\{A\alpha(b\beta + c\gamma - a\alpha)\}} + \sqrt{\{B\beta(a\alpha + c\gamma - b\beta)\}} + \sqrt{\{C\gamma(b\beta + a\alpha - c\gamma)\}} = 0,$$

a curve in general of the fourth degree.

294. The equation of the conic section *inscribed* in a triangle may be written in either of the forms (Art. 108)

$$\sqrt{(l\alpha)} + \sqrt{(m\beta)} + \sqrt{(n\gamma)} = 0,$$

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0.$$



It was proved (Art. 109) that AD, BE, CF meet in a point, their equations being

$$m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0, \quad l\alpha - m\beta = 0;$$

that LP, MQ, NR have for their equations respectively,

$$2m\beta + 2n\gamma - l\alpha = 0, \quad 2n\gamma + 2l\alpha - m\beta = 0, \quad 2l\alpha + 2m\beta - n\gamma = 0,$$

and that PQR is a right line whose equation is

$$l\alpha + m\beta + n\gamma = 0.$$

It is evident likewise that CA, CF, CB, CR form a harmonic pencil, their equations being

$$\beta = 0, \quad l\alpha - m\beta = 0, \quad \alpha = 0, \quad l\alpha + m\beta = 0.$$

To find the equation of a conic inscribed in $\alpha\beta\gamma$, *and having its centre at a given point* $(\alpha'\beta'\gamma')$.

The polar of any point with regard to this conic is (Art. 292)

$$\alpha l(m\beta' + n\gamma' - l\alpha') + \beta m(l\alpha' + n\gamma' - m\beta') + \gamma n(l\alpha' + m\beta' - n\gamma') = 0.$$

Now if it were required to determine l, m, n, so that this polar should coincide with

$$L\alpha + M\beta + N\gamma = 0,$$

we should find

$$l = L(M\beta' + N\gamma' - L\alpha'); \quad m = M(L\alpha' + N\gamma' - M\beta'); \quad n = N(L\alpha' + M\beta' - N\gamma').$$

Hence the locus of the centres of a conic touching three lines, and passing through a given point $\alpha''\beta''\gamma''$, is

$$\sqrt{\{a\alpha''(b\beta + c\gamma - a\alpha)\}} + \sqrt{\{b\beta''(a\alpha + c\gamma - b\beta)\}} + \sqrt{\{c\gamma''(a\alpha + b\beta - c\gamma)\}} = 0,$$

the equation of a conic touching the lines joining the middle points of the sides of the triangle formed by the given tangents.

If the conic touch a fourth given line, $A\alpha + B\beta + C\gamma = 0$, we must (p. 237) have the relation

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0;$$

the locus of the centre is, therefore,

$$\frac{a(b\beta + c\gamma - a\alpha)}{A} + \frac{b(a\alpha + c\gamma - b\beta)}{B} + \frac{c(a\alpha + b\beta - c\gamma)}{C} = 0,$$

the equation of a right line.*

Thus too we may easily form the equation of a conic touching five given right lines, viz. α, β, γ, $A\alpha + B\beta + C\gamma$, $A'\alpha + B'\beta + C'\gamma$; for we have the two equations

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0, \quad \frac{l}{A'} + \frac{m}{B'} + \frac{n}{C'} = 0,$$

from which we can determine $l : m$ and $l : n$.

Ex. 1. Find the equation of the conic touching the five lines, α, β, γ, $\alpha + \beta + \gamma$, $2\alpha + \beta - \gamma$.

We have $l + m + n = 0$, $\frac{1}{2}l + m - n = 0$: hence the required equation is

$$2(-\alpha)^{\frac{1}{2}} + (3\beta)^{\frac{1}{2}} + (\gamma)^{\frac{1}{2}} = 0.$$

Ex. 2. Find the equation of the conic touching α, β, γ, at their middle points.

Ans. $(a\alpha)^{\frac{1}{2}} + (b\beta)^{\frac{1}{2}} + (c\gamma)^{\frac{1}{2}} = 0.$

Ex. 3. Find the condition that $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$ should represent a parabola.

Ans. The curve touches the line at infinity when $\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0$.

* The condition that a conic circumscribed about the triangle ($\alpha\beta\gamma$),

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

should touch another inscribed in it,

$$\sqrt{(L\alpha)} + \sqrt{(M\beta)} + \sqrt{(N\gamma)} = 0,$$

is (note, p. 236)

$$(lL)^{\frac{1}{3}} + (mM)^{\frac{1}{3}} + (nN)^{\frac{1}{3}} = 0;$$

hence we can find the locus of the centre of the conic inscribed in a given triangle, and touching another circumscribed to the same triangle, or *vice versâ* (Hearn, p. 50).

Ex. 4. To find the locus of the focus of a parabola touching α, β, γ.

Generally, if the co-ordinates of one focus of a conic inscribed in the triangle $\alpha\beta\gamma$ be $\alpha'\beta'\gamma'$, the lines joining it to the vertices of the triangle will be

$$\frac{\alpha}{\alpha'} = \frac{\beta}{\beta'}, \quad \frac{\beta}{\beta'} = \frac{\gamma}{\gamma'}, \quad \frac{\gamma}{\gamma'} = \frac{\alpha}{\alpha'};$$

and since the lines to the other focus make equal angles with the sides of the triangle (Art. 194), these lines will be (Art. 57)

$$\alpha'\alpha = \beta'\beta, \quad \beta'\beta = \gamma'\gamma, \quad \gamma'\gamma = \alpha'\alpha;$$

and the co-ordinates of the other focus may be taken $\frac{1}{\alpha'}, \frac{1}{\beta'}, \frac{1}{\gamma'}$.

Hence, if we are given the equation of any locus described by one focus, we can at once write down the equation of the locus described by the other; and if the second focus be at infinity, that is, if $\alpha'' \sin A + \beta'' \sin B + \gamma'' \sin C = 0$, the first must lie on the circle $\frac{\sin A}{\alpha'} + \frac{\sin B}{\beta'} + \frac{\sin C}{\gamma'} = 0$. The co-ordinates of the focus of a parabola at infinity are $\frac{l}{\sin^2 A}, \frac{m}{\sin^2 B}, \frac{n}{\sin^2 C}$, since (remembering the relation in Ex. 3) these values satisfy both the equations, $\alpha \sin A + \beta \sin B + \gamma \sin C = 0$, $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$.

The co-ordinates, then, of the finite focus are $\frac{\sin^2 A}{l}, \frac{\sin^2 B}{m}, \frac{\sin^2 C}{n}$.

Ex. 5. To find the equation of the directrix of this parabola.

Forming, by Art. 294, the equation of the polar of the point whose co-ordinates have just been given, we find

$$l\alpha (\sin^2 B + \sin^2 C - \sin^2 A) + m\beta (\sin^2 C + \sin^2 A - \sin^2 B) + n\gamma (\sin^2 A + \sin^2 B - \sin^2 C) = 0,$$

or $$l\alpha \sin B \sin C \cos A + m\beta \sin C \sin A \cos B + n\gamma \sin A \sin B \cos C = 0.$$

Substituting for n from Ex. 3, the equation becomes

$$l \sin B \sin C (\alpha \cos A - \gamma \cos C) + m \sin C \sin A (\beta \cos B - \gamma \cos C) = 0;$$

hence the directrix always passes through the intersection of the perpendiculars of the triangle (see Ex. 3, p. 54).

DISCRIMINANTS.

295. The condition that an equation of the second degree should represent two right lines, is called the *discriminant* of that equation. When a conic breaks up into two right lines, the polar of any point passes through the intersection of the two lines; being the fourth harmonic to the two lines and the line joining their intersection to the given point. Now the line $\alpha' \frac{dS}{d\alpha} + \beta' \frac{dS}{d\beta} + \gamma' \frac{dS}{d\gamma}$ will always pass through a fixed point, provided that $\frac{dS}{d\alpha}, \frac{dS}{d\beta}, \frac{dS}{d\gamma}$ represent lines meeting in a point. If then we form the condition that

$$a\alpha + b''\beta + b'\gamma = 0, \quad a'\beta + b\gamma + b''\alpha = 0, \quad a''\gamma + b'\alpha + b\beta = 0,$$

should represent lines meeting in a point; by eliminating α, β, γ between these equations, we obtain the discriminant of the given equation, viz.,

$$ab^2 + a'b'^2 + a''b''^2 - aa'a'' - 2bb'b'' = 0,$$

which only differs in notation from what we have obtained already by other methods (see pp. 67, 139).*

296. Given the equations of two conics,

$$(S) \quad a\alpha^2 + a'\beta^2 + a''\gamma^2 + 2b\beta\gamma + 2b'\gamma\alpha + 2b''\alpha\beta = 0,$$
$$(S') \quad A\alpha^2 + A'\beta^2 + A''\gamma^2 + 2B\beta\gamma + 2B'\gamma\alpha + 2B''\alpha\beta = 0,$$

if it were required to form the equation of their chords of intersection, we have only to form the discriminant of $kS + S'$, by writing $ka + A$ for a, $kb + B$ for b, &c. in the discriminant of S; and putting this discriminant = 0, it will be found that we have a cubic to determine k. It is geometrically evident that this must be the case, since if the two conics intersect in the points ABCD, there can be drawn through these four points any of the three pairs of right lines, AB, CD; AC, BD; AD, BC. If then the roots of the cubic be k', k'', k''', the equations of the pairs of right lines will be $k'S + S' = 0$, $k''S + S' = 0$, $k'''S + S' = 0$. The cubic in question actually is

$$\begin{aligned} k^3(ab^2 + a'b'^2 + a''b''^2 - aa'a'' - 2bb'b'') + k^2\{&A(b^2 - a'a'') + A'(b'^2 - a''a) \\ + A''(b''^2 - aa') + 2B(ab - b'b'') &+ 2B'(a'b' - b''b) + 2B''(a''b'' - bb')\} \\ + k\{a(B^2 - A'A'') + a'(B'^2 - A''A) &+ a''(B''^2 - AA') \\ + 2b(AB - B'B'') + 2b'(A'B' - B''B) &+ 2b''(A''B'' - BB')\} \\ + \{AB^2 + A'B'^2 + A''B''^2 - AA'A'' &- 2BB'B''\} = 0. \end{aligned}$$

If we call the discriminant of S, ∇; then it is plain that the co-

* The condition that an algebraic equation should have equal roots is also called the discriminant of that equation. For if the equation be made homogeneous by the introduction of a variable y, the condition that the equation should have equal roots is obtained by eliminating x and y between $\frac{dS}{dx} = 0$, $\frac{dS}{dy} = 0$. And, in general, if a homogeneous function of any number of variables be differentiated successively with respect to all these variables, and the variables eliminated between the resulting equations, the result of elimination is called the discriminant of the given function.

efficient of k^3 is ∇, and that the absolute term is ∇'. The coefficient of k^2 is

$$A\frac{d\nabla}{da} + A'\frac{d\nabla}{da'} + A''\frac{d\nabla}{da''} + B\frac{d\nabla}{db} + B'\frac{d\nabla}{db'} + B''\frac{d\nabla}{db''},$$

as is also evident from Taylor's theorem. The coefficient of k is $a\frac{d\nabla'}{dA}$ + &c.

297. *To find the condition that the line $l\alpha + m\beta + n\gamma$ should touch the conic* S.

Form the discriminant of $kS + (l\alpha + m\beta + n\gamma)^2$, and it will be found to be

$$k^3\nabla + k^2\left\{l^2\frac{d\nabla}{da} + m^2\frac{d\nabla}{da'} + n^2\frac{d\nabla}{da''} + mn\frac{d\nabla}{db} + nl\frac{d\nabla}{db'} + lm\frac{d\nabla}{db''}\right\},$$

the coefficient of k and the absolute term vanishing identically. It is easy to see the geometrical reason why this should be the case. For if S′ represent the two right lines AB, CD, we have $\nabla' = 0$, and one root of the cubic is $k = 0$, as it plainly ought to be. But suppose that S′ is a perfect square, and represent two coincident lines, then the points A, C; B, D; coincide, and the pair of lines AD, BC is also represented by S′. We must have then *two* roots of the cubic, $k = 0$, or the equation must be divisible by k^2. In this case the third pair of lines AC, BD is the pair of tangents to the conic at the points where it is met by $l\alpha + m\beta + n\gamma$; and substituting in $kS + S'$ the value of k obtained by putting the discriminant = 0, the equation of this pair of tangents is found to be

$$\left(l^2\frac{d\nabla}{da} + \text{\&c.}\right)S - \nabla(l\alpha + m\beta + n\gamma)^2 = 0.$$

But suppose now that the line $l\alpha + m\beta + n\gamma$ touches S, then it is plain that the pair of tangents AC, BD also coincides with S′; we must therefore have the three roots of the cubic $k = 0$, or the equation must be divisible by k^3. Hence we obtain the same condition that $l\alpha + m\beta + n\gamma$ should touch S, as was otherwise obtained (Art. 154), viz.,

$$l^2\frac{d\nabla}{da} + m^2\frac{d\nabla}{da'} + n^2\frac{d\nabla}{da''} + mn\frac{d\nabla}{db} + nl\frac{d\nabla}{db'} + lm\frac{d\Delta}{db''} = 0.$$

We may sometimes write this condition $\Sigma = 0$.

The condition that $kS + S'$ should touch $l\alpha + m\beta + n\gamma$ is immediately obtained by writing $ka + A$ for a, $kb + B$ for b, &c. in Σ; and the result will obviously contain k in the second degree. Hence the problem to describe a conic through four points to touch a given line admits of two solutions.

Ex. Find the condition that $l\alpha + m\beta + n\gamma$ should touch $S + (l'\alpha + m'\beta + n'\gamma)^2$.

Ans. $\Sigma + K = 0$, where K is the result of writing $mn' - nm'$, $nl' - ln'$, $lm' - nl'$ for α, β, γ in S.

298. *To find the condition that two conics* S *and* S′ *should touch each other.*

When two points A, B, of the four points of intersection of two conics coincide, then it is plain that the pair of lines AC, BD is identical with the pair AD, BC. In this case, then, the cubic in k (Art. 296) must have two equal roots. Now it can readily be proved that the discriminant of the cubic

$$Lk^3 + Mk^2 + Nk + P = 0$$

is $$(MN - 9LP)^2 = 4(M^2 - 3LN)(N^2 - 3MP).$$

Substituting then for L, M, N, P, their values given in Art. 296, we obtain the required condition, which will be of the sixth degree in the coefficients of each equation. And it may be inferred, as at the close of the last Article, that the problem "to describe a conic through four given points to touch a conic," is one which admits in general of six solutions.

299. *To find the co-ordinates of the pole with regard to* S *of the right line* $l\alpha + m\beta + n\gamma$.

If these co-ordinates be α', β', γ', we must have

$$\alpha\frac{dS}{d\alpha'} + \beta\frac{dS}{d\beta'} + \gamma\frac{dS}{d\gamma'} \text{ identical with } l\alpha + m\beta + n\gamma,$$

or $$a\alpha' + b''\beta' + b'\gamma' = l, \quad a'\beta' + b\gamma' + b''\alpha' = m, \quad a''\gamma' + b'\alpha' + b\beta' = n;$$

whence we get α' proportional to

$$l(b^2 - a'a'') + m(a''b'' - bb') + n(a'b' - b''b), \text{ \&c.},$$

values which may be written

$$\alpha' = \frac{d\Sigma}{dl}, \quad \beta' = \frac{d\Sigma}{dm}, \quad \gamma' = \frac{d\Sigma}{dn}.$$

CHAPTER XV.

GEOMETRICAL METHODS.

300. Having in the previous chapters sufficiently illustrated the use of the method of co-ordinates, we purpose to occupy the present chapter with some important geometrical methods, an account of which must form an essential part of any work devoted to the theory of curves.

THE METHOD OF RECIPROCAL POLARS.*

301. Being given a fixed conic section (Σ) and any curve (S), we can generate another curve (s) as follows: draw any tangent to S, and take its pole with regard to Σ; the locus of this pole will be a curve s, which is called the *polar curve* of S with regard to Σ. The conic Σ, with regard to which the pole is taken, is called the *auxiliary* conic.

We have already met with a particular example of polar curves (Ex. 20, p. 190), where we proved that the polar curve of one conic section with regard to another is always a curve of the second degree.

We shall for brevity say that a point *corresponds* to a line when we mean that the point is the pole of that line with regard to Σ; thus, since it appears from our definition that every point of s is the pole with regard to Σ of some tangent to S, we shall briefly express this relation by saying that every point of s *corresponds* to some tangent of S.

302. *The point of intersection of two tangents to* S *will correspond to the line joining the corresponding points of s.*

This follows from the property of the conic Σ, that the point

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* This beautiful method was introduced by M. Poncelet, whose account of it will be found at the commencement of the fourth volume of Crelle's Journal. The reader will find the principle of duality, which is involved in this method, treated of from a purely analytical point of view in the author's work on the *Higher Plane Curves*, chap. I.

of intersection of any two lines is the pole of the line joining the poles of these two lines (Art. 146).

Let us suppose that in this theorem the two tangents to S are indefinitely near, then the two corresponding points of *s* will also be indefinitely near, and the line joining them will be a tangent to *s* (Art. 81); it also easily follows, from our definition of a tangent, that any tangent to a curve intersects the consecutive tangent at its point of contact (see Art. 142); hence for this case the last theorem becomes: *If any tangent to* S *correspond to a point on s, the point of contact of that tangent to* S *will correspond to the tangent through the point on s.*

Hence we see that the relation between the curves is *reciprocal*, that is to say, that the curve S might be generated from *s* in precisely the same manner that *s* was generated from S; hence the name "reciprocal polars."

303. We are now able, being given any theorem *of position* (Art. 1) concerning any curve S, to deduce another concerning the curve *s*. Thus, for example, if we know that a number of points connected with the figure S lie on one right line, we learn that the corresponding lines connected with the figure *s* meet in a point (Art. 146), and *vice versâ;* if a number of points connected with the figure S lie on a conic section, the corresponding lines connected with *s* will touch the polar of that conic with regard to Σ; or, in general, if the *locus* of any point connected with S be any curve S′, the *envelope* of the corresponding line connected with *s* is *s*′, the reciprocal polar of S′.

304. *The degree of the polar reciprocal of any curve is equal to the number of tangents which can be drawn from any point to that curve.*

For the degree of *s* is the same as the number of points in which any line cuts *s*; and to a number of points on *s*, lying on a right line, correspond *the same number* of tangents to S passing through the point corresponding to that line. Thus, if S be a conic section, two, and only two, tangents, real or imaginary, can be drawn to it from any point (Art. 142); therefore, any line meets *s* in two, and only two points, real or imaginary; we may thus infer, independently of Ex. 20, p. 190, that the reciprocal of any conic section is a curve of the second degree.

305. We shall exemplify, in the case where S and *s* are conic sections, the mode of obtaining one theorem from another by this method.

We know (Art. 268) that "if a hexagon be *in*scribed in S, whose *sides* are A, B, C, D, E, F, then the *points* of intersection, AD, BE, CF, are *in one right line*. Hence we infer, that "if a hexagon be *circum*scribed about *s*, whose *vertices* are *a*, *b*, *c*, *d*, *e*, *f*, then the *lines ad*, *be*, *cf*, will *meet in a point*" (Art. 266). Thus we see that Pascal's theorem and Brianchon's are reciprocal to each other, and it was thus, in fact, that the latter was first obtained.

In order to give the student an opportunity of rendering himself expert in the application of this method, we shall write in parallel columns some theorems, together with their reciprocals. The beginner ought carefully to examine the force of the argument by which the one is inferred from the other, and he ought to attempt to form for himself the reciprocal of each theorem before looking at the reciprocal we have given. He will soon find that the operation of forming the reciprocal theorem will reduce itself to a mere mechanical process of interchanging the words "point" and "line," "inscribed" and "circumscribed," "locus" and "envelope," &c.

If two vertices of a triangle move along fixed right lines, while the sides pass each through a fixed point, the locus of the third vertex is a conic section. (Ex. 4, p. 230.)

If, however, the points through which the sides pass lie in one right line, the locus will be a right line. (p. 40.)

In what other case will the locus be a right line? (p. 41.)

If two sides of a triangle pass through fixed points, while the vertices move on fixed right lines, the envelope of the third side is a conic section.

If the lines on which the vertices move meet in a point, the third side will pass through a fixed point.

In what other case will the third side pass through a fixed point? (p. 47.)

If two conics touch, their reciprocals will also touch; for the first pair have a point common, and also the tangent at that point common, therefore the second pair will have a tangent common and its point of contact also common. So likewise if two conics have double contact their reciprocals will have double contact.

If a triangle be circumscribed to a conic section, two of whose vertices move on fixed lines, the locus of the third vertex is a conic section, having double contact with the given one. (Ex. 1, p. 229.)

If a triangle be inscribed in a conic section, two of whose sides pass through fixed points, the envelope of the third side is a conic section, having double contact with the given one. (Ex. 2, p. 229.)

306. We proved (Art. 302, see figure, p. 258) if to two points P, P′, on S, correspond the tangents pt, $p't'$, on s, that the tangents at P and P′ will correspond to the points of contact p, p', and therefore Q, the intersection of these tangents, will correspond to the chord of contact pp'. Hence we learn that *to any point* Q, *and its polar* PP′, *with respect to* S, *correspond a line* pp' *and its pole* q *with respect to* s.

Given two points on a conic, and two of its tangents, the line joining the points of contact of those tangents passes through a fixed point. (Art. 264.)

Given two tangents and two points on a conic, the point of intersection of the tangents at those points will move along a fixed right line.

Given four points on a conic, the polar of a fixed point passes through a fixed point. (Ex. 3, p. 137.)

Given four tangents to a conic, the locus of the pole of a fixed right line is a right line.

Given four points on a conic, the locus of the pole of a fixed right line is a conic section. (Ex. 2, p. 241.)

Given four tangents to a conic, the envelope of the polar of a fixed point is a conic section.

The lines joining the vertices of a triangle to the opposite vertices of its polar triangle with regard to a conic, meet in a point. (Ex. 2, p. 243.)

The points of intersection of each side of any triangle, with the opposite side of the polar triangle, lie in one right line. (Ex. 3, p. 243.)

Inscribe in a conic a triangle whose sides pass through three given points. (Ex. 6, p. 244.)

Circumscribe about a conic a triangle whose vertices rest on three given lines.

307. Given two conics, S and S′, and their two reciprocals, s and s'; to any point common to S and S′ will correspond a tangent common to s and s', and to any chord of intersection of S and S′ will correspond an intersection of common tangents to s and s'.

If three conics have two points common, and, therefore, one common chord, their other three common chords will meet in a point. (Art. 267.)

If three conics have two tangents common, the points of intersection of the other three pairs of common tangents lie on one right line.

If three conics have two common tangents, or if they have each double

If three conics have two points common, or if they have each double contact with a fourth, their six chords of intersection will pass three by three through the same points. (Art. 265.)

Or, in other words, three conics, having each double contact with a fourth, may be considered as having four radical centres. (p. 105.)

contact with a fourth, the six points of intersection of common tangents lie three by three on the same right lines.

Or, three conics, having each double contact with a fourth, may be considered as having four axes of similitude. (See Art. 122, of which this theorem is an extension.)*

If through the point of contact of two conics which touch, any chord be drawn, tangents at its extremities will meet on the common chord of the two conics.

If from any point on the tangent at the point of contact of two conics which touch, a tangent be drawn to each, the line joining their points of contact will pass through the intersection of common tangents to the conics.

If, through the intersection of common tangents of two conics any two chords be drawn, lines joining their extremities will intersect on one or other of the common chords of the two conics. (p. 226.)

If, on a common chord of two conics, any two points be taken, and from these tangents be drawn to the conics, the diagonals of the quadrilateral so formed will pass through one or other of the intersections of common tangents to the conics.

If A and B be two conics having each double contact with S, the chords of contact of A and B with S, and their chords of intersection with each other, meet in a point, and form a harmonic pencil. (Art. 264.)

If A and B be two conics having each double contact with S, the intersections of the tangents at their points of contact with S, and the intersections of tangents common to A and B, lie in one right line, which they divide harmonically.

If A, B, C, be three conics, having each double contact with S, and if A and B both touch C, the tangents at the points of contact will intersect on a common chord of A and B.

If A, B, C, be three conics, having each double contact with S, and if A and B both touch C, the line joining the points of contact will pass through an intersection of common tangents of A and B.*

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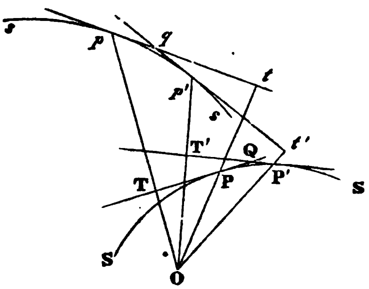
* The reader will take notice that we have now proved that every theorem used in Art. 127, in the theory of three circles, has a theorem corresponding in the theory of three conics which are each inscribed in the same given conic; and hence that, given three such conics, we can find a fourth inscribed in the same conic, and such as to touch the three given conics. The learner will do well to refer to Art. 127, and to examine for himself how the demonstration there given is to be extended to the case of three conics inscribed in a given conic. The chief difference occurs in (5) of that Article, for the line $a''b''$ is now constructed by joining the pole of $SS'S''$ to *any one of the four* radical centres of

308. We have hitherto supposed the auxiliary conic Σ to be any conic whatever. It is most common, however, to suppose this conic a circle; and hereafter, when we speak of polar curves, we intend the reader to understand polars *with regard to a circle*, unless we expressly state otherwise.

We know (Art. 86) that the polar of any point with regard to a circle is perpendicular to the line joining this point to the centre, and that the distances of the point and its polar are, when multiplied together, equal to the square of the radius; hence the relation between polar curves with regard to a circle is often stated as follows: *Being given any point* O, *if from it we let fall a perpendicular* OT *on any tangent to a curve* S, *and produce it until the rectangle* OT.O*p* *is equal to a constant* k^2, *then the locus of the point p is a curve s, which is called the polar reciprocal of* S. For this is evidently equivalent to saying that p is the pole of PT, with regard to a circle whose centre is O and radius k. We see, therefore (Art. 302), that the tangent pt will correspond to the point of contact P, that is to say, that OP will be perpendicular to pt, and that OP.O$t = k^2$.



It is easy to show that a change in the magnitude of k will affect only the *size* and not the *shape* of s, which is all that in most cases concerns us. In this manner of considering polars, all mention of the circle may be suppressed, and s may be called the

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the three conics. The problem therefore admits of thirty-two solutions instead of eight, as in the case of the three circles. The theorems which answer to (6) of the same Article are the following:

The chord of contact of the required conic with S passes through the intersection of one of the axes of similitude of the three given conics with the polar of one of their radical centres with regard to S.

The pole of this chord, with regard to S, lies on the line joining one of their radical centres with the pole, with regard to S, of one of their axes of similitude.

The reader will find a very able investigation of this whole problem in a memoir published by Mr. Cayley in vol. xxxix. of Crelle's Journal.

reciprocal of S *with regard to the point* O. We shall call this point the *origin.*

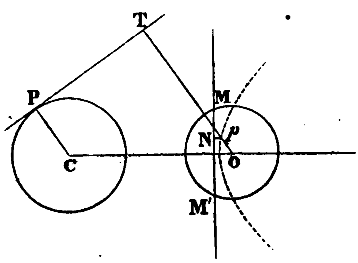
The *advantage* of using the circle for our auxiliary conic chiefly arises from the two following theorems, which are at once deduced from what has been said, and which enable us to transform, by this method, not only theorems of position, but also theorems involving the magnitude of lines and angles:

The distance of any point P *from the origin is the reciprocal of the distance of the corresponding line pt.*

The angle TQT′ *between any two lines* TQ, T′Q, *is equal to the angle pOp′ subtended at the origin by the corresponding points* p, p', for Op is perpendicular to TQ, and Op' to T′Q.

We shall give some examples of the application of these principles when we have first investigated the following problem:

309. *To find the polar reciprocal of one circle with regard to another.* That is to say, to find the locus of the pole p with regard to the circle (O) of any tangent PT to the circle (C). Let MN be the polar of the point C with regard to O, then having the points C, p, and their polars MN, PT, we have by Art. 98, the ratio $\frac{OC}{CP} = \frac{Op}{pN}$, but the first ratio is constant, since both OC and CP are constant; hence the distance of p from O is to its distance from MN in the constant ratio OC : CP, its locus is therefore a conic, of which O is a focus, MN the corresponding directrix, and whose eccentricity is OC divided by CP. Hence the eccentricity is greater, less than, or = 1, according as O is without, within, or on the circle C.



Hence the *polar reciprocal of a circle is a conic section, of which the origin is the focus, the line corresponding to the centre is the directrix, and which is an ellipse, hyperbola, or parabola, according as the origin is within, without, or on the circle.*

310. We shall now deduce some properties concerning angles, by the help of the theorem given in Art. 308.

Any two tangents to a circle make equal angles with their chord of contact.

The line drawn from the focus to the intersection of two tangents bisects the angle subtended at the focus by their chord of contact. (Art. 196.)

For the angle between one tangent PQ (see fig. p. 258) and the chord of contact PP′ is equal to the angle subtended at the focus by the corresponding points p, q; and similarly, the angle QP′P is equal to the angle subtended by p', q; therefore, since QPP′ = QP′P, $pOq = p'Oq$.

Any tangent to a circle is perpendicular to the line joining its point of contact to the centre.

Any point on a conic, and the point where its tangent meets the directrix, subtend a right angle at the focus.

This follows as before, recollecting that the directrix of the conic answers to the centre of the circle.

Any line is perpendicular to the line joining its pole to the centre of the circle.

Any point and the intersection of its polar with the directrix subtend a right angle at the focus.

The line joining any point to the centre of a circle makes equal angles with the tangents through that point.

If the point where any line meets the directrix be joined to the focus, the joining line will bisect the angle between the focal radii to the points where the given line meets the curve.

The locus of the intersection of tangents to a circle, which cut at a given angle, is a concentric circle.

The envelope of a chord of a conic, which subtends a given angle at the focus, is a conic having the same focus and the same directrix.

The envelope of the chord of contact of tangents which cut at a given angle is a concentric circle.

The locus of the intersection of tangents, whose chord subtends a given angle at the focus, is a conic having the same focus and directrix.

If from a fixed point tangents be drawn to a series of concentric circles, the locus of the points of contact will be a circle passing through the fixed point, and through the common centre.

If a fixed line intersect a series of conics having the same focus and same directrix, the envelope of the tangents to the conics, at the points where this line meets them, will be a conic having the same focus, and touching both the fixed line and the common directrix.

In the latter theorem, if the fixed line be at infinity, we find the envelope of the asymptotes of a series of hyperbolæ having the same focus and same directrix, to be a parabola having the same focus and touching the common directrix.

If two chords at right angles to each other be drawn through any point on a circle, the line joining their extremities passes through the centre.

The locus of the intersection of tangents to a parabola which cut at right angles is the directrix.

We say a parabola, for, the point through which the chords of the circle are drawn being taken for origin, the polar of the circle is a parabola (Art. 309).

The envelope of a chord of a circle which subtends a given angle at a given point on the curve is a concentric circle.

The locus of the intersection of tangents to a parabola, which cut at a given angle, is a conic having the same focus and the same directrix.

Given base and vertical angle of a triangle, the locus of vertex is a circle passing through the extremities of the base.

Given in position two sides of a triangle, and the angle subtended by the base at a given point, the envelope of the base is a conic, of which that point is a focus, and to which the two given sides will be tangents.

The locus of the intersection of tangents to an ellipse or hyperbola which cut at right angles is a circle.

The envelope of any chord of a conic which subtends a right angle at any fixed point is a conic, of which that point is a focus.

"If from any point on the circumference of a circle perpendiculars be let fall on the sides of any inscribed triangle, their three feet will lie in one right line" (Art. 106).

If we take the fixed point for origin, to the triangle *inscribed* in a *circle* will correspond a triangle *circumscribed* about a *parabola;* again, to the foot of the perpendicular on any line corresponds a line through the corresponding point perpendicular to the radius vector from the origin. Hence, "If we join the focus to each vertex of a triangle circumscribed about a parabola, and erect perpendiculars at the vertices to the joining lines, those perpendiculars will pass through the same point." If, therefore, a circle be described, having for diameter the radius vector from the focus to this point, it will pass through the vertices of the circumscribed triangle. Hence, *Given three tangents to a parabola, the locus of the focus is the circumscribing circle* (p. 187).

The locus of the foot of the perpendicular (or of a line making a

If from any point a radius vector be drawn to a circle, the envelope of constant angle with the tangent) from the focus of an ellipse or hyperbola on the tangent is a circle.

a perpendicular to it at its extremity (or of a line making a constant angle with it) is a conic having the fixed point for its focus.

311. Having sufficiently exemplified in the last Article the method of transforming theorems involving angles, we proceed to show that theorems involving the magnitude of lines *passing through the origin* are easily transformed by the help of the first theorem in Art. 308. For example, the sum (or, in some cases, the difference, if the origin be without the circle) of the perpendiculars let fall from the origin on any pair of parallel tangents to a circle is constant, and equal to the diameter of the circle.

Now, to two parallel lines correspond two points on a line passing through the origin. Hence, "the sum of the reciprocals of the segments of any focal chord of an ellipse is constant."

We know (p. 169) that this sum is the reciprocal of the semi-parameter of the ellipse, and since we learn from the present example that it only depends on the diameter, and not on the position of the reciprocal circle, we infer that *the reciprocals of equal circles, with regard to any origin, have the same parameter.*

The rectangle under the segments of any chord of a circle through the origin is constant.

The rectangle under the perpendiculars let fall from the focus on two parallel tangents is constant.

Hence, given the tangent from the origin to a circle, we are given the conjugate axis of the reciprocal hyperbola.

Again, the theorem, that the sum of the focal distances of any point on an ellipse is constant, may be expressed thus:

The sum of the distances from the focus of the points of contact of parallel tangents is constant.

The sum of the reciprocals of perpendiculars let fall from any point on two tangents to a circle, whose chord of contact passes through the point, is constant.

312. Many relations involving the magnitude of lines *not* passing through the origin may be transformed by the help of the theorem of Art. 98. Thus we know, that if PA, PB, PC, PD, be the perpendiculars let fall from any point of a conic on the sides of an inscribed quadrilateral, $PA \cdot PC = kPB \cdot PD$ (Art. 260); now we may write this relation, $\frac{PA}{OP} \cdot \frac{PC}{OP} = k \cdot \frac{PB}{OP} \cdot \frac{PD}{OP}$,

but if a, b, c, d, be the points corresponding to the lines A, B, C, D, and ap the perpendicular let fall from a on the line corresponding to P we have (Art. 98) $\frac{PA}{OP} = \frac{ap}{Oa}$. Similarly for the other sides; and Oa, Ob, Oc, Od, being constant, we infer that *if a fixed quadrilateral be circumscribed to a conic, the product of the perpendiculars let fall from two opposite vertices on any variable tangent is in a constant ratio to the product of the perpendiculars let fall from the other two vertices.*

The product of the perpendiculars from any point of a conic on two fixed tangents, is in a constant ratio to the square of the perpendicular on their chord of contact. (Art. 260.)

The product of the perpendiculars from two fixed points of a conic on any tangent, is in a constant ratio to the square of the perpendicular on it, from the intersection of tangents at those points.

If, however, the origin be taken on the chord of contact, the reciprocal theorem is, "the intercepts, made by any variable tangent on two parallel tangents have a constant rectangle."

The product of the perpendiculars on any tangent of a conic from two fixed points (the foci) is constant.

The square of the radius vector from a fixed point to any point on a conic, is in a constant ratio to the product of the perpendiculars let fall from that point of the conic on two fixed right lines.

313. Very many theorems concerning magnitude may be reduced to theorems concerning lines cut harmonically or anharmonically, and are transformed by the following principle: *To any four points on a right line correspond four lines passing through a point, and the anharmonic ratio of this pencil is the same as that of the four points.*

This is evident, since each leg of the pencil drawn from the origin to the given points is perpendicular to one of the corresponding lines. We may thus derive the anharmonic properties of the conics in general from that of the circle.

The anharmonic ratio of the pencil joining four points on a conic to a variable fifth is constant.

The anharmonic ratio of the points in which four fixed tangents to a conic cut any variable fifth is constant.

The first of these theorems is true for the circle, since all the angles of the pencil are constant, therefore the second is true for all the conics. The second theorem is true for the circle, since the angles which the four points subtend at the centre are constant, therefore the first theorem is true for all the conics. By observing the angles which correspond in the reciprocal figure to the angles which are constant in the case of the circle, the student will perceive that the angles which the four points of the variable tangent subtend at either focus are constant, and that the angles are constant which are subtended at the focus by the four points in which any inscribed pencil meets the directrix.

In like manner, the theorem of Art. 149 is the reciprocal of that in Art. 147, and both, being true for the circle, must be true for all the conics.

314. The anharmonic ratio of a line is not the only relation concerning the magnitude of lines which can be expressed in terms of the angles subtended by the lines at a fixed point. For, if there be any relation which by substituting (as in Art. 54) for each line AB involved in it, $\frac{OA \cdot OB \cdot \sin AOB}{OP}$ can be reduced to a relation between the sines of angles subtended at a given point O, this relation will be equally true for any transversal cutting the lines joining O to the points A, B, &c.; and by taking the given point for origin a reciprocal theorem can be easily obtained. For example, the following theorem, due to Carnot, is an immediate consequence of Art. 151: "If any conic meet the side AB of any triangle in the points c, c'; BC in a, a'; AC in b, b'; then the ratio

$$\frac{Ac \cdot Ac' \cdot Ba \cdot Ba' \cdot Cb \cdot Cb'}{Ab \cdot Ab' \cdot Bc \cdot Bc' \cdot Ca \cdot Ca'} = 1."$$

Now, it will be seen that this ratio is such that we may substitute for each line Ac the sine of the angle AOc, which it subtends at any fixed point; and if we take the reciprocal of this theorem, we obtain the theorem given already at p. 240.

315. Having shown how to form the reciprocals of particular theorems, we shall add some general considerations respecting reciprocal conics.

We proved (Art. 309) that the reciprocal of a circle is an ellipse, hyperbola, or parabola, according as the origin is within, without, or on the curve; we shall now extend this conclusion to all the conic sections. It is evident that, the nearer any line or point is to the origin, the farther the *corresponding* point or line will be; that if any line passes through the origin, the corresponding point must be at an infinite distance; and that the line corresponding to the origin itself must be altogether at an infinite distance. To two tangents, therefore, through the origin on one figure, will correspond two points at an infinite distance on the other; hence, if two *real* tangents can be drawn from the origin, the reciprocal curve will have two *real* points at infinity, that is, it will be a hyperbola; if the tangents drawn from the origin be imaginary, the reciprocal curve will be an ellipse; if the origin be on the curve, the tangents from it coincide (p. 130), therefore the points at infinity on the reciprocal curve coincide, that is, the reciprocal curve will be a parabola. Since the line at infinity corresponds to the origin, we see that, if the origin be a point on one curve, the line at infinity will be a tangent to the reciprocal curve; and we are again led to the theorem (Art. 255) that *every parabola has one tangent situated at an infinite distance.*

Hence Ex. 2, p. 160, is the reciprocal of the theorem, Art. 226.

316. To the points of contact of two tangents through the origin must correspond the tangents at the two points at infinity on the reciprocal curve, that is to say, the asymptotes of the reciprocal curve. The eccentricity of the reciprocal hyperbola depending solely on the angle between its asymptotes, depends, therefore, on the angle between the tangents drawn from the origin to the original curve.

Again, the intersection of the asymptotes of the reciprocal curve (i. e. its centre) corresponds to the chord of contact of tangents from the origin to the original curve. We met with a particular case of this theorem when we proved that to the centre of a circle corresponds the directrix of the reciprocal conic, for the directrix is the polar of the origin which is the focus of that conic.

We can thus, likewise, find the *axes* of the reciprocal curve, for they must be lines drawn through its centre parallel to the internal and external bisectors of the angle between the tangents drawn from the origin. This may otherwise be expressed (by the help of the theorem, Art. 194), that if through the origin we draw a conic confocal to the given one, the axes of the reciprocal conic will be parallel to the tangent and normal at the origin to the confocal conic. This latter statement is preferable, because it holds when the origin is within the curve.

317. Hence, given two circles, we can find a point such that the reciprocals of both shall be *confocal* conics. For, since the reciprocals of all circles must have one focus (the origin) common; in order that the other focus should be common, it is only necessary that the two reciprocal curves should have the same centre, that is, that the polar of the origin with regard to both circles should be the same, or that the origin should be one of the two points determined in Art. 116. Hence, given a *system* of circles, as in Art. 114, their reciprocals with regard to one of these limiting points will be a *system* of confocal conics. Theorems, therefore, concerning confocal conics, are at once transformed into theorems relating to the system of circles, e. g., the theorem of Art. 192 corresponds to "the common tangent to two circles subtends a right angle at either of the limiting points." The theorem of Art. 194 corresponds to—"if any line intersect two circles, its two intercepts between the circles subtend equal angles at either limiting point." Or, again, by Ex. 3, Art. 231, any fixed point, and the fixed point through which (Art. 115) its polar must pass, subtend a right angle at the limiting points.

We may mention here that the method of reciprocal polars affords a simple solution of the problem, "to describe a circle touching three given circles." The locus of the centre of a circle touching *two* of the given circles (1), (2), is evidently a hyperbola, of which the centres of the given circles are the foci, since the problem is at once reduced to—"Given base and difference of sides of a triangle." Hence (Art. 309) the polar of the centre with regard to either of the given circles (1) will always touch a circle which can be easily constructed. In like manner, the polar of the centre of any circle touching (1) and (3) must also touch a given circle. Therefore, if we draw a common tangent to the two circles thus determined, and take the pole of this line with respect to (1), we have the centre of the circle touching the three given circles.*

318. Given any two conics; there are three points such that their reciprocals with regard to any of them will be concentric curves. For *there are three points whose polars with regard to the two conics are the same*, namely, if we form the common inscribed quadrilateral by joining the four points in which the curves intersect, the three points E, F, O (see Art. 149, Ex. 1). These three points may be real, even when the conics cut in imaginary points.

319. *To find the equation of the reciprocal of a conic with regard to its centre.*

We found, in Art. 182, that the perpendicular on the tangent could be expressed in terms of the angles it makes with the axes

$$p^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

Hence the polar equation of the reciprocal curve is

$$\frac{k^4}{\rho^2} = a^2\cos^2\theta + b^2\sin^2\theta,$$

or

$$\frac{a^2x^2}{k^4} + \frac{b^2y^2}{k^4} = 1,$$

a concentric conic, whose axes are the reciprocals of the axes of the given conic.

320. *To find the equation of the reciprocal of a conic with regard to any point* $(x'y')$.

The length of the perpendicular from any point is (Art. 182)

$$p = \frac{k^2}{\rho} = \sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)} - x'\cos\theta - y'\sin\theta\ ;$$

therefore, the equation of the reciprocal curve is

$$(xx' + yy' + k^2)^2 = a^2x^2 + b^2y^2.$$

321. *To find the reciprocal of the conic*

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = 0.$$

For symmetry we shall write $k^2 = -z^2$, and look for the reciprocal

* This solution is taken from *Gergonne's Annales.*

with regard to $x^2+y^2+z^2=0$. Then the polar with regard to this of any point on the reciprocal curve will touch the given curve. But the equation of the polar is $xx'+yy'+zz'=0$; and expressing (Art. 154) the condition that this line should touch the given conic, the equation of the reciprocal is found to be

$$(a'a''-b^2)x^2+(a''a-b'^2)y^2+(aa'-b''^2)z^2$$
$$+2(b'b''-ab)yz+2(b''b-a'b')zx+2(bb'-a''b'')xy=0.$$

We have seen (Art. 296) that the coefficients in this equation are equal to $\frac{d\nabla}{da}, \frac{d\nabla}{da'}$, &c. We shall denote these coefficients by $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}'', \mathfrak{B}, \mathfrak{B}', \mathfrak{B}''$. It is easy to deduce from this equation the properties which we have already obtained geometrically, such as, that if the curve be a parabola, the origin will be a point on the reciprocal curve, &c.

Ex. 1. To find the equation of the reciprocal of the reciprocal of a given conic. This must evidently represent the given curve itself. The equation is

$$(\mathfrak{A}'\mathfrak{A}''-\mathfrak{B}^2)x^2+\&c.=0;$$

and writing for $\mathfrak{A}'$, &c., their values, this is found equal to the given equation multiplied by ∇. In like manner the discriminant of the reciprocal is found $=\nabla^2$.

Ex. 2. To find the reciprocal of a system of conics which pass through four points. The equation of any conic of the system being $S+kS'=0$, the equation of the reciprocal is found by writing $a+kA$ for a, $a'+kA'$ for a', &c., in the equation of the reciprocal. It is easy to see that the result will contain k in the second degree. We may write it $\Sigma+k\Phi+k^2\Sigma'=0$, where Σ and Σ' are the reciprocals of S and S', while

$$\Phi=(a'A''+a''A'-2bB)x^2+(a''A+aA''-2b'B')y^2+(aA'+a'A-2b''B'')z^2$$
$$+2(b'B''+b''B'-aB-bA)yz+2(b''B+bB''-a'B'-b'A')zx$$
$$+2(bB'+b'B-a''B''-b''A'')xy.$$

Now, since the original system of conics passes through four fixed points, the reciprocal system always touches four fixed right lines. But the form of the equation shows that the reciprocal always touches $4\Sigma\Sigma'=\Phi^2$. This, then, is the equation of the four lines which are common tangents to Σ, Σ', and the other conics of the reciprocal system. But the form of $4\Sigma\Sigma'=\Phi^2$, the equation to these four lines, shows that Σ is touched by them, and that Φ passes through the points of contact. In like manner, Φ passes through the four points where Σ' is touched by the common tangents. Hence the eight points of contact of common tangents to the two conics Σ, Σ', all lie on the same conic Φ.

Ex. 3. To find the equation of the common tangents to S and S'.

The system reciprocal to the system of conics which have the same common tangents will pass through four fixed points, and will be $\Sigma+k\Sigma'=0$. Forming, then, the reciprocal of this latter system, we find $\nabla S+kF+k^2\nabla'S'=0$, where F is what Φ becomes when the coefficients $\mathfrak{A}$, $\mathfrak{A}'$, &c. of the reciprocal are written for A, A', &c. The equation, then, of the common tangents will be $F^2=4\nabla\nabla'SS'$.

Ex. 4. To find the envelope of a system of confocal conics.

The equation of such a system is $\frac{x^2}{a^2-k^2}+\frac{y^2}{b^2-k^2}=1$. The reciprocal of this is (Art. 319) $(a^2-k^2)x^2+(b^2-k^2)y^2=1$; and as this denotes a system of conics through the four points of intersection of $(a^2x^2+b^2y^2-1)$ and (x^2+y^2), it follows that the system of confocal conics touches four fixed right lines. Arranging their equation,

$$(b^2x^2+a^2y^2-a^2b^2)+k^2(a^2+b^2-x^2-y^2)-k^4=0,$$

they always touch

$$(a^2+b^2-x^2-y^2)^2+4(b^2x^2+a^2y^2-a^2b^2)=0,$$

which will be found to be equivalent to

$$\{y^2+(x-c)^2\}\{y^2+(x+c)^2\}=0,$$

a result in accordance with Art. 282.

Ex. 5. The equation of the pair of tangents from any point $x'y'z'$ to S is found by substituting $yz'-zy'$, $zx'-xz'$, $xy'-yx'$ for x, y, z in the equation of the reciprocal curve.

Any point on either tangent through $x'y'z'$ evidently possesses the property that the line joining it to $x'y'z'$ touches the curve. In order, then, to find the equation of the pair of tangents, we have only to express (Art. 154) the condition that the line joining two points

$$x(y'z''-y''z')+y(z'x''-z''x')+z(x'y''-x''y')=0$$

should touch the curve, and to consider then $x''y''z''$ as variable. And remembering (Art. 321) that the coefficients are the same in the condition that a line should touch the curve, and in the equation of the reciprocal curve, the truth of the theorem is manifest. As we have already (Art. 150) obtained the equation of the pair of tangents in another form, it follows (as may easily be verified) that

$$(ax^2+a'y^2+\&c.)(ax'^2+a'y'^2+\&c.)-(axx'+a'yy'+\&c.)^2=\mathfrak{A}(yz'-zy')^2+\mathfrak{A}'(zx'-xz')^2+\&c.$$

In like manner,

$$(\mathfrak{A}x^2+\&c.)(\mathfrak{A}x'^2+\&c.)-(\mathfrak{A}xx'+\&c.)^2=\nabla\{a(yz'-zy')^2+\&c.\}.$$

Ex. 6. To verify that, if two conics have double contact with each other, their reciprocals have double contact with each other (Art. 294).

The reciprocal of $S+(lx+my+nz)^2$ is (Art. 297) $\Sigma+\{a(mz-ny)^2+\&c.\}$.

But since (Ex. 5)

$$\nabla\{a(mz-ny)^2+\&c.\}=\Sigma(\mathfrak{A}l^2+\&c.)-(\mathfrak{A}lx+\&c.)^2.$$

The reciprocal is

$$\{\nabla+(\mathfrak{A}l^2+\&c.)\}\Sigma-(\mathfrak{A}lx+\&c.)^2=0,$$

a conic evidently having double contact with Σ.

322. Given the reciprocal of a curve with regard to the origin of co-ordinates, to find the equation of its reciprocal with regard to any point $(x'y')$.

If the perpendicular from the origin on the tangent be P, the perpendicular from any other point is (Art. 27)

$$P-x'\cos\theta-y'\sin\theta,$$

and, therefore, the polar equation of the locus is

$$\frac{k^2}{\rho} = \frac{k^2}{R} - x'\cos\theta - y'\sin\theta;$$

hence

$$\frac{k^2}{R} = \frac{x'x + y'y + k^2}{\rho} \text{ and } \frac{R\cos\theta}{k^2} = \frac{\rho\cos\theta}{xx' + yy' + k^2};$$

we must, therefore, substitute, in the equation of the given reciprocal, $\frac{k^2x}{xx' + yy' + k^2}$ for x, and $\frac{k^2y}{xx' + yy' + k^2}$ for y.

The effect of this substitution may be very simply written as follows: Let the equation of the reciprocal with regard to the origin be $u_n + u_{n-1} + u_{n-2}$, &c. (see Art. 271),

then the reciprocal with regard to any point is

$$u_n + u_{n-1}\left(\frac{xx' + yy' + k^2}{k^2}\right) + u_{n-2}\left(\frac{xx' + yy' + k^2}{k^2}\right)^2 + \text{\&c.}$$

a curve of the same degree as the given reciprocal.

323. Before quitting the subject of reciprocal polars, we wish to mention a class of theorems, for the transformation of which M. Chasles has proposed to take as the auxiliary conic a *parabola* instead of a *circle*. We proved (Art. 216) that the intercept made on the axis of the parabola between any two lines is equal to the intercept between perpendiculars let fall on the axis from the poles of these lines. This principle, then, enables us readily to transform theorems which relate to the magnitude of lines measured parallel to a fixed line. We shall give one or two specimens of the use of this method, premising that to two tangents parallel to the axis of the auxiliary parabola correspond the two points at infinity on the reciprocal curve, and that, consequently, the curve will be a hyperbola or ellipse, according as these tangents are real or imaginary. The reciprocal will be a parabola if the axis pass through a point at infinity on the original curve.

"Any variable tangent to a conic intercepts portions on two parallel tangents whose rectangle is constant."

To the two points of contact of parallel tangents answer the asymptotes of the reciprocal hyperbola, and to the intersections of those parallel tangents with any other tangent answer parallels to the asymptotes through any point; and we obtain, in the first instance, that the asymptotes and parallels to them through any point on the curve intercept portions on any fixed line whose rectangle is constant. But this is plainly equivalent to the theorem: "The rectangle under parallels drawn to the asymptotes from any point on the curve is constant."

Chords drawn from two fixed points of a hyperbola to a variable third point, intercept a constant length on the asymptote.

If any tangent to a parabola meet two fixed tangents, perpendiculars from its extremities on the tangent at the vertex will intercept a constant length on that line.

This method of parabolic polars is plainly much more limited in its application than the method of circular polars, whose resources in transforming theorems of magnitude M. Chasles has possibly underrated.

HARMONIC AND ANHARMONIC PROPERTIES OF CONICS.*

324. The harmonic and anharmonic properties of conic sections admit of so many applications in the theory of these curves, that we think it not unprofitable to spend a little time in pointing out to the student the number of particular theorems either directly included in the general enunciations of these properties, or which may be inferred from them without much difficulty.

The cases which we shall most frequently consider are, when one of the four points of the right line, whose anharmonic ratio we are examining, is at an infinite distance. The anharmonic ratio of four points, A, B, C, D, being in general $= \frac{AB \cdot CD}{AD \cdot BC}$, if D be at an infinite distance, the ratio $\frac{CD}{AD}$ is ultimately $= 1$, and the anharmonic ratio becomes simply $\frac{AB}{BC}$. If the line be cut harmonically, its anharmonic ratio $= 1$, and if D be at an infinite distance AC is bisected. The reader is supposed to be acquainted with the geometric investigation of these and the other fundamental theorems connected with anharmonic section.

* The discovery of the anharmonic properties of conics is due to M. Chasles, from the notes to whose *History of Geometry* the following pages have been developed.

325. We shall commence with the theorem (Art. 147): "If any line through a point O meet a conic in the points R′, R″, and the polar of O in R, the line ORR′R″ is cut harmonically."

First. Let R″ be at an infinite distance; then the line OR must be bisected at R′; that is, *if through a fixed point a line be drawn parallel to an asymptote of an hyperbola, or to a diameter of a parabola, the portion of this line between the fixed point and its polar will be bisected by the curve* (Art. 216).

Secondly. Let R be at an infinite distance, and R′R″ must be bisected at O; that is, *if through any point a chord be drawn parallel to the polar of that point, it will be bisected at the point.*

If the polar of O be at infinity, every chord through that point meets the polar at infinity, and is therefore bisected at O. Hence this point is the centre, or *the centre may be considered as a point whose polar is at infinity* (p. 139).

Thirdly. Let the fixed point itself be at an infinite distance, then all the lines through it will be parallel, and will be bisected on the polar of the fixed point. Hence *every diameter of a conic may be considered as the polar of the point at infinity in which its ordinates are supposed to intersect* (p. 241).

This also follows from the equation of the polar of a point (Art. 144),

$$(2Ax + By + D) + (2Cy + Bx + E)\frac{y'}{x'} + \frac{Dx + Ey + 2F}{x'} = 0.$$

Now, if $x'y'$ be a point at infinity on the line $my = nx$, we must make $\frac{y'}{x'} = \frac{n}{m}$, and x' infinite, and the equation of the polar becomes

$$m(2Ax + By + D) + n(2Cy + Bx + E) = 0,$$

a diameter conjugate to $my = nx$ (Art. 139).

326. We may, in like manner, make particular deductions from the theorem (Art. 149), that the two tangents through any point, any other line through the point, and the line to the pole of this last line, form an harmonic pencil.

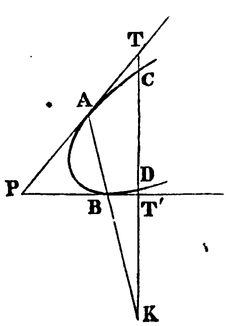
Thus, if one of the lines through the point be a diameter, the other will be parallel to its conjugate, and since the polar of any point on a diameter is parallel to its conjugate, we learn that the portion between the tangents of any line drawn parallel to the polar of the point is bisected by the diameter through it.

Again, let the point be the centre, the two tangents will be the asymptotes. Hence *the asymptotes, together with any pair of conjugate diameters, form an harmonic pencil,* and the portion of any tangent intercepted between the asymptotes is bisected by the curve (Art. 201).

327. The anharmonic property of the points of a conic (Art. 260) gives rise to a much greater variety of particular theorems. For, the four points on the curve may be any whatever, and either one or two of them may be at an infinite distance; the fifth point O, to which the pencil is drawn, may be also either at an infinite distance, or may coincide with one of the four points, in which latter case one of the legs of the pencil will be the tangent at that point; then, again, we may measure the anharmonic ratio of the pencil by the segments on *any* line drawn across it, which we may, if we please, draw parallel to one of the legs of the pencil, so as to reduce the anharmonic ratio to a simple ratio.

The following examples being intended as a practical exercise to the student in developing the consequences of this theorem, we shall merely state the points whence the pencil is drawn, the line on which the ratio is measured, and the resulting theorem, recommending to the reader a closer examination of the manner in which each theorem is inferred from the general principle.

We use the abbreviation $\{O\cdot ABCD\}$ to denote the anharmonic ratio of the pencil OA, OB, OC, OD.



Ex. 1. $\{A.ABCD\} = \{B.ABCD\}$.

Let these ratios be estimated by the segments on the line CD; let the tangents at A, B meet CD in the points T, T′, and let the chord AB meet CD in K, then the ratios are

$$\frac{TK.CD}{TD.KC} = \frac{KT'.CD}{KD.T'C},$$

that is, if any chord CD meet two tangents in T, T′, and their chord of contact in K,

$$KC.KT'.DT = KD.KT.CT'.$$

(The reader must be careful, in this and the following examples, to take the points of the pencil *in the same order* on both sides of the equation. Thus, on the left-hand side of this equation we took K second, because it answers to the leg OB of the pencil; on the right-hand we take K first, because it answers to the leg OA).

Ex. 2. Let T and T' coincide, then

$$KC.DT = KD.CT,$$

or, any chord through the intersection of two tangents is cut harmonically by the chord of contact (Art. 147).

Ex. 3. Let T' be at an infinite distance, or the secant CD drawn parallel to PT', and it will be found that the ratio will reduce to

$$TK^2 = TC.TD.$$

Ex. 4. Let one of the points be at an infinite distance, then $\{O.ABC\infty\}$ is constant. Let this ratio be estimated on the line $C\infty$. Let the lines AO, BO, cut $C\infty$ in a, b; then the ratio of the pencil will reduce to $\frac{Ca}{Cb}$; and we learn, that if two fixed points, A, B, on a hyperbola or parabola, be joined to any variable point O, and the joining line meet a fixed parallel to an asymptote (if the curve be a hyperbola), or a diameter (if the curve be a parabola), in a, b, then the ratio $Ca : Cb$ will be constant.

Ex. 5. If the same ratio be estimated on any other parallel line, lines inflected from any three fixed points to a variable point cut a fixed parallel to an asymptote or diameter, so that $ab : ac$ is constant.

Ex. 6. It follows from Ex. 4, that if the lines joining AB to any fourth point O' meet $C\infty$ in $a'b'$, we must have

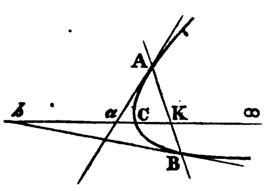
$$\frac{ab}{a'b'} = \frac{aC}{a'C}.$$

Now let us suppose the point C to be also at an infinite distance, the line $C\infty$ becomes an asymptote, the ratio $ab : a'b'$ becomes one of equality, and lines joining two fixed points to any variable point on the hyperbola intercept on either asymptote a constant portion (p. 173).

Ex. 7. $\{A.ABC\infty\} = \{B.ABC\infty\}$.

Let these ratios be estimated on $C\infty$; then if the tangents at A, B, cut $C\infty$ in a, b, and the chord of contact AB in K, we have

$$\frac{Ca}{CK} = \frac{CK}{Cb}$$

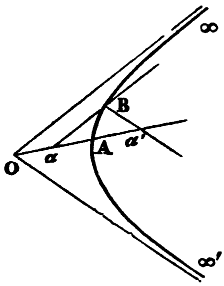


(observing the caution in Ex. 1). Or, if any parallel to an asymptote of an hyperbola, or a diameter of a parabola, cut two tangents and their chord of contact, the intercept from the curve to the chord is a geometric mean between the intercepts from the curve to the tangents. Or, conversely, if a line ab, parallel to a given one, meet the sides of a triangle in the points $a\ b$ K, and there be taken on it a point C such that $CK^2 = Ca : Cb$, the locus of C will be a parabola, if Cb be parallel to the bisector of the base of the triangle (Art. 216), but otherwise an hyperbola, to an asymptote of which ab is parallel.

Ex. 8. Let two of the fixed points be at infinity,

$$\{\infty.AB\infty\infty'\} = \{\infty'.AB\infty\infty'\};$$

the lines $\infty\infty$, $\infty'\infty'$, are the two asymptotes, while $\infty\infty'$ is altogether at infinity.



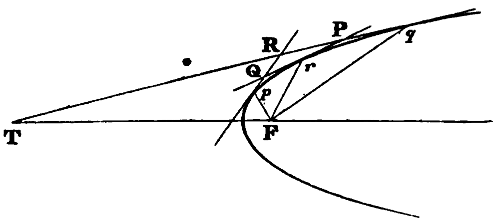
Let these ratios be estimated on the diameter OA; let this line meet the parallels to the asymptotes B∞, B∞′, in a and a'; then the ratios become $\frac{OA}{Oa} = \frac{Oa'}{OA}$. Or, parallels to the asymptotes through any point on a hyperbola cut any semidiameter, so that it is a mean proportional between the segments on it from the centre.

Hence, conversely, if through a fixed point O a line be drawn cutting two fixed lines, Ba, Ba', and a point A taken on it so that OA is a mean between Oa, Oa', the locus of A is a hyperbola, of which O is the centre, and Ba, Ba', parallel to the asymptotes.

Ex. 9. $\{\infty . AB\infty\infty'\} = \{\infty' . AB\infty\infty'\}$.

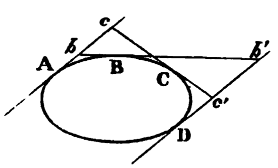
Let the segments be measured on the asymptotes, and we have $\frac{Oa}{Ob} = \frac{Ob'}{Oa'}$, (O being the centre), or the rectangle under parallels to the asymptotes through any point on the curve is constant (we invert the second ratio for the reason given in Ex. 1).

328. We next proceed to examine some particular cases of the anharmonic property of the tangents to a conic (Art. 274).



Ex. 1. This property assumes a very simple form, if the curve be a parabola, for one tangent to a parabola is always at an infinite distance (Art. 255). Hence three fixed tangents to a parabola cut any fourth in the points A, B, C, so that AB : AC is always constant. If the variable tangents coincide in turn with each of the given tangents, we obtain the theorem,

$$\frac{pQ}{QR} = \frac{RP}{Pq} = \frac{Qr}{rP}.$$



Ex. 2. Let two of the four tangents to an ellipse or hyperbola be parallel to each other, and let the variable tangent coincide alternately with each of the parallel tangents. In the first case the ratio is

$$\frac{Ab}{Ac}, \text{ and in the second } \frac{Dc'}{Db'}.$$

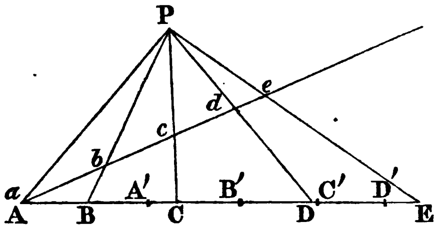
Hence the rectangle $Ab \cdot Db'$ is constant.

It may be deduced from the anharmonic property of the points of a conic, that if the lines joining any point on the curve O to A, D, meet the parallel tangents in the points b, b', then the rectangle $Ab \cdot Db'$ will be constant.

INVOLUTION.

329. If we have a system of points on a right line ABCDE, &c., and another system A′B′C′D′E′, &c., either on the same or

on a different right line, the systems are said to be ***similar***, if the anharmonic ratio of any four points of the first system be equal to that of the four corresponding points of the second system. Thus, if we join the points A, B, C, &c., to any point P, and cut by any transversal the pencil so formed, we obtain a system *abc*, &c., which is obviously similar to the given one. In the figure the transversal is drawn through A, so that the points *a* and A coincide.



It is always possible to construct a system similar to a given one, and such that three arbitrary points A′, B′, C′, shall correspond to three points A, B, C, of the given system. For draw through A any line making an angle with AB; measure on it from the point A, $ab = A'B'$, $ac = A'C'$: then the intersection of *b*B, *c*C determines the point P, by joining which to the points D, E, &c., we obtain the corresponding points *de*, &c. And if we take $C'D' = cd$, $D'E' = de$, &c., we have a system A′B′C′D′E′, &c. similar to ABCDE.

330. When two similar systems form part of the same right line, it will not in general happen that any point will have the same point corresponding to it when it is considered as belonging to the first and as belonging to the second system. Thus if in the figure we consider the point A′ as belonging to the first system, and construct the corresponding point by joining PA′, &c., the point so determined would not in general coincide with A. If, however, it should happen that the points A, A′ mutually correspond, whether A be considered as belonging to the first or to the second system, then *every* pair of points which correspond will correspond, no matter to which system they are considered to belong, and the whole series of points is said to form a system ***in involution***.

Supposing, for instance, that to the points ABB′A′ of the first system there correspond A′B′*b*A of the second, we say that the points *b* and B must coincide. For we have

$$\{ABB'A'\} = \{A'B'bA\},$$

or

$$\frac{AB \cdot B'A'}{AA' \cdot BB'} = \frac{A'B' \cdot bA}{A'A \cdot B'b},$$

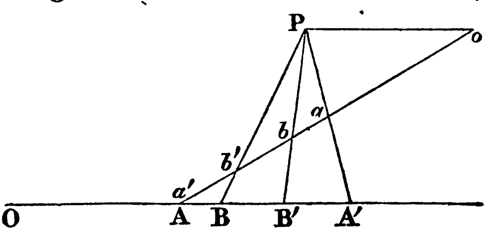
or

$$AB : BB' :: Ab : bB'.$$

The line AB' then is cut at b, and at B into parts which are in the same ratio, and therefore b and B must coincide.

331. *Two pairs of points* AA', BB' *determine a system in involution.*

It would in fact only be a particular case of Art. 329 to determine a system similar to AA'BB'CD, &c., and such that to A, A', B should correspond A', A, B'. We need only draw through A any line making an angle with AB; measure off from A, $a'a = A'A$, $a'b' = A'B'$; then draw $A'a'$, Bb' intersecting in P, and the point P joined to B', C, D, &c. will determine the corresponding points of the second system. N.B.—The points A, A' are said to be conjugate to each other.



332. We recommend the reader to make a table of the different relations of magnitude between three pairs of points in involution, inferred from the identity of their anharmonic ratios. For instance, from $\{ABCA'\} = \{A'B'C'A\}$ we have

$$\frac{AB \cdot CA'}{AA' \cdot BC} = \frac{A'B' \cdot C'A}{AA' \cdot B'C'},$$

or

$$AB \cdot CA' \cdot B'C' = A'B' \cdot C'A \cdot BC.$$

As the development of these relations can present no difficulty to the reader, the only case on which we think it necessary to dwell is when one of the points has its conjugate at an infinite distance. This would happen if we draw Po parallel to AB, and meeting $a'b'$ in o; and measure $A'O = a'o$. The point O will then have its conjugate at infinity, and is called the *centre* of the system of points in involution.

Now the relation between the points takes in this case a very simple form; for we have

$$\{ABOO'\} = \{A'B'O'O\},$$

or

$$\frac{AO \cdot BO'}{AO' \cdot BO} = \frac{A'O' \cdot B'O}{A'O \cdot B'O'};$$

let O′ be at an infinite distance, and this equation becomes

$$OA \cdot OA' = OB \cdot OB';$$

or, *the product of the distances from the centre of any two conjugate points is constant.* It is plain that the construction given in this Article enables us, "being given two pairs of points of a system in involution, to find the centre."

333. Some writers have founded their definition of involution on the property just proved, and have defined a system of points in involution as a series of points so taken that

$$OA \cdot OA' = OB \cdot OB' = \&c. = c^2.$$

It can at once then be proved that the anharmonic ratio of any four points of such a system is equal to that of their four conjugates, since the anharmonic ratio $\dfrac{(r-r')(r''-r''')}{(r-r'')(r'-r''')}$ (where r is the distance of any of the points from O) remains unchanged, if we substitute for each of the distances r, its reciprocal.

334. A point which coincides with its conjugate has been called by Mr. Davies (see *The Mathematician*, vol. i. pp. 169, 243) a *focus* of the system of points in involution. It is plain that there are two foci equidistant from the centre on either side of it, and whose common distance is given by the equation $OF^2 = OA \cdot OA'$. When A and A′ both lie on the same side of the centre we have $OF^2 = + c^2$, and the foci are real; but if A and A′ lie on different sides of the centre $OF^2 = - c^2$, and the foci are imaginary.

Any two conjugate points of the system, together with the two foci, form four points of a line cut harmonically. For the relation $\{AFF'A'\} = \{A'FF'A\}$ gives us

$$\frac{AF \cdot A'F'}{AA' \cdot FF'} = \frac{AF' \cdot A'F}{AA' \cdot FF'}, \text{ or } \frac{FA}{F'A} = \frac{FA'}{F'A'};$$

or the distance between the foci FF′ is divided internally and externally at A and A′ into parts which are in the same ratio.

Cor. When one focus is at infinity, the other bisects the distance between two conjugate points, and it follows hence that in this case the distance AB between any two points of the system is equal to A′B′, the distance between their conjugates.

335. *Given two pairs of points of the system, we can find the foci:* either by first finding the centre (Art. 332), or directly as follows:—Since F is conjugate to itself, we have

$$\{AFBA'\} = \{A'FB'A\},$$

or

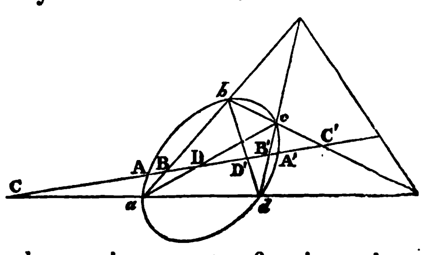
$$\frac{AF \cdot BA'}{A'F \cdot BA} = \frac{A'F \cdot B'A}{AF \cdot B'A'}.$$

Hence $$AF^2 : A'F^2 :: AB \cdot AB' : A'B \cdot A'B';$$

or F is the point where the line AA′ is cut, either internally or externally, in a certain given ratio.

It is important to observe that the relation between six points in involution is of the class noticed in Art. 314, and is such that the same relations will subsist between the sines of the angles subtended by them at any point as subsist between the segments of the lines themselves. Consequently, *if a pencil be drawn from any point to six points in involution, any transversal cuts this pencil in six points in involution.* Again, *the reciprocal of six points in involution is a pencil in involution.*

336. We proceed to mention the most important application of these principles to the theory of conic sections.



If a quadrilateral *abcd* be inscribed in a conic section, and any transversal cut the conic in A, A′, the sides *ab*, *cd*, in B, B′, and the sides *ad*, *bc* in C, C′, then the points AA′BB′CC′ are in involution, for by the anharmonic property of conic sections,

$$\{a \cdot AdbA'\} = \{c \cdot AdbA'\};$$

but if we observe the points in which these pencils cut AA′, we get

$$\{ACBA'\} = \{AB'C'A'\} = \{A'C'B'A\}.$$

Since two pairs of points BB′, CC′ determine a system in involution, the points EE′, in which any other conic through the points *abcd* meet the transversal, belong to the same system in involution. Hence *a system of conics circumscribing the same quadrilateral meet any transversal in a system of points in involution.*

Reciprocally, *if a system of conics be inscribed in the same quadrilateral, the pairs of tangents drawn to them from any point will form a system in involution.*

337. Since the diagonals *ac*, *bd* may be considered as a conic through the four points, it follows as a particular case of the last Article that any transversal cuts the four sides, and the diagonals of a quadrilateral in points BB′, CC′, DD′, which are in involution. This property enables us, being given two pairs of points BB′, DD′ of a system in involution, to construct the point conjugate to any other C. For take any point at random, *a*; join *a*B, *a*D, *a*C; construct any triangle *bcd*, whose vertices rest on these three lines, and two of whose sides pass through B′D′, then the remaining side will pass through C′, the point conjugate to C. The point *a* may be taken at infinity, and the lines *a*B, *a*D, *a*C will then be parallel to each other. If the point C be at infinity the same method will give us the centre of the system. The simplest construction for this case is,—"Through B, D, draw any pair of parallel lines B*b*, D*c*; and through B′D′ a different pair of parallels D′*b*, B′*c*; then *bc* will pass through the centre of the system."

Ex. 1. If three conics circumscribe the same quadrilateral, the common tangent to any two is cut harmonically by the third. For the points of contact of this tangent are the foci of the system in involution.

Ex. 2. If through the intersection of the common chords of two conics we draw a tangent to one of them, this line will be cut harmonically by the other. For in this case the points D and D′ in the last figure coincide, and will therefore be a focus.

Ex. 3. If two conics have double contact with each other, or if they have a contact of the third order, any tangent to the one is cut harmonically at the points where it meets the other, and where it meets the chord of contact. For in this case the common chords coincide, and the point where any transversal meets the chord of contact is a focus.

Ex. 4. To describe a conic through four points *abcd* to touch a given right line. The point of contact must be one of the foci of the system BB′CC′, &c., and these points can be determined by Art. 334. This problem, therefore, admits of two solutions.

Ex. 5. If a parallel to an asymptote meet the curve in C, and any inscribed quadrilateral in points *abcd*; $Ca \cdot Cc = Cb \cdot Cd$. For C is the centre of the system.

Ex. 6. Solve the examples, p. 273, &c., as cases of involution.

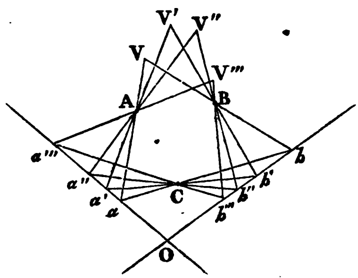
In Ex. 1, K is a focus: in Ex. 2, T is also a focus: in Ex. 3, T is a centre, &c.

Ex. 7. The intercepts on any line between a hyperbola and its asymptotes are equal. For in this case one focus of the system is at infinity (Art. 335).

338. We now proceed to give some examples of problems easily solved by the help of the anharmonic properties of conics.

Ex. 1. To prove Mac Laurin's method of generating conic sections (p. 230), viz.,—To find the locus of the vertex V of a triangle whose sides pass through the points A, B, C, and whose base angles move on the fixed lines O*a*, O*b*.

Let us suppose four such triangles drawn, then since the pencil $\{C.aa'a''a'''\}$ is the same pencil as $\{C.bb'b''b'''\}$, we have



$$\{aa'a''a'''\} = \{bb'b''b'''\},$$

and, therefore,

$$\{A.aa'a''a'''\} = \{B.bb'b''b'''\};$$

or, from the nature of the question,

$$\{A.VV'V''V'''\} = \{B.VV'V''V'''\};$$

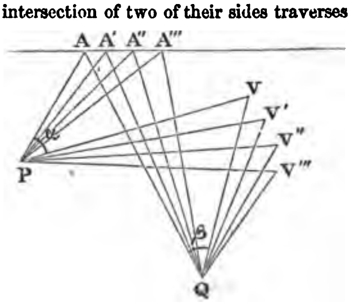
and therefore A, B, V, V', V'', V''' lie on the same conic section. Now if the first three triangles be fixed, it is evident that the locus of V''' is the conic section passing through ABVV'V''.

Ex. 2. M. Chasles has showed that the same demonstration will hold if the side *ab*, instead of passing through the fixed point C, touch any conic which touches O*a*, O*b*, for then any four positions of the base cut O*a*, O*b*, so that

$$\{aa'a''a'''\} = \{bb'b''b'''\} \text{ (Art. 274)},$$

and the rest of the proof proceeds the same as before.

Ex. 3. Newton's method of generating conic sections:—Two angles of constant magnitude move about fixed points P, Q; the intersection of two of their sides traverses the right line AA'; then the locus of V, the intersection of their other two sides, will be a conic passing through P, Q.



For, as before, take four positions of the angles, then

$$\{P.AA'A''A'''\} = \{Q.AA'A''A'''\};$$

but $\{P.AA'A''A'''\} = \{P.VV'V''V'''\}$,

$$\{Q.AA'A''A'''\} = \{Q.VV'V''V'''\},$$

since the angles of the pencils are the same;

therefore $$\{P.VV'V''V'''\} = \{Q.VV'V''V'''\};$$

and, therefore, as before, the locus of V''' is a conic through P, Q, V, V', V''.

Ex. 4. M. Chasles has extended this method of generating conic sections, by supposing the point A, instead of moving on a right line, to move on any conic passing through the points P, Q, for we shall still have

$$\{P.AA'A''A'''\} = \{Q.AA'A''A'''\}.$$

Ex. 5. The demonstration would be the same if, in place of the angles APV, AQV being constant, APV and AQV cut off constant intercepts each on one of two fixed lines, for we should then prove the pencil

$$\{P.AA'A''A'''\} = \{P.VV'V''V'''\},$$

because both pencils cut off intercepts of the same length on a fixed line.

Thus, also, given base of a triangle and the intercept made by the sides on any fixed line, we can prove that the locus of vertex is a conic section.

Ex. 6. We may also extend Ex. 1, by supposing the extremities of the line ab to move on any conic section passing through the points AB, for, taking four positions of the triangle, we have, by Art. 275,

$$\{aa'a''a'''\} = \{bb'b''b'''\};$$

therefore, $$\{A.aa'a''a'''\} = \{B.bb'b''b'''\},$$

and the rest of the proof proceeds as before.

Ex. 7. The base of a triangle passes through C, the intersection of common tangents to two conic sections; the extremities of the base ab lie one on each of the conic sections, while the sides pass through fixed points AB, one on each of the conics: the locus of the vertex is a conic through A, B.

The proof proceeds exactly as before, depending now on the last theorem proved, Art. 275.

We may mention that the theorem of Art. 275 admits of a simple geometrical proof. Let the pencil $\{O.ABCD\}$ be drawn from points corresponding to $\{o.abcd\}$. Now, the lines OA, oa, intersect at r on one of the common chords of the conics; in like manner, BO, bo, intersect in r' on the same chord, &c.; hence $\{rr'r''r'''\}$ measures the anharmonic ratio of both these pencils.

Ex. 8. In Ex. 6 the base, instead of passing through a fixed point C, may be supposed to touch a conic having double contact with the given conic (see Art. 278).

Ex. 9. If a polygon be inscribed in a conic, all whose sides but one pass through fixed points, the envelope of that side will be a conic having double contact with the given one.

For, take any four positions of the polygon, then, if a, b, c, &c., be the vertices of the polygon, we have

$$\{aa'a''a'''\} = \{bb'b''b'''\} = \{cc'c''c'''\}, \text{ \&c.}$$

The problem is, therefore, reduced to that of Art. 278,—"Given three pairs of points, $aa'a''$, $dd'd''$, to find the envelope of $a'''d'''$, such that

$$\{aa'a''a'''\} = \{dd'd''d'''\}."$$

Ex. 10. To inscribe a polygon in a conic section, all whose sides pass through fixed points.

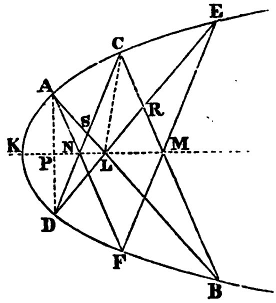
If we assume any point (a) at random on the conic for the vertex of the polygon, and form a polygon whose sides pass through the given points, the point z, where the last side meets the conic, will not, in general, coincide with a. If we make four such attempts to inscribe the polygon, we must have, as in the last example,

$$\{aa'a''a'''\} = \{zz'z''z'''\}.$$

Now, if the last attempt were successful, the point a''' would coincide with z''', and the problem is reduced to,—"Given three pairs of points, $aa'a''$, $zz'z''$, to find a point K such that

$$\{Kaa'a''\} = \{Kzz'z''\}."$$

Now, if we make $az''a'za''z'$ the vertices of an inscribed hexagon (in the order here given, taking an a and z alternately, and so that az, $a'z$, $a''z''$, may be opposite vertices), then either of the points in which the line joining the intersections of opposite sides meets the conic may be taken for the point K. For, in the figure, the points ACE are $aa'a''$, DFB are $zz'z''$; and if we take the sides in the order ABCDEF, L, M, N are the intersections of opposite sides. Now, since {KPNL} measures both {D . KACE} and {A.KDFB}, we have



$$\{KACE\} = \{KDFB\} \quad Q. E. D.*$$

It is easy to see, from the last example, that K is a point of contact of a conic having double contact with the given conic, to which az, $a'z'$, $a''z''$ are tangents, and that we have therefore just given the solution of the question, "To describe a conic touching three given lines, and having double contact with a given conic."

Ex. 11. The anharmonic property affords also a simple proof of Pascal's theorem, alluded to in the last example.

We have {E.CDFB} = {A.CDFB}. Now, if we examine the segments made by the first pencil on BC, and by the second on DC, we have

$$\{CRMB\} = \{CDNS\}.$$

Now, if we draw a pencil from the point L to each of these points, both pencils will have the three legs, CL, DE, AB, common, therefore the fourth legs, NL, LM, must form one right line.

Ex. 12. Pascal's theorem leads at once to Mac Laurin's method of generating conic sections, for if we suppose the five points ABCDE given, and F variable, then F will be the vertex of a triangle FMN, whose sides pass through the fixed points L, A, E, and whose base angles move on the fixed lines CD, CB. We see, therefore, that, given five points on a conic, we can determine as many other points on the conic as we please. By the same construction, given five points on a conic, ABCDE, we can determine the point where any line AN through one of them meets the conic again. So also, given five points on a conic, we can find its centre. For we may draw parallels through A to BC, BD, and determine the points where they meet the conic again, and then find the centre by note, p. 128.

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* This construction for inscribing a polygon in a conic is due to M. Poncelet (*Traité des Propriétés Projectives*, p. 351). The demonstration here used, which was communicated to me by Mr. Townsend, seems to me more simple than that employed by M. Poncelet. The proof here used shows that Poncelet's construction will equally solve the problem, "To inscribe a polygon in a conic, each of whose sides shall touch a conic having double contact with the given conic." The conics touched by the sides may be all different.

Ex. 13. Given four points on a conic, ADFB, and two fixed lines through any one of them, DC, DE, to find the envelope of the line CE joining the points where those fixed lines again meet the curve.

The vertices of the triangle CEM move on the fixed lines DC, DE, NL, and two of its sides pass through the fixed points, B, F, therefore, the third side envelopes a conic section touching DC, DE (by the reciprocal of Mac Laurin's mode of generation).

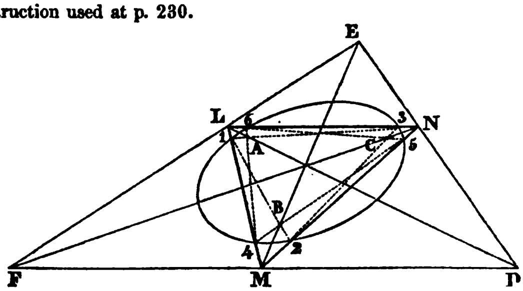
Ex. 14. Given four points on a conic ABDE, and two fixed lines, AF, CD, passing each through a different one of the fixed points, the line CF joining the points where the fixed lines again meet the curve will pass through a fixed point.

For the triangle CFM has two sides passing through the fixed points B, E, and the vertices move on the fixed lines AF, CD, NL, which fixed lines meet in a point, therefore (p. 255) CF passes through a fixed point.

The reader will find, in the section on Projections, how the last two theorems are suggested by other well-known theorems.

Ex. 15. To inscribe a triangle in a conic whose three sides pass through three given points.

This is of course a particular case of Ex. 10, but our present object is to give a geometrical proof of the construction used at p. 230.



If we consider the quadrilateral of which E, L, N are vertices, and D, F the intersections of opposite sides; by the harmonic properties of a quadrilateral, ML, ME, MN, MD form a harmonic pencil, and therefore the line B1 is cut harmonically in the points where it meets these four lines. But since B is the pole of MD, B1 is also cut harmonically in the points where it meets the conic and where it meets MD; hence it appears that B1 and MN must intersect on the conic, or that 1, 2, B lie on one right line. In the same manner it is proved that 13 passes through A, and 32 through C.

339. It was proved (Ex. 4, p. 244) that the anharmonic ratio of four points on a right line is the same as that of their polars with respect to any conic. A particular case of this theorem is, *the anharmonic ratio of any four diameters is equal to that of their four conjugates.* We might also prove this directly, from the consideration that the anharmonic ratio of four chords proceeding from any point of the curve is equal to that of the supplemental chords (Art. 183).

A conic circumscribes a given quadrilateral, to find the locus of its centre.

Draw diameters of the conic bisecting the sides of the quadrilateral, their anharmonic ratio is equal to that of their four conjugates, but this last ratio is given, since the conjugates are parallel to the four given lines; hence the locus is a conic passing through the middle points of the given sides. If we take the cases where the conics break up into two right lines, we see that the intersections of the diagonals, and also those of the opposite sides, are points in the locus, and, therefore, that these points lie on a conic passing through the middle points of the sides and of the diagonals. When the given quadrilateral has a re-entrant angle it is easy to see that such a quadrilateral cannot be inscribed in a closed figure of the shape of the ellipse or parabola, and that the circumscribing conic must therefore be a hyperbola, which may have some of the vertices in opposite branches. But since the centre of an hyperbola is never at infinity, the locus of centres must in this case be an *ellipse*. Through four points not so disposed, in general, two parabolæ can be drawn, for (Art. 255) this is a particular case of Ex. 4, p. 280. The locus of centres will in this case be a *hyperbola*, having for asymptotes lines parallel to the diameters of these two parabolæ. The locus of centres will be a *parabola* when one of the given points is at an infinite distance; that is, when it is required, "Given three points and a parallel to an asymptote, to find the locus of centre."

It is very easy to show, by the same method, that the locus of the pole of any given right line is a conic section.

340. We think it unnecessary to go through the theorems, which are only the polar reciprocals of those investigated in the last examples; but we recommend the student to form the polar reciprocal of each of these theorems, and then to prove it directly by the help of the anharmonic property of the *tangents* of a conic. A single example will suffice.

Any transversal through a fixed point P *meets two fixed lines* OA, OB, *in the points* A, B, *and portions of given lengths* AC, BD, *are taken on those lines: to find the envelope of* CD.

Take any four positions of the transversal, and we have

$$\{AA'A''A'''\} = \{BB'B''B'''\},$$

but $\{AA'A''A'''\} = \{CC'C''C'''\}$, and $\{BB'B''B'''\} = \{DD'D''D'''\}$;

therefore, the four lines, CD, C′D′, C″D″, C‴D‴, cut the two lines OC, OD, so that

$$\{CC'C''C'''\} = \{DD'D''D'''\},$$

and, therefore, the envelope of CD is a conic touching OA, OB.

341. Generally when the envelope of a moveable line is found by this method to be a conic section, it is useful to take notice whether in any particular position the moveable line can be altogether at an infinite distance, for if it can, the envelope is a parabola (Art. 255). Thus, in the last example the line CD cannot be at an infinite distance, unless in some position AB can be at an infinite distance, that is, unless P is at an infinite distance. Hence we see that in the last example if the transversal, instead of passing through a fixed point, were parallel to a given line, the envelope would be a parabola. In like manner, the nature of the locus of a moveable point is often at once perceived by observing particular positions of the moveable point, as we have exemplified in Art. 339.

342. Given three points on a right line, a, b, c, and three points on another right line, A, B, C, if we take dD so that

$$\{abcd\} = \{ABCD\},$$

it is evident, from the preceding Articles, that the envelope of dD is a conic section, and that the lines pd, PD, joining dD to two fixed points, will intersect on a conic passing through these points.* Let us examine the most general relation between d and D that this should be the case. If we denote the distances of $abcd$ from any fixed point o on the same line by r, r', r'', r''', and the distances of ABCD from a fixed point O on the other right line by R, R′, R″, R‴, we have

$$\frac{(r-r')(r''-r''')}{(r-r'')(r'-r''')} = \frac{(R-R')(R''-R''')}{(R-R'')(R'-R''')};$$

* We saw, p. 229, that it is also true, if ABCD, $abcd$, be points on the same conic section, that Dd will envelope a conic if $\{ABCD\} = \{abcd\}$, and the intersection of PD, pd, will in this case be on a conic if P, p be points on the conic. Again, any two conics will be cut by four tangents to any conic having double contact with both, so that $\{ABCD\} = \{abcd\}$ (Art. 278); but it will not be true conversely, that, if this relation holds, the envelope of Dd will be a conic, unless the points ABC, abc, be so taken that Aa, Bb, Cc, may all touch the same conic having double contact with both.

and if we suppose r and R alone variable, this gives a relation of the form

$$kRr + lR + mr + n = 0 \quad \text{(compare Art. 278)}.$$

This relation containing three independent constants is, therefore, the most general connexion between od and OD if dD envelope a conic touching od, OD.

If $k = 0$, dD will envelope a parabola, since then R and r will become infinite at the same time.

M. Chasles has given this relation in a different form. Let there be given two other points e and E, then if $\lambda . \frac{ed}{od} + \mu . \frac{ED}{OD} = 1$, dD will envelope a conic; for if the distances eo, EO, be called a, A, this relation may be written

$$\lambda . \frac{r-a}{r} + \mu . \frac{R-A}{R} = 1,$$

an equation included in the general form we have given.

343. The distances from the origin of a pair of points on the axis being given by the equation $Ax^2 + 2Bx + C = 0$, and those of another pair of points by the equation $A'x^2 + 2B'x + C' = 0$, to find the condition that the four should form a harmonic system.

The roots of the first equation being α, α', and of the second β, β', the required condition is

$$(\beta - \alpha)(\beta' - \alpha') + (\beta - \alpha')(\beta' - \alpha) = 0,$$

which, expressed in terms of the coefficients, is

$$AC' + A'C - 2BB' = 0.$$

N. B. It can be proved that the condition that the anharmonic ratio of the system shall be given is, that $(2BB' - AC' - A'C)^2$ shall be in a given ratio to $(B^2 - AC)(B'^2 - A'C')$.

344. The pair of points given by any equation of the form $(Ax^2 + 2Bx + C) + l(A'x^2 + 2B'x + C') = 0$ is in involution with the points given by $Ax^2 + 2Bx + C = 0$, $A'x^2 + 2B'x + C' = 0$.

For, let the foci of the system determined by the latter two pairs of points be given by the equation $ax^2 + 2bx + c = 0$, then we must have (Art. 343)

$$aC + cA - 2bB = 0, \quad aC' + cA' - 2bB' = 0;$$

and it is evident that when these conditions are fulfilled we must have

$$a(C + lC') + c(A + lA') - 2b(B + lB') = 0.$$

345. To find the centre and the foci of the system just written.

The foci are found, by solving for a, b, c, from the equations

$$aC + cA - 2bB = 0, \quad aC' + cA' - 2bB' = 0,$$

and substituting the resulting values in $ax^2 + 2bx + c = 0$; when we get

$$(AB' - BA')x^2 + (AC' - CA')x + (BC' - CB') = 0.$$

This may be otherwise written, if we make the equation homogeneous by introducing a new variable y, and write

$$U = Ax^2 + 2Bxy + Cy^2, \quad V = A'x^2 + 2B'xy + C'y^2,$$

then the equation which determines the foci is

$$\frac{dU}{dx}\frac{dV}{dy} - \frac{dU}{dy}\frac{dV}{dx} = 0.$$

The centre is got by determining l so that the equation $U + lV = 0$ shall have one of its roots infinite, or shall have the coefficient of $x^2 = 0$ (Art. 131). The centre therefore is given by the equation

$$2(BA' - B'A)x + (CA' - C'A) = 0.$$

346. *To find the locus of a point such that the tangents from it to two given conics shall form a harmonic pencil.*

For simplicity we shall take the equations of the conics, $Ax^2 + Cy^2 + Fz^2 = 0$, $A'x^2 + C'y^2 + F'z^2 = 0$, which is equivalent to supposing (see Art. 281) that we have chosen for x, y, z the three lines whose poles with regard to both conics are the same (Art. 318). Then the equation of the pair of tangents from any point to the first conic being

$$(Ax^2 + Cy^2 + Fz^2)(Ax'^2 + Cy'^2 + Fz'^2) = (Axx' + Cyy' + Fzz')^2,$$

if we make in this $z = 0$, the points where the line z is met by these tangents is given by the equation

$$A(Cy'^2 + Fz'^2)x^2 - 2ACx'y'xy + C(Ax'^2 + Fz'^2)y^2 = 0,$$

and forming the condition (Art. 345) that this shall form a harmonic system with the corresponding pair of points for the second conic, we find for the equation of the locus,

$$AC'(A'x^2 + F'z^2)(Cy^2 + Fz^2) + A'C(Ax^2 + Fz^2)(C'y^2 + F'z^2) = 2AA'CC'x^2y^2,$$

or $AA'(CF' + C'F)x^2 + CC'(AF' + A'F)y^2 + FF'(AC' + A'C)z^2 = 0.$

And it will be seen that this is identical with the equation (see Ex. 3, p. 268) of the conic F which passes through the eight points of contact of common tangents to the two conics. It is proved in like manner that if the anharmonic ratio of the tangents be given, the locus is a curve of the fourth degree, $F^2 = kSS'$.

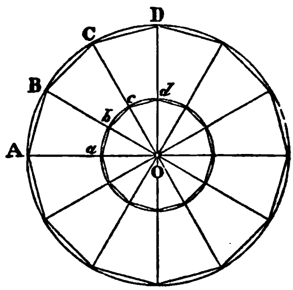
THE METHOD OF INFINITESIMALS.

347. In the next Part we purpose to show how the differential calculus enables us readily to draw tangents to curves, and to determine the magnitude of their areas and arcs. We wish first, however, to give the reader some idea of the manner in which these problems were investigated by geometers before the invention of that method. The geometric methods are not merely interesting in a historical point of view; they afford solutions of some questions more concise and simple than those furnished by analysis, and they have even recently led to a beautiful theorem (Art. 357), which had not been anticipated by those who have applied the integral calculus to the rectification of conic sections.

If a polygon be inscribed in any curve, it is evident that the more the number of the sides of the polygon is increased, the more nearly will the area and perimeter of the polygon approach to equality with the area and perimeter of the curve, and the more nearly will any side of the polygon approach to coincidence with the tangent at the point where it meets the curve. Now, if the sides of the polygon be multiplied *ad infinitum*, the polygon will coincide with the curve, and the tangent at any point will coincide with the line joining two indefinitely near points on the curve. In like manner, we see that the more the number of the sides of a *circumscribing* polygon is increased, the more nearly will its area and perimeter approach to equality with the area and perimeter of the curve, and the more nearly will the intersection of two of its adjacent sides approach to the point of contact of either. Hence, in investigating the area or perimeter of any curve, we may substitute for the curve an inscribed or circumscribing polygon of an indefinite number of sides; we may consider any tangent of the curve as the line joining two indefinitely near points on the curve, and any point on the curve as the intersection of two indefinitely near tangents.

348. Ex. 1. *To find the direction of the tangent at any point of a circle.*

In any isosceles triangle AOB, either base angle OBA is less than a right angle by half the vertical angle; but as the points A and B approach to coincidence, the vertical angle may be supposed less than any assignable angle, therefore the angle OBA which the tangent makes with the radius is ultimately equal to a right angle. We shall frequently have occasion to use the principle here proved, viz., that two indefinitely near lines of *equal length* are at right angles to the line joining their extremities.



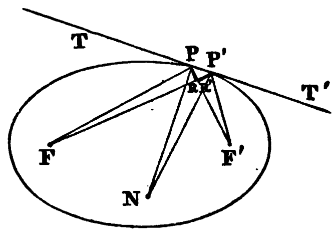
Ex. 2. *The circumferences of two circles are to each other as their radii.*

If polygons of the same number of sides be inscribed in the circles, it is evident, by similar triangles, that the bases *ab*, AB, are to each other as the radii of the circles, and, therefore, that the whole perimeters of the polygons are to each other in the same ratio; and since this will be true, no matter how the number of sides of the polygon be increased, the circumferences are to each other in the same ratio.

Ex. 3. *The area of a circle is equal to the radius multiplied by the semicircumference.*

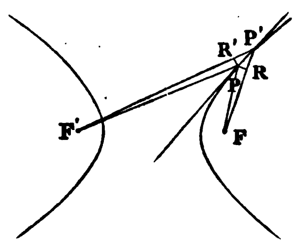
For the area of any triangle OAB is equal to half its base multiplied by the perpendicular on it from the centre; hence the area of any inscribed regular polygon is equal to half the sum of its sides multiplied by the perpendicular on any side from the centre; but the more the number of sides is increased, the more nearly will the perimeter of the polygon approach to equality with that of the circle, and the more nearly will the perpendicular on any side approach to equality with the radius, and the difference between them can be made less than any assignable quantity; hence ultimately the area of the circle is equal to the radius multiplied by the semicircumference; or $= \pi r^2$.

349. Ex. 1. *To determine the direction of the tangent at any point on an ellipse.*



Let P and P′ be two indefinitely near points on the curve, then FP + PF′ = FP′ + P′F′; or, taking FR = FP, F′R′ = F′P′, we have P′R = PR′; but in the triangles PRP′, PR′P′, we have also the base PP′ common, and (by Ex. 1, Art. 348) the angles PRP′, PR′P′ right; hence the angle PP′R = P′PR′. Now TPF is ultimately equal to PP′F, since their difference PFP′ may be supposed less than any given angle; hence TPF = P′PF′, or the focal radii make equal angles with the tangent.

Ex. 2. *To determine the direction of the tangent at any point on a hyperbola.*



We have

$$F'P' - F'P = FP' - FP,$$

or, as before,

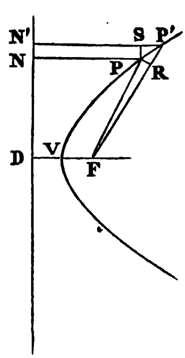
$$P'R = P'R'.$$

Hence the angle

$$PP'R = PP'R',$$

or, the tangent is the internal bisector of the angle FPF′.

Ex. 3. *To determine the direction of the tangent at any point of a parabola.*



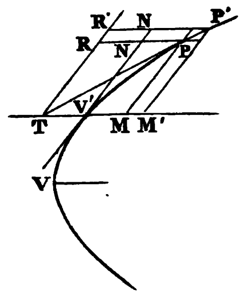
We have FP = PN, and FP′ = P′N′; hence P′R = P′S, or the angle N′P′P = FP′P. The tangent, therefore, bisects the angle FPN.

350. Ex. 1. *To find the area of the parabolic sector* FVP.

Since PS = PR, and PN = FP, we have the triangle FPR half the parallelogram PSNN′. Now if we take a number of points P′P″, &c.

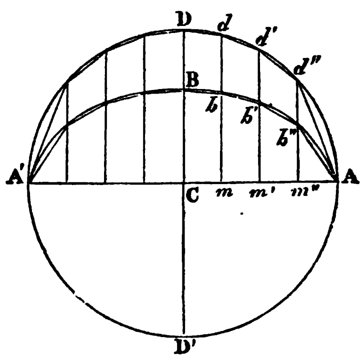
between V and P, it is evident that the closer we take them, the more nearly will the sum of all the parallelograms PSNN', &c., approach to equality with the area DVPN, and the sum of all the triangles PFR, &c., to the sector VFP; hence ultimately the sector PFV is half the area DVPN, and therefore one-third of the quadrilateral DFPN.

Ex. 2. *To find the area of the segment of a parabola cut off by any right line.*



Draw the diameter bisecting it, then the parallelogram PR' is equal to PM', since they are the complements of parallelograms about the diagonal; but since TM is bisected at V', the parallelogram PN' is half PR'; if, therefore, we take a number of points P, P', P'', &c., it follows that the sum of all the parallelograms PM' is double the sum of all the parallelograms PN', and therefore ultimately that the space V'PM is double V'PN; hence the area of the parabolic segment V'PM is to that of the parallelogram V'NPM in the ratio 2 : 3.

351. Ex. 1. *The area of an ellipse is equal to the area of a circle whose radius is a geometric mean between the semiaxes of the ellipse.*



For if the ellipse and the circle on the transverse axis be divided by any number of lines parallel to the axis minor, then since $mb : md :: m'b' : m'd' :: b : a$, the quadrilateral $mbb'm'$ is to $mdd'm'$ in the same ratio, and the sum of all the one set of quadrilaterals, that is, the polygon $Bbb'b''A$ inscribed in the ellipse is to the corresponding polygon $Ddd'd''A$ inscribed in the circle, in the same ratio. Now this will be true whatever be the number of the sides of the polygon: if we suppose them, therefore, increased indefinitely, we learn that the area of the ellipse is to the area of the circle as b to a; but the area of the circle being $= \pi a^2$, the area of the ellipse $= \pi ab$.

Cor. It can be proved, in like manner, that if any two figures be such that the ordinate of one is in a constant ratio to the corresponding ordinate of the other, the areas of the figures are in the same ratio.

Ex. 2. *Every diameter of a conic bisects the curve.*

For if we suppose a number of ordinates drawn to this diameter, since the diameter bisects them all, it also bisects the trapezium formed by joining the extremities of any two adjacent ordinates, and by supposing the number of these trapezia increased without limit, we see that the diameter bisects the curve.

352. Ex. 1. *The area of the sector of a hyperbola made by joining any two points of it to the centre, is equal to the area of the segment made by drawing parallels from them to the asymptotes.*

For since the triangle PKC = QLC, the area PQC = PQKL.

Ex. 2. *Any two segments,* PQKL, RSMN, *are equal, if*

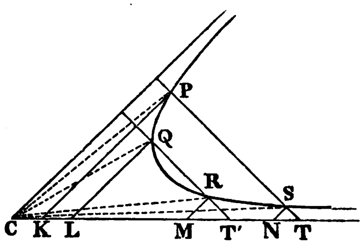
$$PK : QL :: RM : SN.$$

For

$$PK : QL :: CL : CK,$$

but (Art. 202)

$$CL = MT', \; CK = NT;$$



we have, therefore,

$$RM : SN :: MT' : NT,$$

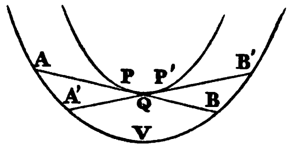
and therefore QR is parallel to PT. We can now easily prove that the sectors PCQ, RCS are equal, since the diameter bisecting PS, QR will bisect both the hyperbolic area PQRS, and also the triangles PCS, QCR.

If we suppose the points Q, R to coincide, we see that we can bisect any area PKNS by drawing an ordinate QL, a geometric mean between the ordinates at its extremities.

Again, if a number of ordinates be taken, forming a continued geometric progression, the area between any two is constant.

353. *The tangent to the interior of two similar, similarly placed, and concentric conics cuts off a constant area from the exterior conic.*

For we proved (p. 203) that this tangent is always bisected at the point of contact; now if we draw any two tangents, the angle AQA′ will be equal to BQB′, and the nearer we suppose the point Q to P, the more nearly will the sides AQ, A′Q approach to equality with the sides BQ, B′Q; if, therefore, the two tangents be taken indefinitely near, the triangle AQA′ will be equal to BQB′, and the space AVB will be equal to A′VB′; since, therefore, this space remains constant as we pass from any tangent to the consecutive tangent, it will be constant whatever tangent we draw.



Cor. 1. It can be proved, in like manner, that if a tangent to one curve always cut off a constant area from another, it will be bisected at the point of contact; and, conversely, that if it be always bisected it cuts off a constant area.

Hence we can draw through a given point a line to cut off from a given conic the *minimum* area. If it were required to cut off a *given* area it would be only necessary to draw a tangent through the point to some similar and concentric conic, and the greater the given area, the greater will be the distance between the two conics. The area will therefore evidently be least when this last conic passes through the given point; and since the tangent at the point must be bisected, the line through a given point which cuts off the minimum area is bisected at that point.

In like manner, the chord drawn through a given point which cuts off the minimum or maximum area from any curve is bisected at that point. In like manner can be proved the following two theorems. I am indebted to the late Professor Mac Cullagh for my knowledge of all the theorems of this Article, and I do not remember having seen them elsewhere published.

Ex. 1. *If a tangent* AB *to one curve cut off a constant arc from another, it is divided at the point of contact, so that* AP : PB *inversely as the tangents to the outer curve at* A *and* B.

Ex. 2. *If the tangent* AB *be of a constant length, and if the perpendicular let fall on* AB *from the intersection of the tangents at* A *and* B *meet* AB *in* M, *then* AP *will* = MB.

354. *To find the radius of curvature at any point on an ellipse.*

The centre of the circle circumscribing any triangle is the intersection of perpendiculars erected at the middle points of the sides of that triangle; it follows, therefore, that the centre of the circle passing through three consecutive points on the curve is the intersection of two consecutive normals to the curve.

Now, given any two triangles FPF′, FP′F′, and PN, P′N, the two bisectors of their vertical angles, it is easily proved, by elementary geometry, that twice the angle PNP′ = PFP′ + PF′P′. (See figure, p. 291).

Now, since the arc of any circle is proportional to the angle it subtends at the centre (Euc. VI. 33), and also to the radius (Art. 348), if we consider PP′ as the arc of a circle, whose centre is N, the angle PNP′ is measured by $\frac{PP'}{PN}$. In like manner, taking FR = FP, PFP′ is measured by $\frac{PR}{FP}$, and we have

$$\frac{2PP'}{PN} = \frac{PR}{FP} + \frac{P'R'}{F'P'};$$

but
$$PR = P'R' = PP' \sin PP'F;$$

therefore, denoting this angle by θ, PN by R, FP, F′P, by ρ, ρ', we have

$$\frac{2}{R \sin\theta} = \frac{1}{\rho} + \frac{1}{\rho'}.$$

Hence it may be inferred that *the focal chord of curvature is double the harmonic mean between the focal radii.* Substituting $\frac{b}{b'}$ for $\sin\theta$, $2a$ for $\rho + \rho'$, and b'^2 for $\rho\rho'$, we obtain the known value,

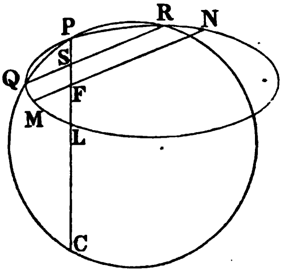
$$R = \frac{b'^3}{ab}.$$

The radius of curvature of the hyperbola or parabola can be investigated by an exactly similar process. In the case of the parabola we have ρ' infinite, and the formula becomes

$$\frac{2}{R \sin\theta} = \frac{1}{\rho}.$$

I owe to Mr. Townsend the following investigation, by a different method, of the length of the focal chord of curvature:

Draw *any* parallel QR to the tangent at P, and describe a circle through PQR meeting the focal chord PL of the conic at C. Then by the circle PS · SC = QS · SR, and by the conic (Ex. 2, p. 170)



$$PS \cdot SL : QS \cdot SR :: PL : MN;$$

therefore, whatever be the circle,

$$SC : SL :: MN : PL;$$

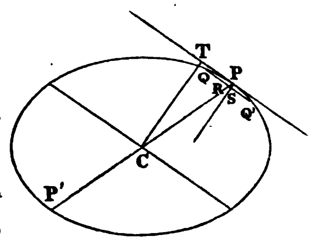
but for the circle of curvature the points S and P coincide, therefore

$$PC : PL :: MN : PL;$$

or, *the focal chord of curvature is equal to the focal chord of the conic drawn parallel to the tangent at the point* (p. 210).

355. The radius of curvature of a central conic may otherwise be found thus:

Let Q be an indefinitely near point on the curve, QR a parallel to the tangent, meeting the normal in S; now, if a circle be described passing through P, Q, and touching PT at P, since QS is a perpendicular let fall from Q on the diameter of this circle, we have PQ^2 = PS multiplied by the diameter; or the radius of curvature $= \frac{PQ^2}{2PS}$. Now, since QR is always drawn parallel to the tangent, and since PQ must ultimately coincide with the tangent, we have PQ ultimately equal to QR; but, by the property of the ellipse (if we denote CP and its conjugate by a', b'),



$$b'^2 : a'^2 :: QR^2 : PR \cdot RP' (= 2a' \cdot PR),$$

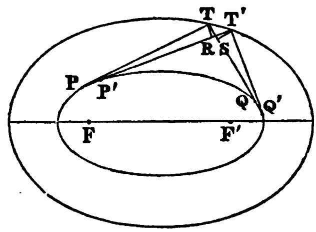
therefore

$$QR^2 = \frac{2b'^2 \cdot PR}{a'}.$$

Hence the radius of curvature $= \frac{b'^2}{a'} \cdot \frac{PR}{PS}$. Now, no matter how small PR, PS are taken, we have, by similar triangles, their *ratio* $\frac{PR}{PS} = \frac{CP}{CT} = \frac{a'}{p}$. Hence radius of curvature $= \frac{b'^2}{p}$.

It is not difficult to prove that *at the intersection of two confocal conics the centre of curvature of either is the pole with respect to the other of the tangent to the former at the intersection.*

356. *If two tangents be drawn to an ellipse from any point of a confocal ellipse, the excess of the sum of these two tangents over the arc intercepted between them is constant.**



For, take an indefinitely near point T', and let fall the perpendiculars TR, T'S, then (Art. 348)

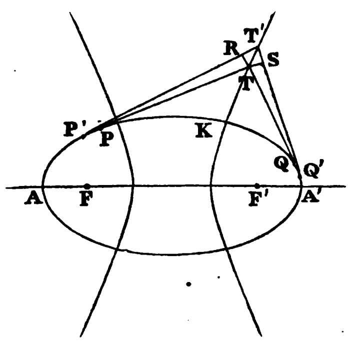
$$PT = PR = PP' + P'R$$

(for P'R may be considered as the continuation of the line PP'); in like manner, $Q'T' = QQ' + QS$.

Again, since, by Art. 194, the angle $TT'R = T'TS$, we have $TS = T'R$; and therefore $PT + TQ' = PT' + T'Q'$. Hence, $(PT + TQ) - (P'T' + T'Q') = PP' - QQ' = PQ - P'Q'$.

Cor. The same theorem will be true of any two curves which possess the property that two tangents, TP, TQ, to the inner one, always make equal angles with the tangent TT' to the outer.

357. *If two tangents be drawn to an ellipse from any point of a confocal hyperbola, the difference of the arcs* PK, QK, *is equal to the difference of the tangents* TP, TQ.†



For it appears, precisely as before, that the excess of $T'P' - P'K$ over $TP - PK = T'R$, and that the excess of $T'Q' - Q'K$ over $TQ - QK$ is $T'S$, which is equal to $T'R$, since (Art. 194) TT' bisects the angle RT'S. The difference, therefore, between the excess of TP over PK, and that of TQ over QK, is constant; but

* This beautiful theorem was discovered by Dr. Graves. See his *Translation of Chasles's Memoirs on Cones and Spherical Conics*, p. 77.

† This extension of the preceding theorem was discovered by Mr. Mac Cullagh. *Dublin Exam. Papers*, 1841, p. 41; 1842, pp. 68, 83. M. Chasles afterwards indepen-

in the particular case where T coincides with K, both these excesses, and consequently their difference, vanish; in every case, therefore, TP − PK = TQ − QK.

Cor. *Fagnani's theorem*, "That an elliptic quadrant can be so divided, that the difference of its parts may be equal to the difference of the semiaxes," follows immediately from this Article, since we have only to draw tangents at the extremities of the axes, and through their intersection to draw a hyperbola confocal with the given ellipse. The co-ordinates of the points where it meets the ellipse are found to be

$$x^2 = \frac{a^3}{a+b}, \quad y^2 = \frac{b^3}{a+b}.$$

358. *If a polygon circumscribe a conic, and if all the vertices but one move on confocal conics, the locus of the remaining vertex will be a confocal conic.*

In the first place, we assert that if the vertex T of an angle PTQ circumscribing a conic, move on a confocal conic (see fig. Art. 356); and if we denote by a, b, the diameters parallel to TP, TQ; and by α, β, the angles TPT′, TQ′T′, made by each of the sides of the angle with its consecutive position, then $a\alpha = b\beta$. For (Art. 356) TR = T′S; but TR = TP·α; T′S = T′Q′·β, and (Art. 152) TP and TQ are proportional to the diameters to which they are parallel.

Conversely, if $a\alpha = b\beta$, T moves on a confocal conic. For by reversing the steps of the proof we prove that TR = T′S; hence that TT′ makes equal angles with TP, TQ, and therefore coincides with the tangent to the confocal conic through T; and therefore that T′ lies on that conic.

If then the diameters parallel to the sides of the polygon be a, b, c, &c., that parallel to the last side being d, we have $a\alpha = b\beta$, because the first vertex moves on a confocal conic; in like manner $b\beta = c\gamma$, and so on until we find $a\alpha = d\delta$, which shows that the last vertex moves on a confocal conic.*

dently noticed the same extension of Dr. Graves's theorem. *Comptes Rendus*, October, 1843, tom. xvii. p. 838.

* This proof is taken from a paper by Dr. Hart; *Cambridge and Dublin Math. Jour.*, iv. 193.

THE METHOD OF PROJECTIONS.*

359. We have already several times had occasion to point out to the reader the advantage gained by taking notice of the number of particular theorems often included under one general enunciation, but we now propose to lay before him a short sketch of a method which renders us a still more important service, and which enables us to tell when from a particular given theorem we can safely infer the general one under which it is contained. The method of projections, indeed, as requiring some knowledge of the geometry of three dimensions, may seem scarcely admissible into a treatise on *plane* geometry; yet, as it is only in laying down its principles that we shall have to use a few not very difficult theorems of solid geometry, and as the applications of the method (the principles being once admitted) do not require any consideration of space of three dimensions, we feel that it could not with propriety be excluded from the present treatise. The reader will have less difficulty in following the demonstrations here given, as in studying spherical trigonometry he has been already introduced to the consideration of space of three dimensions.

360. If all the points of any figure be joined to any fixed point in space (O), the joining lines will form a *cone*, of which the point O is called the *vertex*, and the section of this cone, by any plane, will form a figure which is called the *projection* of the given figure. The plane by which the cone is cut is called the *plane of projection.*

To any point of one figure will correspond *a point in the other.*

For, if any point A be joined to the vertex O, the point a, in which the joining line OA is cut by any plane, will be the projection on that plane of the given point A.

A right line will always be projected into a right line.

For, if all the points of the right line be joined to the vertex,

* This method is the invention of M. Poncelet. See his *Traité des Propriétés Projectives*, published in the year 1822. I shall be glad if the slight sketch here given induces any reader to study a work, from which I have perhaps derived more information than from any other on the theory of curves.

the joining lines will form a plane, and this plane will be intersected by any plane of projection in a right line.

Hence, if any number of points in one figure lie in a right line, so will also the corresponding points on the projection; and if any number of lines in one figure pass through a point, so will also the corresponding lines on the projection.

361. *Any plane curve will always be projected into another curve of the same degree.*

For it is plain that, if the given curve be cut by any right line in any number of points, A, B, C, D, &c., the projection will be cut by the projection of that right line in *the same number* of corresponding points, *a*, *b*, *c*, *d*, &c., but the degree of a curve is estimated geometrically by the number of points in which it can be cut by any right line. If AB meet the curve in some real and some imaginary points, *ab* will meet the projection in the same number of real and the same number of imaginary points.

In like manner, if any two curves intersect, their projections will intersect in the same number of points, and any point common to one pair, whether real or imaginary, must be considered as the projection of a corresponding real or imaginary point common to the other pair.

Any tangent to one curve will be projected into a tangent to the other.

For, any line AB on one curve must be projected into the line *ab* joining the corresponding points of the projection. Now, if the points A, B, coincide, the points *a*, *b*, will also coincide, and the line *ab* will be a tangent.

More generally, if any two curves touch each other in any number of points, their projections will touch each other in the same number of points.

362. If a plane through the vertex parallel to the plane of projection meet the original plane in a line AB, then any pencil of lines diverging from a point on AB will be projected into a system of parallel lines on the plane of projection. For, since the line from the vertex to any point of AB meets the plane of projection at an infinite distance, the intersection of any two lines which meet on AB is projected to an infinite distance on the

plane of projection. Conversely, *any system of parallel lines on the original plane is projected into a system of lines meeting on a point in the line* DF, *where a plane through the vertex parallel to the original plane is cut by the plane of projection.* The method of projections then leads us naturally to the conclusion, that any system of parallel lines may be considered as passing through a point at an infinite distance, for their projections on any plane pass through a point in general at a finite distance; and again, that *all the points at infinity on any plane may be considered as lying on a right line*, since we have showed, that the projection of any point in which parallel lines intersect must lie somewhere on the right line DF in the plane of projection.

363. We see now that if any property of a given curve does not involve the *magnitude* of lines or angles, but merely relates to the *position* of lines as drawn to certain points, or touching certain curves, or to the position of points, &c., then this property will be true for any curve into which the given curve can be projected. Thus, for instance, "if through any point in the plane of a circle a chord be drawn, the tangents at its extremities will meet on a fixed line." Now since we shall presently prove that every curve of the second degree can be projected into a circle, the method of projections shows at once that the properties of poles and polars are true not only for the circle, but also for all curves of the second degree. Again, Pascal's and Brianchon's theorems are properties of the same class, which it is sufficient to prove in the case of the circle, in order to know that they are true for all conic sections.

364. Properties which, if true for any figure, are true for its projection, are called *projective properties*. Beside the classes of theorems mentioned in the last Article, there are many projective theorems which *do* involve the magnitude of lines. For instance, the anharmonic ratio of four points in a right line, {ABCD} being measured by the ratio of the pencil {O.ABCD} drawn to the vertex, must be the same as that of the four points {*abcd*}, where this pencil is cut by any transversal. Again, if there be an equation between the mutual distances of any number of points in a right line, such as

$$AB.CD.EF + k.AC.BE.DF + l.AD.CE.BF + \&c. = 0,$$

where in each term of the equation the same points are mentioned, although in different orders, this property will be projective. For (see Art. 314) if for AB we substitute

$$\frac{OA.OB.\sin AOB}{OP}, \&c.$$

each term of the equation will contain OA.OB.OC.OD.OE.OF in the numerator, and OP^3 in the denominator. Dividing, then, by these, there will remain merely a relation between the sines of angles subtended at O. It is evident that the points A, B, C, D, E, F, need not be on the same right line; or, in other words, that the perpendicular OP need not be the same for all, provided the points be so taken that after the substitution, each term of the equation may contain in the denominator the same product, $OP.OP'.OP''$, &c. Thus, for example, "If lines meeting in a point and drawn through the vertices of a triangle ABC meet the opposite sides in the points a, b, c, then $Ab.Bc.Ca = Ac.Ba.Cb$." This is a relation of the class just mentioned, and which it is sufficient to prove for any projection of the triangle ABC. Let us suppose the point C projected to an infinite distance, then AC, BC, Cc are parallel, and the relation becomes

$$Ab.Bc = Ac.Ba,$$

the truth of which is at once perceived on making the figure.

365. It appears from what has been said, that if we wish to demonstrate any projective property of any figure, it is sufficient to demonstrate it for the *simplest* figure into which the given figure can be projected; e. g. for one in which any line of the given figure is at an infinite distance.

Thus, if it were required to investigate the harmonic properties of a complete quadrilateral ABCD, whose opposite sides intersect in E, F, and the intersection of whose diagonals is G, we may join all the points of this figure to any point in space O, and cut the joining lines by any plane parallel to OEF, then EF is projected to infinity, and we have a new quadrilateral, whose sides ab, cd intersect at e at infinity, that is, are parallel; while ad, bc intersect in a point f at infinity, or are also parallel. We thus see that *any quadrilateral may be projected into a parallelogram.* Now since the diagonals of a parallelogram bisect each other, the diagonal *ac* is cut harmonically in the points *a*, *g*, *c*, and the point where it meets the line at infinity *ef*. Hence AB is cut harmonically in the points A, G, C, and where it meets EF.

Ex. If two triangles ABC, A′B′C′, be such that the points of intersection of AB, A′B′; BC, B′C′; CA, C′A′; lie in a right line, then the lines AA′, BB′, CC′ meet in a point.

Project to infinity the line in which AB, A′B′, &c., intersect; then the theorem becomes: "If two triangles *abc*, *a′b′c′* have the sides of the one respectively parallel to the sides of the other, then the lines *aa′*, *bb′*, *cc′* meet in a point." But the truth of this latter theorem is evident, since *aa′*, *bb′* both cut *cc′* in the same ratio.

366. Before giving examples of the application of the method of projections to curves of the second degree, we wish to examine more particularly than in Art. 361 the nature of the section made by any plane in a cone standing on a circular base. We there proved that the projection of a circle must be always a curve of the second degree, and we wish now to show that the same circle may be projected into any of the three species of curves of the second degree. We commence by proving that *any curve will be projected into a similar curve, on a plane parallel to the plane of the original curve.*

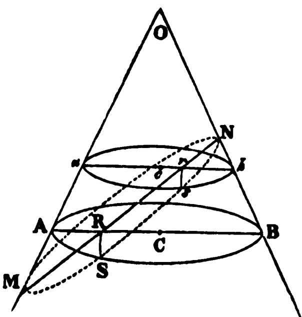
For take any fixed point A in the plane of one of them, and the corresponding point *a* in the plane of the other, and let radii vectores be drawn from them to any corresponding points B, *b*; now from the similar triangles OAB, O*ab*, AB is to *ab* in the constant ratio OA : O*a*; and since *every* radius vector of the one curve is in this constant ratio to the corresponding radius vector of the other, the two curves are similar (Art. 239).

Cor. If a cone standing on a circular base be cut by any plane parallel to the base, the section will be a circle. This is evident as before: we may, if we please, suppose the points A, *a* the centres of the curves.

367. *The section by any plane of a cone standing on a circular base is a curve of the second degree.*

A cone of the second degree is said to be *right* if the line joining the vertex to the centre of the circle which is taken for base be perpendicular to the plane of that circle; in which case this line is called the *axis* of the cone. If this line be not perpendicular to the plane of the base, the cone is said to be *oblique*. The investigation of the sections of an oblique cone is exactly the same as that of the sections of a right cone, but we shall treat them separately, because the figure in the latter case being more simple will be more easily understood by the learner, who may at first find some difficulty in the conception of figures in space.

Let a plane (OAB) be drawn through the axis of the cone OC perpendicular to the plane of the section, so that both the section MS*s*N and the base ASB are supposed to be perpendicular to the plane of the paper: the line RS, in which the section meets the base, is, therefore, also supposed perpendicular to the plane of the paper. Let us first suppose the line MN, in which the section cuts the plane OAB to meet both the sides OA, OB, as in the figure, on the same side of the vertex.

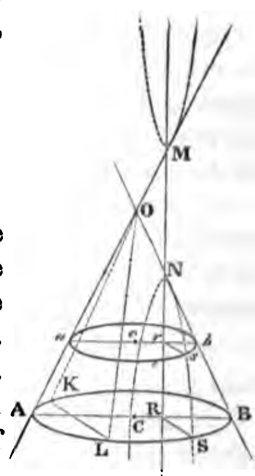


Now let a plane parallel to the base be drawn at any other point *s* of the section. Then we have (Euc. III. 35) the square of RS, the ordinate of the circle, = AR . RB, and in like manner $rs^2 = ar \cdot rb$. But from a comparison of the similar triangles ARM, *ar*M; BRN, *br*N, it can at once be proved that

$$AR \cdot RB : MR \cdot RN :: ar \cdot rb : Mr \cdot rN.$$

Therefore

$$RS^2 : rs^2 :: MR \cdot RN : Mr \cdot rN.$$



Hence the section MS*s*N is such that the square of any ordinate *rs* is to the rectangle under the parts in which it cuts the line MN in the constant ratio $RS^2 : MR \cdot RN$. Hence it can immediately be inferred (Art. 152) that the section is an *ellipse*, of which MN is the axis major, while the square of the axis minor is to MN^2 in the given ratio

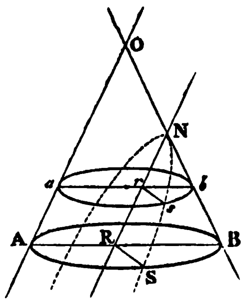
$$RS^2 : MR \cdot RN.$$

Secondly. Let MN meet one of the sides OR *produced*. The proof proceeds exactly as before, only that now we prove the square of the ordinate rs in a constant ratio to the rectangle $Mr \cdot rN$ under the parts into which it cuts the line MN *produced*. The learner will have no difficulty in proving that the locus will in this case be a *hyperbola*, consisting evidently of the two opposite branches NsS, $Ms'S'$.

Thirdly. Let the line MN be *parallel* to one of the sides. In this case, since $AR = ar$, and $RB : rb :: RN : rN$, we have the square of the ordinate rs $(= ar \cdot rb)$ to the abscissa rN in the constant ratio

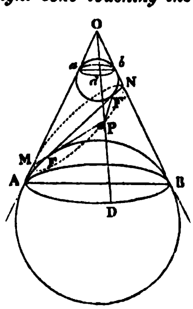
$$RS^2 (= AR \cdot RB) : RN.$$

The section is therefore a *parabola*.*



* Those who first treated of conic sections only considered the case when a right cone is cut by a plane perpendicular to a side of the cone: that is to say, when MN is perpendicular to OB. Conic sections were then divided into sections of a right-angled, acute, or obtuse-angled cone; and according to Eutochius, the commentator on Apollonius, were called parabola, ellipse, or hyperbola, according as the angle of the cone was equal to, less than, or exceeded a right angle. (See the passage cited in full, *Walton's Examples*, p. 428.) It was Apollonius who first showed that all three sections could be made from one cone; and who, according to Pappus, gave them the names parabola, ellipse, and hyperbola, for the reason stated, p. 170. The authority of Eutochius, who was more than a century later than Pappus, may not be very great, but the name parabola was used by Archimedes, who was prior to Apollonius.

It is worth mentioning, that *if a sphere be inscribed in a right cone touching the plane of any section, the point of contact will be a focus of that section, and the corresponding directrix will be the intersection of the plane of the section with the plane of contact of the cone with the sphere.* (*Bp. Hamilton's Conic Sections*, lib. ii. prop. 37.)

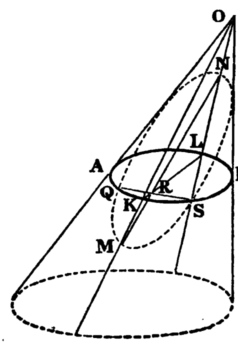


Let a sphere be both inscribed and exscribed between the cone and the plane of the section. Now, if any point P of the section be joined to the vertex, and the joining line meet the planes of contact in Dd, then we have $PD = PF$, since they are tangents to the same sphere, and, similarly, $Pd = PF'$, therefore $PF + PF' = Dd$, which is constant. The point (R) where FF' meets AB produced, is a point on the directrix, for by the property of the circle NFMR is cut harmonically, therefore, R is a point on the polar of F. This is only a particular case of a more general theorem.

It is not difficult to prove that the parameter of the section MPN is constant, if the distance of the plane from the vertex be constant.

368. It is evident that the projections of the tangents at the points A, B of the circle are the tangents at the points M, N of the conic section (Art. 362); now in the case of the parabola the point M and the tangent at it go off to infinity; we are therefore again led to the conclusion that *every parabola has one tangent altogether at an infinite distance.*

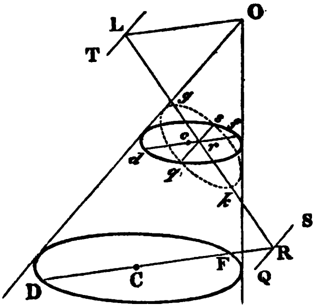
369. Let the cone now be supposed oblique. The plane of the paper is a plane drawn through the line OC, perpendicular to the plane of the circle AQSB. Now let the section meet the base in any line QS, draw a diameter LK bisecting QS, and let the section meet the plane OLK in the line MN, then the proof proceeds exactly as before; we have the square of the ordinate RS equal to the rectangle LR . RK; if we conceive a plane, as before, drawn parallel to the base (which, however, is left out of the figure in order to avoid rendering it too complicated), we have the square of any other ordinate, *rs*, equal to the corresponding rectangle *lr* . *rk*; and we then prove by the similar triangles KRM, *kr*M; LRN, *lr*N, in the plane OLK, exactly as in the case of the right cone, that $RS^2 : rs^2$, as the rectangle under the parts in which each ordinate divides MN, and that therefore the section is a conic of which MN is the diameter bisecting QS, and which is an ellipse when MN meets both the lines OL, OK on the same side of the vertex, an hyperbola when it meets them on different sides of the vertex, and a parabola when it is parallel to either.



In the proof just given QS is supposed to intersect the circle in real points; if it did not, we have only to take, instead of the circle AB, any other parallel circle *ab*, which *does* meet the section in real points, and the proof will proceed as before.

370. *If a circular section be cut by any plane in a line* RS, *the rectangle* DR . RF *of the segments of the diameter of the circle conjugate to* QS *is to the rectangle* gR . Rk *under the segments of the diameter of the section conjugate to* QS *as the square of the diameter of the section parallel to* QS *is to the square of the conjugate diameter* gk.

This has been proved in the last Article, in the case where QS meets the circle in real points, since $rs^2 = dr \,.\, rf$. Now, if the plane meet any other parallel plane in a line QS which does not meet the curve: First, we say that the diameters conjugate to QS with regard to the circle, and with regard to the other section, will meet QS in the same point R; for, by Art. 366, the diameter df, bisecting chords of any circular section parallel to qs, will be projected into a diameter bisecting the parallel chords of any parallel section. The middle points, therefore, of all chords parallel to qs, must lie in the plane Odf, and, consequently, the diameter conjugate to QS, in the section $gqks$, must be the line gk, in which it is met by the plane Odf. DF, therefore, and gk, intersect in the point R, where QS meets the plane Odf.



Now, since we have proved that the lines gk, df, DF, lie in one plane passing through the vertex, the points D, d, are projections of g, that is, they lie in one right line passing through the vertex; we have, therefore, by similar triangles, as in Art. 367, $dr \,.\, rf : DR \,.\, RF :: gr \,.\, rk \,.\, gR \,.\, Rk$; and, since $dr \,.\, rf : gr \,.\, rk$, as the squares of the parallel semidiameters, $DR \,.\, RF : gR \,.\, Rk$ in the same ratio.

371. If a plane be drawn through the vertex parallel to the circular base meeting the section $gqks$ in TL, it follows, as a particular case of the preceding, that $gL \,.\, Lk : OL^2$ in the ratio of the squares of the parallel diameters of the section. Hence we see that, given any conic section and a line, TL, in its plane, it is an indeterminate problem to find O the vertex of a cone such that the section of it, by any plane parallel to OTL, should be a circle. For, draw the diameter of the section conjugate to TL, then the distance of L from the vertex of the cone is determined by the present Article; also OL must lie in the plane perpendicular to TL, since it is parallel to the diameter of a circle perpendicular to TL; O may, therefore, be any point of a certain circle in a plane perpendicular to TL.

Hence, *Given any conic section, and any line* TL *in its plane not cutting it, we can project it, so that the conic section may become a circle; and the line may be projected to infinity,* for we have only to take any point O, such that the plane OTL may be parallel to the planes of circular section, and then any plane parallel to OTL will be a plane of projection fulfilling the required conditions.

372. *Given any conic section and a point in its plane, we can project it into a circle, of which the projection of that point is the centre,* for we have only to project it so that the projection of the polar of the given point may pass to infinity (Art. 157).

Or again, *Any two conic sections may be projected so as both to become circles,* for we have only to project one of them into a circle so as that any of its chords of intersection with the other shall pass to infinity, and then, by Art. 259, the projection of the second conic passing through the same points at infinity as the circle must be a circle also.

Any two conics which have double contact with each other may be projected into concentric circles.

For we have only to project one of them into a circle so that its chord of contact with the other may pass to infinity (Art. 259).

Strictly speaking, all these projections have only been shown to be possible when the line projected to infinity does not meet the conic in real points; but it will be found in practice that this is a limitation which it is unnecessary to attend to, and that a projective proposition once proved true for any state of a figure *may* become *unmeaning*, but will *never* become *false*, when certain lines in that figure have become imaginary. Thus, for example, although the method of projecting into concentric circles only directly proves properties of conics having double contact, *whose chord of contact is imaginary*, we shall not think it necessary to seek for an independent proof of the same properties in the case where the chord of contact is real.

373. We shall now give some examples of the method of deriving properties of conics from those of the circle, or from other more particular properties of conics.

Ex. 1. "A line through any point is cut harmonically by the curve and the polar of that point." This property and its reciprocal are projective properties (Art. 364), and both being true for the circle, are true for every conic. Hence all the properties of the circle depending on the theory of poles and polars are true for all the conic sections.

Ex. 2. The anharmonic properties of the points and tangents of a conic are projective properties, which, when proved for the circle, as in Art. 313, are proved for all the conics. Hence, every property of the circle which results from either of its anharmonic properties is true also for all the conic sections.

Ex. 3. Carnot's theorem (Art. 314), that if a conic meet the sides of a triangle,

$$Ab \cdot Ab' \cdot Bc \cdot Bc' \cdot Ca \cdot Ca' = Ac \cdot Ac' \cdot Ba \cdot Ba' \cdot Cb \cdot Cb',$$

is a projective property which need only be proved in the case of the circle, in which case it is evidently true, since $Ab \cdot Ab' = Ac \cdot Ac'$, &c.

The theorem is evidently true, and can be proved in like manner for any polygon.

Ex. 4. From Carnot's theorem, thus proved, could be deduced the properties of Art. 151, by supposing the point C at an infinite distance; we then have

$$\frac{Ab \cdot Ab'}{Ac \cdot Ac'} = \frac{Ba \cdot Ba'}{Bc \cdot Bc'},$$

where the line Ab is parallel to Ba.

Ex. 5. Given two concentric circles, any chord of one which touches the other is bisected at the point of contact.

Given two conics having double contact with each other, any chord of one which touches the other is cut harmonically at the point of contact, and where it meets the chord of contact of the conics. (Ex. 3, p. 280.)

For the line at infinity in the first case is projected into the chord of contact of two conics having double contact with each other. Ex. 4, p. 203, is only a particular case of this theorem.

Ex. 6. Given three concentric circles, any tangent to one is cut by the other two in four points whose anharmonic ratio is constant.

Given three conics all touching each other in the same two points, any tangent to one is cut by the other two in four points whose anharmonic ratio is constant.

The first theorem is obviously true, since the four lengths are constant. The second may be considered as an extension of the anharmonic property of the tangents of a conic. In like manner, the theorems (in Art. 278) with regard to anharmonic ratios in conics having double contact are immediately proved by projecting the conics into concentric circles.

Ex. 7. We mentioned already, that it was sufficient to prove Pascal's theorem for the case of a circle, but by the help of Art. 362 we may still further simplify our figure, for we may suppose the line joining the intersection of AB, DE, to that of BC, EF, to pass off to infinity; and it is only necessary to prove that, if a hexagon be inscribed in a circle having the side AB parallel to DE, and BC to EF, then CD will be parallel to AF; but the truth of this can be shown from elementary considerations.

Ex. 8. A triangle is inscribed in any conic, two of whose sides pass through fixed points, to find the envelope of the third (p. 229). Let the line joining the fixed points be projected to infinity, and at the same time the conic into a circle, and this property becomes,—"A triangle is inscribed in a circle, two of whose sides are parallel to fixed lines, to find the envelope of the third." But this envelope is a concentric circle, since the vertical angle of the triangle is given; hence, in the general case, the envelope is a conic touching the given conic in two points on the line joining the two given points.

Ex. 9. To investigate the projective properties of a quadrilateral inscribed in a conic. Let the conic be projected into a circle, and the quadrilateral into a parallelogram (Art. 365). Now the intersection of the diagonals of a parallelogram inscribed in a circle is the centre of the circle; hence the intersection of the diagonals of a quadrilateral inscribed in a conic is the pole of the line joining the intersections of the opposite sides. Again, if tangents to the circle be drawn at the vertices of this parallelogram, the diagonals of the quadrilateral so formed will also pass through the centre, bisecting the angles between the first diagonals; hence, "the diagonals of the inscribed and corresponding circumscribing quadrilateral pass through a point, and form an harmonic pencil."

Ex. 10. Given four points on a conic, the locus of its centre is a conic through the middle points of the sides of the given quadrilateral.

Given four points on a conic, the locus of the pole of any fixed line is a conic passing through the fourth harmonic to the point in which this line meets each side of the given quadrilateral.

Ex. 11. The locus of the point where parallel chords of a circle are cut in a given ratio is an ellipse having double contact with the circle. (Art. 166.)

If through a fixed point O a line be drawn meeting the conic in A, B, and on it a point P be taken, such that {OABP} may be constant, the locus of P is a conic having double contact with the given conic.

374. We may project several properties relating to foci by the help of the definition of a focus given, page 233.

Ex. 1. The locus of the centre of a circle touching two given circles is a hyperbola, having the centres of the given circles for foci.

If a conic be described through two fixed points, and touching two conics which also pass through those points, the locus of the pole of the line joining those points is a conic inscribed in the quadrilateral formed by joining the two given points to the poles of the same line with regard to the given conics.

We give this example as worth the learner's study, because it illustrates the different principles that all circles pass through two fixed points at infinity (Art. 259); that the centre is the pole of the line joining them (Art. 157); that a focus is the intersection of tangents passing through these fixed points (Art. 282); and that we are safe in extending our conclusion from imaginary to real points (Art. 372).

Ex. 2. Given the focus and two points of a conic section, the intersection of tangents at those points will be on a fixed line. (Art. 196.)

Given two tangents, and two points on a conic, the locus of the intersection of tangents at those points is a right line.

Ex. 3. Given a focus and two tangents to a conic, the locus of the other focus is a

Given four tangents and a fixed point on each of two of them, the locus of the right line. (This follows from Art. 194.)

intersection of tangents from these points is a right line.

For, the two points at infinity on any circle lie one on each of the tangents from one focus, and the intersection of the other tangents from these two points is the other focus.

Ex. 4. Given three tangents to a parabola, the locus of the focus is the circumscribing circle. (p. 187.)

Given four tangents to a conic, and two fixed points on one of them, the locus of intersection of the other tangents from these points is a conic passing through the two points, and circumscribing the triangle formed by the other three tangents.

For every parabola has one tangent at infinity, and the two points through which every circle must pass lie on this tangent.

Ex. 5. The locus of the centre of a circle passing through a fixed point, and touching a fixed line, is a parabola of which the fixed point is the focus.

Given one tangent, and three points on a conic, the locus of the intersection of tangents at any two of these points is a conic inscribed in the triangle formed by those points.

Ex. 6. Given four tangents to a conic, the locus of the centre is the line joining the middle points of the diagonals of the quadrilateral.

Given four tangents to a conic, the locus of the pole of any line is the line joining the fourth harmonics of the points where the given line meets the diagonals of the quadrilateral.

It follows from our definition of a focus, that if two conics have the same focus, this point will be an intersection of common tangents to them, and will possess the properties mentioned in Art. 265. Also, that if two conics have the same focus and directrix, they may be considered as two conics having double contact with each other, and may be projected into concentric circles.

375. Since angles which are constant in any figure will in general not be constant in the projection of that figure, we proceed to show what property of a projected figure may be inferred when any property relating to the magnitude of angles is given,* and we commence with the case of the right angle.

Let the equations of two lines at right angles to each other be $x = 0$, $y = 0$, then the equation which determines the direction of the points at infinity on any circle is $x^2 + y^2 = 0$, or

$$x + y\sqrt{-1} = 0, \quad x - y\sqrt{-1} = 0.$$

* Some particular cases where constant angles are projected into constant angles have been treated of by M. Poncelet, *Traité des Propriétés Projectives*, p. 257; and by Mr. Mulcahy, *Modern Geometry*, p. 116; who have thus deduced by projection properties relating to angles subtended at the foci of conics from properties of the circle. As these theorems, however, may be more simply obtained otherwise, we have thought it better not to occupy space with this method of obtaining them, and have substituted the general theory of projection of angles given in the text.

Hence (Art. 55) these four lines form an harmonic pencil. Hence, given four points, A, B, C, D, of a line cut harmonically, where A, C may be real or imaginary, if these points be transferred by a real or imaginary projection, so that A, C may become the two imaginary points at infinity on any circle, then any lines through B, D will be projected into lines at right angles to each other. Conversely, *any two lines at right angles to each other will be projected into lines which cut harmonically the line joining the two fixed points which are the projections of the imaginary points at infinity on a circle.*

Ex. 1. The tangent to a circle is at right angles to the radius.

Any chord of a conic is cut harmonically by any tangent, and by the line joining the point of contact of that tangent to the pole of the given chord. (Art. 147.)

For the chord of the conic is supposed to be the projection of the line at infinity on the plane of the circle; the points where the chord meets the conic will be the projections of the imaginary points at infinity on the circle; and the pole of the chord will be the projection of the centre of the circle.

Ex. 2. Any right line drawn through the focus of a conic is at right angles to the line joining its pole to the focus. (Art. 197.)

Any right line through a point, the line joining its pole to that point, and the two tangents from the point, form an harmonic pencil. (Art. 149.)

It is evident that the first of these properties is only a particular case of the second, if we recollect that the tangents from the focus are the lines joining the focus to the two imaginary points in any circle (Art. 282).

Ex. 3. Let us apply Ex. 6 of the last Article to determine the locus of the pole of a given line with regard to a system of confocal conics. Being given the two foci, we are given a quadrilateral circumscribing the conic (Art. 282), one of the diagonals of this quadrilateral is the line joining the foci, therefore (Ex. 6) one point on the locus is the fourth harmonic to the point where the given line cuts the distance between the foci. Again, another diagonal is the line at infinity, and since the extremities of this diagonal are the points at infinity on a circle, by the present Article the locus is perpendicular to the given line. The locus is, therefore, completely determined.

Ex. 4. Two confocal conics cut each other at right angles.

If two conics be inscribed in the same quadrilateral, the two tangents at any of their points of intersection cut any diagonal of the circumscribing quadrilateral harmonically.

The last theorem is a case of the reciprocal of Ex. 1, p. 280.

Ex. 5. The locus of the intersection of two tangents to a central conic, which cut at right angles, is a circle.

If from any two points B, D, which cut a given line AC harmonically, tangents be drawn to a conic, the locus of their intersection O is a conic through the points A, C.

The last theorem may, by Art. 149, be stated otherwise thus: "The locus of a point O, such that the line joining O to the pole of AO may pass through C, is a conic through A, C;" and the truth of it is evident directly, by taking four positions of the line, when we see, by Art. 339, that the anharmonic ratio of four lines, AO, is equal to that of four corresponding lines, CO.

Ex. 6. The locus of the intersection of tangents to a parabola, which cut at right angles, is the directrix.

If in the last example AC touch the given conic, the locus of O will be the line joining the points of contact of tangents from A, C.

Ex. 7. If from any point on a conic two lines at right angles to each other be drawn, the chord joining their extremities passes through a fixed point. (p. 160.)

If a harmonic pencil be drawn through any point on a conic, two legs of which are fixed, the chord joining the extremities of the other legs will pass through a fixed point.

In other words, given two points, a, c, on a conic, and $\{abcd\}$ an harmonic ratio, bd will pass through a fixed point, namely, the intersection of tangents at a, c. But the truth of this may be seen directly: for let the line ac meet bd in K, then since $\{a.abcd\}$ is a harmonic pencil, the tangent at a cuts bd in the fourth harmonic to K: but so likewise must the tangent at c, therefore these tangents meet bd in the same point. As a particular case of this theorem we have the following: "Through a fixed point on a conic two lines are drawn, making equal angles with a fixed line, the chord joining their extremities will pass through a fixed point."

376. *A system of pairs of right lines drawn through a point, every two of which make equal angles with a fixed line, cut the line at infinity in a system of points in involution, of which the two points at infinity on any circle form one pair of conjugate points.* For they evidently cut *any* right line in a system of points in involution, the foci of which are the points where the line is met by the given internal and external bisector of every pair of right lines. The two points at infinity just mentioned belong to the system, since they also are cut harmonically by these bisectors.

The tangents from any point to a system of confocal conics make equal angles with two fixed lines. (Art. 194.)

The tangents from any point to a system of conics inscribed in the same quadrilateral cut any diagonal of that quadrilateral in a system of points in involution of which the two extremities of that diagonal are a pair of conjugate points. (Art. 336.)

377. *Two lines diverging from a fixed point, which contain a constant angle, cut the line joining the two points at infinity on a circle, so that the anharmonic ratio of the four points is constant.*

For the equation of two lines containing an angle θ being $x = 0$, $y = 0$, the direction of the points at infinity on any circle is determined by the equation

$$x^2 + y^2 + 2xy\cos\theta = 0;$$

and, separating this equation into factors, we see, by Art. 55, that the anharmonic ratio of the four lines is constant if θ be constant.

Ex. 1. "The angle contained in the same segment of a circle is constant." We see, by the present Article, that this is the form assumed by the anharmonic property of four points on a circle when two of them are at an infinite distance.

Ex. 2. The envelope of a chord of a conic which subtends a constant angle at the focus is another conic having the same focus and the same directrix.

If tangents through any point O meet the conic in T, T′, and there be taken on the conic two points A, B, such that {O.ATBT′} is constant, the envelope of AB is a conic touching the given conic in the points T, T′.

Ex. 3. The locus of the intersection of tangents to a parabola which cut at a given angle is a hyperbola having the same focus and the same directrix.

If in Art. 375, Ex. 6, the points B, D be so taken that {ABCD} is constant, the locus of O is a conic touching the given conic at the points of contact of tangents from A, C.

Ex. 4. If from the focus of a conic a line be drawn making a given angle with any tangent, the locus of the point where it meets it is a circle.

If a variable tangent to a conic meet two fixed tangents in T, T′, and a fixed line in M, and there be taken on it a point P, such that {PTMT′} may be constant, the locus of P is a conic passing through the points where the fixed tangents meet the fixed line.

A particular case of this theorem is: "The locus of the point where the intercept of a variable tangent between two fixed tangents is cut in a given ratio, is a hyperbola whose asymptotes are parallel to the fixed tangents."

Ex. 5. If from a fixed point O, OP be drawn to a given circle, and the angle TPO be constant, the envelope of TP is a conic having O for its focus.

Given the anharmonic ratio of a pencil three of whose legs pass through fixed points, and whose vertex moves along a given conic, passing through two of the points; the envelope of the fourth leg is a conic touching the lines joining these two to the third fixed point.

A particular case of this is: "If two fixed points A, B, on a conic be joined to a variable point P, and the intercept made by the joining chords on a fixed line be cut in a given ratio at M, the envelope of PM is a conic touching parallels through A and B to the fixed line."

Ex. 6. If from a fixed point O, OP be drawn to a given right line, and the angle TPO be constant, the envelope of TP is a parabola having O for its focus.

Given the anharmonic ratio of a pencil, three of whose legs pass through fixed points, and whose vertex moves along a fixed line, the envelope of the fourth leg is a conic touching the three sides of the triangle formed by the given points.*

* The method of projections can equally be used in obtaining from properties of plane curves properties of other curves not plane, e. g. curves on the surface of a sphere. Mr.

378. We shall conclude this chapter with a brief account of the method of *orthogonal projection*, which, before the publication of M. Poncelet's treatise, was the only method of projection much used by geometers. If from all the points of any figure perpendiculars be let fall on any plane, their feet will trace out a figure which is called the *orthogonal projection* of the given figure. The orthogonal projection of any figure is, therefore, a right section of a *cylinder* passing through the given figure.

All parallel lines are in a constant ratio to their orthogonal projections on any plane.

For (see fig. p. 4) MM′ represents the orthogonal projection of the line PQ, and it is evidently = PQ multiplied by the cosine of the angle which PQ makes with MM′.

All lines parallel to the intersection of the plane of the figure with the plane on which it is projected, are equal to their orthogonal projections.

For, since the intersection of the planes is itself not altered by projection, neither can any line parallel to it.

The area of any figure in a given plane is in a constant ratio to its orthogonal projection on another given plane.

For, if we suppose ordinates of the figure and of its projection to be drawn perpendicular to the intersection of the planes, since every ordinate of the projection is to the corresponding ordinate of the original figure in the constant ratio of the cosine of the

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Mulcahy, some years ago, gave the following method of obtaining the properties of angles subtended at the focus from those of small circles on a sphere. The method depends on the following principle: *the locus of the vertices of all the right cones from which a given ellipse can be cut is a hyperbola passing through the foci of the ellipse.* For, see note, p. 305, the difference of MO and NO is constant, being equal to the difference of MF′ and NF′.

Now, let us take any property of a small circle of a sphere, e. g. if through any point P, on the surface of a sphere, a great circle be drawn, cutting the small circle in the points A, B, then tan ½AP tan ½BP is constant. Now, let us take a cone whose base is the small circle, and whose vertex is the centre of the sphere, and let us cut this cone by any plane, and we learn that "if through a point *p*, in the plane of any conic, a line be drawn cutting the conic in the points *a*, *b*, then the product of the tangents of the halves of the angles which *ap*, *bp* subtend at the vertex of the cone will be constant; this property will be true of the vertex of any right cone, out of which the section can be cut, and, therefore, since the focus is a point in the locus of such vertices, it must be true that tan ½*afp* tan ½*bfp* is constant (see p. 191).

angle between the planes to unity; by Art. 351, Cor., the areas of the figures will be in the same ratio.

Any ellipse can be orthogonally projected into a circle.

For, if we take the intersection of the plane of projection with the plane of the given ellipse parallel to the axis minor of that ellipse, and if we take the cosine of the angle between the planes $= \frac{b}{a}$, then every line parallel to the axis minor will be unaltered by projection, but every line parallel to the axis major will be shortened in the ratio $b:a$, the projection will, therefore (Art. 166), be a circle, whose radius is b.

379. We shall apply the principles laid down in the last Article to investigate the expression for the radius of a circle circumscribing a triangle inscribed in a conic, given Ex. 6, p. 199.*

Let the sides of the triangle be α, β, γ, and its area A, then, by elementary geometry,

$$R = \frac{\alpha\beta\gamma}{4A}.$$

Now let the ellipse be projected into a circle whose radius is b, then, since this is the circle circumscribing the projected triangle, we have

$$b = \frac{\alpha'\beta'\gamma'}{4A'}.$$

But, since parallel lines are in a constant ratio to their projections, we have

$$\alpha' : \alpha :: b : b',$$
$$\beta' : \beta :: b : b'',$$
$$\gamma' : \gamma :: b : b''';$$

and, since (Art. 378) A′ is to A as the area of the circle $(= \pi b^2)$ to the area of the ellipse $(= \pi ab)$, we have

$$A' : A :: b : a.$$

Hence
$$\frac{\alpha'\beta'\gamma'}{4A'} : \frac{\alpha\beta\gamma}{4A} :: ab^2 : b'b''b''',$$

and, therefore,
$$R = \frac{b'b''b'''}{ab}.$$

* This proof of Mr. Mac Cullagh's theorem is due to Dr. Graves.

NOTES.

Pascal's Theorem, Page 222.

M. Steiner was the first who (in *Gergonne's Annales*) directed the attention of geometers to the complete figure obtained by joining in every possible way six points on a conic. M. Steiner's theorems were corrected and extended by M. Plücker (*Crelle's Journal*, vol. v. p. 274), and the subject has been more recently investigated by Messrs. Cayley and Kirkman, the latter of whom, in particular, has added several new theorems to those already known. We shall in this note give a slight sketch of the more important of these, and of the methods of obtaining them. The greater part are derived by joining the simplest principles of the theory of combinations with the following elementary theorems and their reciprocals: "If two triangles be such that the lines joining corresponding vertices meet in a point (which we shall call the *pole* of the two triangles), the intersections of corresponding sides will lie in one right line (which we shall call their *axis*)." "If the intersections of opposite sides of three triangles be for each pair *the same* three points in a right line, the poles of the first and second, second and third, third and first, will lie in a right line."

Now let the six points on a conic be *a*, *b*, *c*, *d*, *e*, *f*, which we shall call the points P. These may be connected by *fifteen* right lines, *ab*, *ac*, &c., which we shall call the lines C. Each of the lines C (for example *ab*) is intersected by the fourteen others; by four of them in the point *a*, by four in the point *b*, and consequently by six in points distinct from the points P (for example the points *ab*, *cd*; &c.) These we shall call the points *p*. There are forty-five such points; for there are six on each of the lines C. To find then the number of points *p*, we must multiply the number of lines C by 6, and divide by 2, since two lines C pass through every point *p*.

If we take the sides of the hexagon in the order *abcdef*, Pascal's theorem is, that the three p points, (ab, de), (cd, fa), (bc, ef), lie in one right line, which we may call either the Pascal *abcdef*, or else we may denote as the Pascal $\begin{Bmatrix} ab.cd.ef \\ de.fa.bc \end{Bmatrix}$, a form which we sometimes prefer, as showing more readily the three points through which the Pascal passes. Through each point p four Pascals can be drawn. Thus through (ab, de) can be drawn *abcdef*, *abfdec*, *abcedf*, *abfedc*. We then find the total number of Pascals by multiplying the number of points p by 4, and dividing by 3, since there are three points p on each Pascal. We thus obtain the number of Pascal's lines $= 60$. We might have derived the same directly by considering the number of different ways of arranging the letters *abcdef*.

Consider now the three triangles whose sides are

$$ab,\ cd,\ ef, \qquad (1)$$
$$de,\ fa,\ bc, \qquad (2)$$
$$cf,\ be,\ ad. \qquad (3)$$

The intersections of corresponding sides of 1 and 2 lie on the same Pascal, therefore the lines joining corresponding vertices meet in a point, but these are the three Pascals,

$$\begin{Bmatrix} ab.de.cf \\ cd.fa.be \end{Bmatrix},\quad \begin{Bmatrix} cd.fa.be \\ ef.bc.ad \end{Bmatrix},\quad \begin{Bmatrix} ef.bc.ad \\ ab.de.cf \end{Bmatrix}.$$

This is Steiner's theorem (p. 222); we shall call this the g point,

$$\begin{Bmatrix} ab.de.cf \\ cd.fa.be \\ ef.bc.ad \end{Bmatrix}$$

The notation shows plainly that on each Pascal's line there is only one g point; for given the Pascal $\begin{Bmatrix} ab.de.cf \\ cd.fa.be \end{Bmatrix}$ the g point on it is found by writing under each term the two letters not already found in that vertical line. Since then three Pascals intersect in every point g, the number of points $g = 20$. If we take the triangles 2, 3; and 1, 3; the lines joining corresponding vertices are the same in all cases: therefore, by the reciprocal of the second preliminary theorem, the three *axes* of the three triangles meet in a point. This, however, is plainly only the g point $\begin{Bmatrix} ab.cd.ef \\ de.fa.bc \\ cf.be.ad \end{Bmatrix}$, and therefore leads us to no new theorem.

Let us now consider the triangles,

$$ab \qquad cd \qquad ef \qquad (1)$$

$$\left.\begin{matrix} ab.ce.df \\ de.bf.ac \end{matrix}\right\},\quad \left.\begin{matrix} cd.bf.ae \\ af.ce.bd \end{matrix}\right\},\quad \left.\begin{matrix} ef.bd.ac \\ bc.ae.df \end{matrix}\right\},\quad (4)$$

$$\left.\begin{matrix} ab.ce.df \\ cf.bd.ae \end{matrix}\right\},\quad \left.\begin{matrix} cd.bf.ae \\ be.ac.df \end{matrix}\right\},\quad \left.\begin{matrix} ef.bd.ac \\ ad.ce.bf \end{matrix}\right\},\quad (5)$$

Now the intersections of corresponding sides of 1 and 4 are three points which lie on the same Pascal; therefore the lines joining corresponding vertices meet in a point. But these are the three Pascals,

$$\left.\begin{matrix} ab.ce.df \\ cd.bf.ae \end{matrix}\right\},\quad \left.\begin{matrix} cd.bf.ae \\ ef.ac.bd \end{matrix}\right\},\quad \left.\begin{matrix} ef.ac.bd \\ ab.df.ce \end{matrix}\right\}.$$

We may denote the point of meeting as the h point, $\left.\begin{matrix} ab.ce.df \\ cd.bf.ae \\ ef.ac.bd \end{matrix}\right\}$.

The notation differs from that of the g points in that only one of the vertical columns contains the six letters without omission or repetition. On every Pascal there are three h points, viz., there are on

$$\left.\begin{matrix} ab.cd.ef \\ de.af.bc \end{matrix}\right\};\quad \left.\begin{matrix} \overline{ab}.cd.ef \\ de.af.bc \\ cf.bd.ae \end{matrix}\right\},\quad \left.\begin{matrix} ab.\overline{cd}.ef \\ de.af.bc \\ ac.be.df \end{matrix}\right\},\quad \left.\begin{matrix} ab.cd.\overline{ef} \\ de.af.bc \\ bf.ce.ad \end{matrix}\right\},$$

where the bar denotes the complete vertical column. We obtain then Mr. Kirkman's extension of Steiner's theorem:—*The Pascals intersect three by three, not only in Steiner's twenty points g, but also in sixty other points h.* The demonstration of Art. 269 applies alike to Mr. Kirkman's and to Steiner's theorem.

In like manner if we consider the triangles 1 and 5, the lines joining corresponding vertices are the same as for 1 and 4; therefore the corresponding sides intersect on a right line, as they manifestly do on a Pascal. In the same manner the corresponding sides of 4 and 5 must intersect on a right line, but these intersections are the three h points,

$$\left.\begin{matrix} \overline{ab}.ce.df \\ de.bf.ac \\ cf.ae.bd \end{matrix}\right\},\quad \left.\begin{matrix} ae.\overline{cd}.bf \\ bd.af.ce \\ ac.be.df \end{matrix}\right\},\quad \left.\begin{matrix} ac.bd.\overline{ef} \\ df.ae.bc \\ ce.bf.ad \end{matrix}\right\}.$$

Moreover, the axis of 4 and 5 must pass through the intersection of the axes of 1, 4, and 1, 5, namely, through the g point, $\left.\begin{matrix} ab.cd.ef \\ de.af.bc \\ cf.be.ad \end{matrix}\right\}$.

In this notation the g point is found by combining the complete vertical columns of the three *h* points. Hence we have the theorem: "*There are twenty lines x, each of which passes through one g and three h points.*" The existence of these lines was observed independently by Mr. Cayley and myself. The proof here given is Mr. Cayley's.

Again, let us take three Pascals meeting in a point *h*. For instance,

$$\left.\begin{matrix} ab.ce.df \\ de.bf.ac \end{matrix}\right\}, \quad \left.\begin{matrix} de.bf.ac \\ cf.ae.bd \end{matrix}\right\}, \quad \left.\begin{matrix} cf.ae.bd \\ ab.df.ce \end{matrix}\right\}.$$

We may, by taking on each of these a point *p*, form a triangle whose vertices are (*df*, *ac*), (*bf*, *ae*), (*bd*, *ce*), and whose sides are, therefore,

$$\left.\begin{matrix} ac.bf.de \\ df.ae.cb \end{matrix}\right\}, \quad \left.\begin{matrix} bf.ce.ad \\ ae.bd.cf \end{matrix}\right\}, \quad \left.\begin{matrix} bd.ac.ef \\ ce.df.ab \end{matrix}\right\}.$$

Again, we may take on each a point *h*, by writing under each of the above Pascals *af*.*cd*.*be*, and so form a triangle whose sides are

$$\left.\begin{matrix} ac.bf.de \\ be.cd.af \end{matrix}\right\}, \quad \left.\begin{matrix} cf.ae.bd \\ be.cd.af \end{matrix}\right\}, \quad \left.\begin{matrix} df.ab.ce \\ be.cd.af \end{matrix}\right\}.$$

But the intersections of corresponding sides of these triangles, which must therefore be on a right line, are the three *g* points,

$$\left.\begin{matrix} be.cd.af \\ ac.bf.de \\ df.ae.bc \end{matrix}\right\}, \quad \left.\begin{matrix} be.cd.af \\ cf.ae.bd \\ ad.bf.ce \end{matrix}\right\}, \quad \left.\begin{matrix} be.cd.af \\ df.ab.ce \\ ac.ef.bd \end{matrix}\right\}, \quad \left.\begin{matrix} be.cd.af \\ cf.ab.de \\ ad.ef.bc \end{matrix}\right\}.$$

I have added a fourth *g* point, which the symmetry of the notation shows must lie on the same right line; these being all the *g* points into the notation of which *be*.*cd*.*af* can enter. Now there can be formed, as may readily be seen, fifteen different products of the form *be*.*cd*.*af*; we have then Steiner's theorem, *The g points lie four by four on fifteen right lines* I.

My limits do not allow me to do more than add the enunciations of a few more theorems (principally Mr. Kirkman's), but the preceding examples are sufficient to show how they may be demonstrated, and how any reader who chooses to prosecute the study of the figure may find other theorems in great abundance: "*The twenty lines x pass four by four through fifteen points y.*" The four lines *x* whose *g* points in the preceding notation have a common vertical column will pass through the same point. "*There are sixty lines* J, *each of which passes through one point p and two points h.*" "*The lines* J *again pass three by three through sixty points j, three of which lie on each of the lines x.*" Mr. Kirkman calls points *m* the intersections of two Pascals, corresponding to hexagons which have four common sides, no opposite pairs being the same for both; for example, *abcdef*, *abcfed*; and points *r*, those corresponding to hexagons which have three common sides, two of which are contiguous; for example, *abcdef*, *abcefd*. "*The ninety points m lie three by three on sixty lines* M." "*There are sixty lines* R, *each containing six points r, and also one of the six points* P, *and which pass in threes through twenty points q.*" (See *Cambridge and Dublin Math. Jour.*, vol. v. p. 185).

Art. 296, Page 250.

Dr. Boole's method (p. 143) may be applied to find the relations between the coefficients of the equations of two conics, which remain unaltered when we transform from one set of trilinear co-ordinates to another. Thus, if we form the condition that $kS + S' = 0$ shall represent two right lines, it is plain that the values of k determined by putting this condition $= 0$, must be the same no matter in what system of co-ordinates S is expressed. Hence then the ratio between any two coefficients in the cubic for k (Art. 296) remains unaltered when we transform from one set of trilinear co-ordinates to another. Several theorems may hence be easily proved. For instance, let us define a *self-conjugate* triangle, one such that any side is the polar of the opposite vertex with regard to a given conic; and let it be required to prove that *the vertices of any two self-conjugate triangles all lie on the same conic* (see Ex. 2, p. 195). Let the sides of the first triangle be x, y, z; those of the second u, v, w; then supposing these quantities to include constants implicitly, the equation of the conic can (Art. 281) be expressed in either of the forms $x^2 + y^2 + z^2 = 0$, or $u^2 + v^2 + w^2 = 0$. And let the equation of any other conic expressed in terms of the sides of the first triangle be

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy = 0,$$

and of those of the second be

$$au^2 + a'v^2 + a''w^2 + 2bvw + 2b'wu + 2b''uv = 0;$$

then we have

$$Ax^2 + \&c. + k(x^2 + y^2 + z^2) = au^2 + \&c. + k(u^2 + v^2 + w^2).$$

Forming then the discriminant of each side of this equation, and equating corresponding coefficients of k, we find

$$A + A' + A'' = a + a' + a'';$$

$$(AA' - B''^2) + (A'A'' - B^2) + (A''A - B'^2) = (aa' - b''^2) + (a'a'' - b^2) + (a''a - b'^2).$$

If now a conic be described passing through three vertices of the first triangle and two of the second, we must have the five quantities A, A', A'', a, a', all $= 0$, and therefore by the first equation $a'' = 0$. Again, if a conic be described to touch the three sides of the first triangle and two of the second, we must have five of the six members of the second equation = 0, and therefore also the sixth, or *the six sides of the two triangles all touch the same conic.* In the same manner it is proved that if two triangles be both inscribed in the same conic, their sides will touch the same conic, and *vice versâ.*

On the Problem to describe a Conic under certain Conditions.

We saw (p. 119) that five conditions determine a conic; we can, therefore, in general describe a conic being given m points and n tangents where $m + n = 5$. We shall not think it worth while to treat separately the cases where any of these are at an infinite distance, for which the constructions for the general case only require to be suitably modified. Thus to be given *a parallel to an asymptote* is equivalent to one condition, for we are then given a point of the curve, namely, the point at infinity on the given parallel. If, for example, we were required to describe a conic, given four points and a parallel to an asymptote, the only change to be made in the construction (p. 283) is to suppose the point E at infinity, and the lines DE, ME therefore drawn parallel to a given line.

To be given *an asymptote* is equivalent to two conditions, for we are then given a tangent and its point of contact, namely, the point at infinity on the given asymptote. To be given *that the curve is a parabola* is equivalent to one condition, for we are then given a tangent, namely, the line at infinity. To be given *that the curve is a circle* is equivalent to two conditions, for we are then given two points of the curve at infinity. To be given *a focus* is equivalent to two conditions, for we are then given two tangents to the curve (p. 233), or we may see otherwise that the focus and any three conditions will determine the curve; for by taking the focus as origin, and reciprocating, the problem becomes, to describe a circle, three conditions being given; and the solution of this, obtained by elementary geometry, may be again reciprocated for the conic. Again, to be given *the pole, with regard to the conic, of any given right line,* is equivalent to two conditions; for three more will determine the curve. For (see figure, p. 132) if we know that P is the polar of R′R″, and that T is a point on the curve, T′, the fourth harmonic, must also be a point on the curve: or if OT be a tangent, OT′ must also be a tangent; if then, in addition to a line and its pole, we are given three points or tangents, we can find three more, and thus determine the curve. Hence, to be given *the centre* (the pole of the line at infinity) is equivalent to two conditions. It may be seen likewise that to be given a point on the polar of a given point is equivalent to one condition. For example, when we are given that the curve is an equilateral hyperbola, this is the same as saying that the two points at infinity on any circle lie each on the polar of the other with respect to the curve.

Given five points.—We have shown (Ex. 12, p. 283) how by the ruler alone we may determine as many other points of the curve as we please. We may also find the polar of any given point with regard to the curve; for by the help of the same Example we can perform the construction of Ex. 2, Art. 149. Hence too we can find the pole of any line, and therefore also the centre.

Five tangents.—We may either reciprocate the constructions of Ex. 12, p. 283, or reduce this question to the last by Art. 266.

Four points and a tangent.—We have already given one method of solving this question, p. 280. As the problem admits of two solutions, of course we cannot expect a construction by the ruler only. We may therefore apply Carnot's theorem (Art. 314),

$$Ac \,.\, Ac' \,.\, Ba \,.\, Ba' \,.\, Cb \,.\, Cb' = Ab \,.\, Ab' \,.\, Bc \,.\, Bc' \,.\, Ca \,.\, Ca'.$$

Let the four points a, a', b, b' be given, and let AB be a tangent, the points c, c' will coincide, and the equation just given determines the ratio $Ac^2 : Bc^2$, everything else in the equation being known. This question may also be reduced, if we please, to those which follow; for given four points, there are (Art. 318) three points whose polars are given; having also then a tangent, we can find three other tangents immediately, and thus have four points and four tangents.

Four tangents and a point.—This is either reduced to the last by reciprocation, or by the method just described; for given four tangents, there are three points whose polars are given (p. 134).

Three points and two tangents.—It is a particular case of Art. 337 that the two points where any line meets a conic, and where it meets two of its tangents, belong to a system in involution of which the point where the line meets the chord of contact is one of the foci. If, therefore, the line joining two of the fixed points a, b, be cut by the two tangents in the points A, B, the chord of contact of those tangents passes through one or other of the fixed points F, F′, the foci of the system (a, b, A, B), (see Art. 264). In like manner the chord of contact must pass through one or other of two fixed points G, G′ on the line joining the given points a, c. The chord must therefore be one or other of the four lines, FG, FG′, F′G, F′G′; the problem, therefore, has four solutions.

Two points and three tangents.—The triangle formed by the three chords of contact has its vertices resting one on each of the three given tangents; and by the last case the sides pass each through a fixed point on the line joining the two given points: therefore this triangle can be constructed.

To be given two points or two tangents to a conic is a particular case of being given that the conic has double contact with a given conic. For the problem to describe a conic having double contact with a given one, and touching three lines, or else passing through three points, see p. 283. Having double contact with two, and passing through a given point, or touching a given line, see p. 237. Having double contact with a given one, and touching three other such conics, see p. 257.

We have already alluded (p. 252) to the problem, "to describe a conic through four points to touch a given conic." Let the required conic be $S + kS'$, which is to touch S''. Then the polar of the point of contact, with regard to S'', is the tangent at the point, and is also its polar for $S + kS'$, and therefore passes through the intersection of the polars with regard to S and S'. Now let it be required to find the locus of a point such that its polars, with regard to S, S', S'', should meet in a point. If ξ, η, ζ be the current co-ordinates, we have to eliminate these between the equations of the three polars,

$$\xi\frac{dS}{dx}+\eta\frac{dS}{dy}+\zeta\frac{dS}{dz}=0,\quad \xi\frac{dS'}{dx}+\eta\frac{dS'}{dy}+\zeta\frac{dS'}{dz}=0,\quad \xi\frac{dS''}{dx}+\eta\frac{dS''}{dy}+\zeta\frac{dS''}{dz}=0;$$

and the result is,

$$\frac{dS}{dx}\left(\frac{dS'}{dy}\cdot\frac{dS''}{dz}-\frac{dS'}{dz}\cdot\frac{dS''}{dy}\right)+\frac{dS}{dy}\left(\frac{dS'}{dz}\cdot\frac{dS''}{dx}-\frac{dS'}{dx}\cdot\frac{dS''}{dz}\right)$$
$$+\frac{dS}{dz}\left(\frac{dS'}{dx}\cdot\frac{dS''}{dy}-\frac{dS'}{dy}\cdot\frac{dS''}{dx}\right)=0,$$

a curve of the third degree, whose intersections with S'' give the six solutions sought.

If S, S', S'' all pass through the same two points A, B, the locus reduces to a line and a conic: for the line joining those points must be a factor in the locus, since the polar of any point C on that line must pass through D, the fourth harmonic to A, B, C. If S, S', S'' represent circles, the equation just written represents the circle cutting all those at right angles.

The locus will also break up into a line and conic, if one of the quantities S' be a perfect square L^2; since L will then be a factor in the locus. Hence we can describe a conic to touch a given conic S at two given points (S, L), and also touching S''; for the intersection of the locus with S'' determines the points of contact with S'' of conics of the form $S + L^2$.

THE END.

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