A

TREATISE

ON

ALGEBRAIC GEOMETRY.

BY THE

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PREFACE.

THE first thirteen sections of the following work were written immediately after I obtained my degree. Sensible how imperfectly qualified I must have then been for the execution of a work of such extent, I laid aside the manuscript, in expectation that some one of more years, experience, and talent, would supply what was, and has continued to be, a desideratum in science—a complete and uniform system of Algebraic Geometry. After the lapse of several years, no work of this description having been announced, I again resumed my labours with increased experience and knowledge, and therefore with increased confidence.

The part of the work now published has been submitted to the best test by which an elementary treatise can be estimated, the purposes of instruction. Such alterations have been made as were suggested, and it is hoped that the treatise, as it now stands, will be found properly adapted to initiate students in the elements of the science.

Such principles of algebra as are assumed in the text, and not to be found in the common elementary treatises in our language, have been explained and proved in the notes. In these the student will also find a considerable portion of historical information respecting the invention and progressive improvement of the different parts of geometry which fall under his consideration throughout the work, and other matter which, if introduced into the text, would have broken its uniformity.

Those who are acquainted with foreign works on this subject will easily estimate the extent of my claims on the score of originality. Much new matter is not to be looked for at this period in any elementary work, and therefore one may justly assume a double portion of credit for whatever may be found. Considerable improvement will be perceived in the method and arrangement. The formulæ which have been given by other writers are rendered more general, and therefore more prolific in results, and more symmetrical in form. A very considerable portion of the examples and illustrations, both geometrical and physical, will, I believe, be found to be original. The transformation of co-ordinates, which is in general so operose, and which is the mean ordinarily used for discovering the properties of curves, is very sparingly introduced, most of the properties being discovered without it with more clearness and facility.

I have been very attentive in supplying a defect which exists in every treatise on the subject which I have ever seen, a total want of examples illustrative of the application of the abstract rules and principles of the science. This deficiency prevails, without a single exception, in all the continental writers. Some will, perhaps, be of opinion that I have fallen into the opposite extreme, and given too much illustration. To this I have only to answer, that in this science the illustrations and examples are not confined in their effect merely to the practice they afford in the analytic art, but that they also store the mind with independent geometrical and physical knowledge. Besides, it should be considered, that the only effectual method of impressing abstract formulæ and rules upon the memory, and, indeed, of making them fully and clearly apprehended by the understanding, is by examples of their practical application. The quantity of examples necessary to make the mind grasp any general principle is different according to the various degrees of talent. A sufficiency, at least, should be given for students of very moderate capacity. It will be much more easy for those of superior parts to omit what they shall feel superfluous, than for those, whose talents are of a lower standard, to supply what they might find deficient.

The title "Algebraic Geometry" has been preferred to either of the titles, "Analytic Geometry" or "the Application of Algebra to Geometry," because the one is equivocal, and the other circumlocutory. The use of the transcendental analysis has been brought as an objection to the present title. I do not, however, think this a sufficient reason for rejecting it.

It is but justice to myself to state a circumstance which, though it cannot affect the intrinsic excellence of this work, if it have any, yet must materially influence the estimation in which its author will be held by the reader. During most of the period in which I have been employed upon the present treatise, from eight to ten hours daily of my time were occupied in the labours of instruction: so that this work may truly be said to be the result of a few spare hours, and these always hours of fatigue both of body and mind. This I hope will plead my apology for any oversights which may be found throughout the work, of which probably there are not a few.

The typographical errors have been very carefully collected in the errata. Their number has been principally caused by the circumstance of my residence in a different country, and nearly four hundred miles from London, where I have found it expedient to publish the work. The difficulties of transmitting the proof sheets for correction with sufficient punctuality and despatch were very great. These difficulties would have been nearly insurmountable, owing to the enormous charges of the post-office, were it not for the kindness and attention of some members of parliament, through whom the necessary correspondence with the publishers in London was conducted.

INTRODUCTION.

Geometry, in its most extensive sense, is the science whose object is the investigation of the properties of figure. Figure* is the mutual relation of the limits of space among each other. It is therefore an affection of lines and surfaces; lines being the limits of superficial, and surfaces those of solid space. The ideas expressed by the terms line and surface admit no definition, and for the same cause require none†. They are con-

* "Figure is the relation which the parts of the termination of extension, or circumscribed space, have amongst themselves."—Locke.

† Although the abstract terms line, and surface, admit no definition, yet their species, with the exception of right lines and plane surfaces, do; these, being simple ideas, are undefinable.

"The several terms of a definition, signifying several ideas, they can altogether by no means represent an idea, which has no composition at all; and therefore a definition, which is properly nothing but the showing the meaning of one word by several others, not signifying each the same thing, can, in the names of simple ideas, have no place."—Locke.

D'Alembert entertains a different opinion on the necessity of defining those terms, and yet, at the same time, seems to admit ceptions so simple and obvious, that any necessity of explaining them is superseded by an appeal to the senses. It is only necessary to observe, that the term *solid* is used without any reference to body, merely to signify the space which a solid body might occupy. Lines and surfaces are subdivided into numerous classes, marked by various characteristic properties.

The first division of lines is into straight lines and curves, and of surfaces into plane and curved. Straight lines and plane surfaces admit no further subdivision, for they are without any variety. One indefinite straight line is so applicable to any

its impossibility. He, however, thinks a bad definition better than none.

"Nous ne pretendons pas pour cela qu'on doive supprimer des elemens de geometrie les definitions de la surface plane et de la ligne droite. Ces definitions sont necessaires; car on ne sauroit connoître les propriétés des lignes droites et des surfaces planes sans parler de quelque propriété simple des ces lignes et des ces surfaces qui puisse être apperçue à la premiere vue de l'esprit, et par consequent être prise pour leur definition. Ainsi on definit la ligne droite, la ligne la plus courte qu'on puisse mener d'un point à un autre; et la surface plane, celle à laquelle une ligne droite se peut appliquer en tout sens. Mais ces deux definitions, quoique peut-être preferable à toutes celles qu'on pourroit imaginer, ne renferment pas l'idée primitive que nous nous formons de la ligne droite et de la surface plane, l'idée si simple et pour ainsi dire, si indivisible et si une, qu'une definition ne peut la rendre plus claire, soit par la nature de cette idée même, soit par l'imperfection du langage."—D'ALEMBERT.

other as perfectly to coincide with it, so that, in effect, the two lines will become one. Straight lines, then, can differ one from another only in magnitude and position; but the *figure* of all straight lines must be the same, and they must therefore possess the same properties. Similar observations apply with equal force to plane surfaces. This, however, is not the case with curves and curved surfaces. Each of these classes contains an endless variety of species, the investigation of the properties of which is the business of the geometer. A more particular subdivision will, however, be necessary before proceeding to the discovery of these properties.

Lines may always be conceived to be described upon surfaces. Under this point of view, curves resolve themselves into two classes. The first embraces those whose points, all situate in the same plane, may be conceived to be described upon a plane surface; and the second, those whose points not lying in the same plane, can only be conceived to be described upon a curved surface. The former are called plane curves, and the latter curves of double curvature. The investigation of curved surfaces involves necessarily the nature and properties of curves of double curvature, and therefore the whole range of geometry may be divided into two principal parts:

THE GEOMETRY OF PLANE CURVES, and

THE GEOMETRY OF CURVED SURFACES.

In conformity with this, the following treatise is divided into two parts, under these denominations.

The first part might naturally be called plane geometry. Names, however, are invented, not after knowledge has reached its full extent, but in its progress to that state. After the limits of a science have been extended by the gradual accession of discoveries, terms are always to be found which are used in a much more confined sense than they might admit of; because their inventors, unacquainted with the extent which lay undiscovered, only applied them to the parts then known; and the difficulty and inconvenience which always attend the alteration of received names induced their successors to invent new terms rather than disturb the accepted sense of the old ones. To this cause the very limited sense of the term, plane geometry, must be attributed.

In the earliest infancy of the science, its limits were confined to the properties of rectilinear figures, or rather to the properties of triangles, into which all rectilinear figures may be resolved. The circle probably served at first as a mere instrument in the construction of rectilinear problems. The properties of this curve, however, soon became the object of investigation, and were discovered in a very early stage of the science. The right line and circle terminated the inquiries of the first geometers with respect to lines. They next turned their views to surfaces, and in these they confined themselves to those generated by the revolution of an angle round the line which bisects it, a rectangle round one of its sides, and a circle round one of its diameters. They thus acquired the notions and investigated the properties of cones, cylinders, and spheres. They accordingly divided their geometry into two parts, called plane and solid geometry.

The term plane geometry is still used in the same sense, and is so much of the geometry of plane curves as includes the right line and circle. In plane geometry, treated according to the ancient method, nothing is permitted to be done but what may be effected by a rule and compass, and nothing is allowed to be true without proof, except a few simple and general propositions called axioms, and prefixed by Euclid to his Elements. On these axioms, and on the definitions, the whole structure of plane geometry rests.

The science continued within these limits until the time of Plato, about four hundred years before the Christian era. The institution of the Platonic School forms a most striking epoch in the progress of geometry. In it originated the conic sections, the geometrical analysis, geometric loci, and the discussion of the celebrated problems of the duplication of the cube, and the trisection of an angle. The geometers of this school, finding that the ingenuity of their predecessors had nearly exhausted plane and solid geometry as they had descended to them, contrived, by the combination of these sciences, to produce new subjects for speculation. They conceived a conical surface intersected by a plane, and a line traced upon the plane by the points common to it, and the surface of the cone. Hence arose the conic sections, the properties of which have employed the talents of geometers from that time to the present, and which have been since discovered to be the lines traced by the planets and comets in their revolutions round the sun, their common centre of attraction. These are the first curves to which the attention of the student is directed in the following work, though defined in a different manner, and conformably to the general system which has been adopted.

The invention of the geometric analysis, besides its intrinsic excellence, has the additional interest arising from our knowledge that it is the invention of Plato himself. The other discoveries are known to have originated in the Platonic school, but we have no authentic record to prove their particular inventors. It does not appear that Plato wrote any work purely mathematical. The authority of Proclus, however, proves him the inventor of the geometrical analysis. Any geometrical question, whether problem or theorem, being submitted to analysis, is assumed as solved if it be a problem, and as true if a theorem. From this assumption a chain of consequences is drawn, which, by the ingenuity of the geometer, is continued until he arrives at some proposition known to be true or false, if the question be a theorem; possible or impossible if it be a problem. The final consequence points out whether the question be true or possible, and, by retracing the steps, a synthetical proof or solution may be found.

Geometric loci in the Platonic school were conceived to be produced by indeterminate geometrical problems in the manner explained in the commencement of the following treatise. The principal use to which they were applied by the ancients was the solution of determinate problems, by the intersection of two loci determined by indeterminate problems. To give a very simple instance; if the problem to be solved be the determination of a triangle, whose base, area, and ratio of sides are given, the problem is resolved by the intersection of a right line and circle; the former the locus of the vertex, where the base and area are given, and the latter its locus when the base and ratio of sides are given.

The celebrated problem of the duplication of the cube was solved mechanically by Plato. Menechme, a pupil of his, solved the same problem by the intersection of two parabolæ *, and by the

* See art. 585.

intersection of a parabola and hyperbola. This was one of the first applications of geometric loci to the solution of determinate problems.

Geometers next began to extend their investigations to the discovery of the lengths of curves, and the areas contained by them. This gave birth to the Method of Exhaustions, the most refined and subtle invention of the ancients. In this method, which was employed with such admirable ingenuity and address by Archimedes, and by the use of which he effected most of his discoveries in geometry, we may, by minute attention, observe the germ of the differential and integral calculus. This, however, must only be understood of the metaphysical principle of that wonderful science; for in their application to geometry, to say nothing of the physical and algebraical sciences, the powers of the calculus are far beyond those of the ancient method.

By the Method of Exhaustions, the lengths and areas of curves were compared, by comparing those of inscribed and circumscribed rectilinear figures. As the number of sides are increased, the differences between the figures, and therefore, *a fortiori*, between each of them and the curve, are continually diminished. It is always possible so to multiply the number of sides, that these differences shall be made less than any assignable magnitude. Under these circumstances, any property of the rectilinear figures which is independent of the number of their sides will be also a property of the curves. This would appear to a modern geometer sufficiently evident, but the ancients were more fastidious, and to remove all possibility of objection, they confirmed the proof in every particular instance, by an argument *ex absurdo*.

Although the ancients passed the limits of plane geometry, yet, from the nature of the method of exhaustions, all their demonstrations were tedious and elaborate. When we enter upon the investigation of any curve beyond the circle, by this method, we are perpetually embarrassed, not with the difficulties of the subject, but with the inadequacy of the method, the insufficiency of which is supplied at the expense of an immense quantity of valuable time and talent.

From the time of Archimedes, Apollonius, Conon, Nicomedes, and Diocles, who lived about three centuries before the Christian era, until the seventeenth century, an interval of two thousand years, geometry made no considerable progress. In the year 1637 Descartes published his *Geometrie*. This work disclosed to the world his discovery of the application of algebra to geometry, which vanquished a great number of the difficulties which had so long impeded the progress of that science. In assigning to Descartes the entire honour of the invention of Algebraic Geometry, it is not meant that no mathematician before him had applied algebra to the resolution of geometrical questions. On the contrary, we find many such applications in the algebra of Bombelli, an algebraist, nearly contemporary with Cardan, and also in the works of Tartaglia, a mathematician of the early part of the sixteenth century, and even so far back as the time of Regiomontanus; but more particularly in the works of Vieta. The general method of representing curves by equations between two unknown quantities, and thence deducing their various properties by algebraic operations performed upon these equations, was, however, unquestionably the invention of Descartes.

This discovery suggested itself to Descartes in the investigation of the following problem, which had been attempted without success by several ancient geometers; among others by Euclid, Apollonius, and Pappus. "To determine a point upon a given plane, from which, if a number of right lines be drawn, inclined at given angles, to as many right lines given in position, the continued product of half the number of lines so drawn, shall bear a given ratio to the continued product of the remaining lines, if their number be even, and so that the continued product of half their number diminished by one, shall bear a given ratio to the continued product of the remaining lines, if their number be odd. Thus, if n be the number of lines so drawn, the continued product of $\frac{n}{2}$ of these shall bear a given ratio to the product of the remaining lines, if n be even, and so that the continued product of $\frac{n-1}{2}$ of them shall bear a given ratio to the continued product of the remaining lines, if n be odd." Descartes observed that the problem was indeterminate, and that an infinite number of such points might be found; in other words, that the solution of the problem was not effected by a point, but by a curve which might be considered as the locus of the sought point. He also found that all these points were related to the lines given in position, and to the given angles by one common relation, which he expressed by an equation composed of constant quantities, representing the several data of the proposed problem, and which therefore are supposed to remain the same, however the sought point may vary its position, and two variable quantities representing lines, the magnitudes of which depending on the position of the sought point, change as it changes. The sought point passing through its various positions being supposed to describe the locus, he assumed this equation to represent the curve; for, any value being assigned to one of the variables, the equation solved for the other determines a point of the locus. It is not difficult to conceive one of the variables uniformly and continually to change its magnitude, and the other at the same time to undergo such a continuous change of magnitude, that the condition of the equation will always continue to be satisfied; the generating point will thus, by continued motion, trace out the locus.

Descartes perceiving the importance and power of the principle which he used in this solution, immediately conceived the notion of founding upon it the whole geometry of curve lines. By this felicitous application of equations of two unknown quantities, the science of geometry was utterly revolutionised. Every curve described by any given law being expressed by an equation between two variables deducible from that law, was thus brought under the dominion of algebra. This equation, including the essence of the curve, its various properties flowed from it; its different branches, the limits of its course; its asymptotes, diameters, centres; inflections, cusps, and, in a word, all its affections he found to be algebraically deducible from its equation. Thus the equation may be considered as a short formula in which all the properties of the curve are embodied, and from which the analyst is always able to deduce them by fixed and general rules, which are not peculiar to the equation of any particular curve, but indifferently applicable to those of all curves.

The immediate consequence of this memorable discovery was, that geometry at once oversprang the narrow limits which had circumscribed it for ages, and took a range, the extent of which is literally infinite. Instead of a few simple and particular curves, which had hitherto constituted the only objects of the science, the geometer discussed the properties of whole classes of curves, distinguished and arranged according to the degrees of the equations which represent them. The variety of curves thus became as infinite as that of equations. The ancient geometry proceeded upon no general methods. It consisted of scattered propositions arbitrarily put together, connected by no necessary tie or general law. The discovery of each particular property therefore cost the geometer a distinct effort of invention, and demanded a separate expenditure of intellectual energy; and, even when successful, he was as often indebted to chance as to his own sagacity. Thus, for example, their method of drawing a tangent to one curve furnished no clue which could lead to the solution of the same problem with another curve, and therefore the geometer was beset with the same difficulties every new curve he approached. The application of algebra at once removed these defects. It determined uniform and general rules for the investigation of the properties of every curve whatever. Nay, it did not alone assist the operation of the reasoning faculty, but actually supplied the place of invention by furnishing means of discovering curves in infinite variety. No equation between two unknown quantities can be proposed but a corresponding curve is immediately discoverable, whose nature and properties afford matter for geometrical speculation.

To algebra we are indebted for the classification of curves in different orders, forming, says Cramer, a sort of geometrical arsenal, where the implements of the science are so arranged, that, without hesitation, we can choose whichever may be best adapted to the resolution of any proposed problem.

Notwithstanding the extent and importance of the invention of Descartes, something still remained to be done before geometry could be considered to have reached that perfection of which it seemed susceptible. No method had been given by Descartes for the discovery of the lengths and areas of curves; problems, known by the names *rectification* and *quadrature*. Rectification had even been by some geometers considered impossible. Quadrature had been effected only in a very few instances. Archimedes had effected that of the parabola, and given an approximation to that of the circle. Besides these deficiencies, the method of drawing tangents, given by Descartes, although general, was, in many cases, attended with considerable difficulties, and required frequently the resolution of equations of the higher orders. A very short period, however, gave to the world a science which removed these difficulties, and may justly be considered to have brought geometry to a state little short of positive perfection.

The investigations which had arisen from the invention of Descartes, directed the attention of all the great geometers of the world to the discovery of a general method of drawing tangents to curves, which should be free from the objections to which both the methods * which Descartes had delivered were liable. Fermat, Roberval, Barrow, Sluze, and others, severally attempted the general solution of this problem without complete success. Their methods were operose, frequently impracticable, and never applicable to transcendental curves in general. Although the essays of these geometers did not subdue the difficulties of the problem, yet every new attempt shed additional light upon the subject, and gradually facilitated the solution. At length attentive consideration of the subject conducted two great geometers to the discovery of the true and general principles upon which all such problems depended.

Newton and Leibnitz each claim the honour of

* See note on art. 132.

the discovery of the Fluxionary or Differential Calculus, which at once presented easy and general methods for the solution of all problems of tangents, rectification, and quadrature. The invention of this science, unquestionably the most splendid conception the human mind ever entertained, whether we regard the nature of the science itself, or the extent, variety, and importance of its applications, was too grand an achievement of genius not to rouse the ambition even of the greatest men to claim the credit of it. The mathematicians of the continent, on the part of Leibnitz, and those of England, on the part of Newton, each advanced their claims, and hence arose the greatest and most protracted contest which ever agitated the philosophical world. With the exception of Newton himself, the parties displayed on both sides a degree of asperity and personal acrimony very inconsistent with the dignity of the prize for which they contended.

Without entering into any detail of the particulars of this memorable scientific war, we shall merely observe, that in its commencement, Leibnitz appealed to the Royal Society for justice for the injuries done to his fame by the British mathematicians; upon which the Society appointed a committee to examine into and report upon the rights of the illustrious candidates for the invention of the Calculus. Their report was published in 1712, under the title "Commercium

Epistolicum D. Johannis Collins et aliorum de Analysi promotâ." The principal part of this publication consists of extracts from a correspondence between Newton, Barrow, Gregory, Wallis, Keil, Collins, Leibnitz, Oldenburg, Sluze, and others. Upon this correspondence, the committee reported as follows :—

I. "That Mr. Leibnitz was in London in the beginning of the year 1673; and went thence in or about March to Paris, where he kept a correspondence with Mr. Collins, by means of Mr. Oldenburg, till about September, 1676, and thence returned by London and Amsterdam to Hanover: and that Mr. Collins was very free in communicating to able mathematicians what he had received from Mr. Newton and Mr. Gregory.

II. "That when Mr. Leibnitz was the first time in London, he contended for the invention of another differential method, properly so called; and, notwithstanding that he was shown by Dr. Pell that it was Mouton's method, persisted in maintaining it to be his own invention, by reason that he had found it by himself without knowing what Mouton had done before, and had much improved it. And we find no mention of his having any other differential method than Mouton's before his letter of the 21st of June, 1677, which was a year after a copy of Mr. Newton's letter of the 10th of December, 1672, had been sent to Paris to be communicated to him, and above four years after Mr. Collins began to communicate that letter to his correspondents; in which letter the method of Fluxions was sufficiently described to any intelligent person.

III. "That by Mr. Newton's letter of the 13th of June, 1676, it appears that he had the method of Fluxions above five years before the writing of that letter. And by his *Analysis per equationes numero terminorum infinitas*, communicated by Dr. Barrow to Mr. Collins in July, 1669, we find that he had invented the method before that time.

IV. "That the differential method is one and the same with the method of Fluxions, excepting the name and mode of notation; Mr. Leibnitz calling those quantities differences, which Mr. Newton calls Moments or Fluxions, and marking them with the letter *d*, a mark not used by Mr. Newton. And therefore we take the proper question to be, not who invented this or that method, but who was the first inventor of the method. And we believe that those who have reputed Mr. Leibnitz the first inventor knew little or nothing of his correspondence with Mr. Collins and Mr. Oldenburg long before; nor of Mr. Newton's having that method above fifteen years before Mr. Leibnitz began to publish it in the *Acta Eruditorum* of Leipsic.

"For which reason we reckon Mr. Newton the first inventor; and are of opinion that Mr. Kiel, in asserting the same, has been no ways injurious to Mr. Leibnitz. And we submit to the judgment of the Society, whether the extract and papers now presented to you, together with what is extant to the same purpose in Dr. Wallis's third volume, may not deserve to be made public."

The foreign mathematicians, as might be expected, were by no means satisfied of the justice of this decision, in which it was more than insinuated that Leibnitz was guilty of a disgraceful theft. Even to the present day a difference of opinion on the subject exists, and the fire of party zeal is far from being extinct. The foreign writers generally contend that Leibnitz has the merit of the invention, though some of them, at the same time, allow that Newton was acquainted with its principles first, although he did not disclose them to the world. Bossut insinuates that Newton, being president of the Royal Society, must necessarily have had a strong influence on this report; also, that it was made *ex parte*, and that its publication was hastened to avoid introducing a defence which Leibnitz had in preparation. The foreign writers, in general, strongly deny the fact, that the principles of Newton's method, or any hints which could lead to them, are contained in the letters and papers alluded to in the report, and published with it. Montucla, one of the most candid of the French writers on the subject, says, "On ne peut douter que Newton ne soit le premier inventeur des calculs dont il s'agit. Les preuves en sont plus claires que le jour; mais Leibnitz est-il coupable d'avoir publié comme sienne une decouverte qu'il auroit puisée dans les écrits même de Newton." At the same time he insists upon the injustice of Newton to Leibnitz, in suppressing in the edition of the Principia, published in 1726, a scholium which appeared in the former edition, in which Newton is alleged to have allowed Leibnitz the merit of the invention. He also accuses Newton of having been secretly the author of the notes which accompany the Commercium Epistolicum.

One of the latest attempts to keep the discord of the scientific world alive upon this subject, is the preface to the last edition of Lacroix's *Traité du Calcul Differentiel et Integral*, repeating again all the former arguments on the subject, except those on which the claims of Newton are founded. He observes, "L'exposé fidèle que je viens de faire de la naissance du Calcul Differentiel, d'apres le Commercium Epistolicum, imprimé par ordre de la Société Royale de Londres, ne peut laisser aucune doute sur les droits incontestable de Leibnitz à la decouverte de ce calcul; et comme il est le premier qui l'ait rendue publique, tandis que Newton, preferant son repos à sa gloire et à l'interêt de ses contemporains, semblait avoir oublié sa methode, n'est-il pas aussi celui qu'on doit nommer le premier dans cette decouverte?"

Although there certainly still exists a difference of opinion as to the proportion of the merit of the discovery to be allotted to each of these illustrious claimants, yet, it seems to be generally agreed, that a proportion is due to each. It is generally acknowledged that, although Newton did not promulge the method of Fluxions, yet that he has the priority as to the invention. Even some of the partisans of Leibnitz do not dispute this. On the other hand, Leibnitz first gave formal publication to the calculus. His notation also is very superior to that of fluxions—so much so, that even in these countries it has nearly superseded it.

The first subject on which this surprising science began to work its wonders was geometry. Problems, which solved by the ancient methods, or even by those of Descartes, were tedious and embarrassing, were solved by the dash of a pen. Problems which had foiled the talents of Archimedes, eluded the sagacity of Apollonius, and under which, even the method of Descartes sunk powerless, yielded with the utmost facility to the new calculus. By the uniformity and generality of its processes, it rendered geometry at once an imposing and magnificent edifice, raised upon a solid foundation, displaying an unity of design, a justness of proportion, and a stability of structure, which would strike an ancient geometer with astonishment and admiration, were he to rise from the tomb to behold it *.

* Si les deux plus grands géomètres de l'antiquité, Archi-

From the date of its discovery to the present day, the calculus has been rapidly advancing towards perfection under the hands of the great mathematicians of Europe, who have devoted their talents to its improvement. Every impulse given to the advancement of this science has produced a corresponding impression upon the other parts of mathematics and physics, but on none more perceptibly than Geometry. This branch of mathematics is largely indebted to the calculus. It owes to the integral calculus all solutions relative to rectification and quadrature, and to the differential calculus, the general method of tangents, the general principles of contact and osculation, the methods of detecting singular points, and its entire power over transcendental curves.

One of the most remarkable circumstances attending the progress of Geometry is the different routes pursued by the British and foreign geometers since the time of Newton. That great man entertained a strong predilection in favour of the ancient geometrical methods. A stronger proof of this cannot be offered than his having discovered, by the aid of the modern analytic and fluxionary calculus, most of those wonderful

mèdes et Apollonius, pouvaient revivre, ils seraient eux-mêmes frappés d'etonnement et de l'admiration, en contemplant les progrès que les sciences exactes ont faits depuis leur temps jusq'au notre à travers des siecles barbares qui ont tant de fois interrompu la marche du genie.—BOSSUT.

truths communicated to the world in his PRINCIPIA; and yet presented them in all the repelling tediousness and circuitous complexity of the ancient geometry.

The method followed throughout this stupendous work is such, as to induce foreign mathematicians to ascribe the adoption of it to motives which could never have influenced a mind like Newton's. "La clef des plus difficiles problemes," says Bossut, "qui y sont resolus est la methode des fluxions ou l'analyse infinitesimale, mais presentée sous un forme moins simple qui rendait l'ouvrage penible à suivre. Aussi n'eut il d'abord tout les succes qu'il meritait; on y trouva de l'obscurité des demonstrations puisées dans des sources trop detournées, un usage trop affecté de la methode synthetique des anciens tandis que l'analyse aurait beaucoup mieux fait connaitre l'esprit et le progres de l'invention. L'extreme concision de quelques endroits fit penser ou que Newton doué d'un sagacité extraordinaire avoit un peu trop presumé de la penetration de ses lecteurs; ou que par une faiblesse dont les plus grandes hommes ne sont pas toujours exempts il avoit cherché à surprendre un admiration qui le vulgaire accord facilement aux choses qui passent ou fatiguent son intelligence." Alluding to the superiority of the modern analysis over the ancient methods, D'Alembert says, "Peut être serons nous contredits ici par les Anglois grands partisans de la Geometrie

Ancienne sur la foi de Newton qui la louoit et qui s'en servoit pour cacher sa route en employant l'analyse pour se conduire lui même." Though no one knowing the character of Newton can, for a moment, assent to these imputations; yet, it is much to be regretted, that through an ill-founded prejudice, he should ever have given occasion to them.

In the hands of Newton the powers of the ancient geometry were extended to their extreme limit. Supplying their inadequacy by his own sagacity, Archimedes had previously astonished the scientific world by what he made them effect. But even Archimedes would shrink from the competition, if he beheld the miracles wrought by the more than human genius of Newton, with the same feeble instruments, very little improved. Deeply impressed with the wonders they thus beheld effected and guided by his avowed judgment, the English schools of science, until a few years since, have uniformly pursued the ancient geometrical methods. The consequence has been, that the progress of mathematical science has been much slower in Great Britain than elsewhere. At the death of Newton, Geometry had done all that geometry could do, and the highest efforts of human talent could stretch its powers no farther. The students at our universities have traversed the same ground in every direction again and again. Ingenuity has been exhausted in supplying them with employment by the invention of collections of contemptible geometrical quibbles for their solution, which possess no other excellence than their difficulty. Instead of expanding the mind and invigorating the intellect, presenting enlarged views, extended and general theories, and storing the memory with useful and elevating knowledge, they confer very little benefit but what may justly be called geometrical trick.

While the schools of Great Britain were thus wasting the splendid abilities by which they have ever been distinguished, on objects so unworthy of them, and throwing away the golden opportunities of honour which the progressive improvement of analysis each year presented, far different were the objects which exercised the rest of the learned world. The Algebraic and Transcendental Analysis were embraced with eagerness, and promoted with rapidity. Every year witnessed new accessions to these sciences, and consequent advancements in geometrical and physical knowledge. Impelled by these powerful engines, the Newtonian philosophy, which at home stood nearly where its illustrious founder had left it, abroad advanced with the speed of light, and we find the result of the various improvements it has received up to the present day in the great work of Laplace.

The immense advantage thus gained upon us by the philosophers of Europe in mathematical and physical science became at length too apparent to be longer overlooked. The university of Cambridge was the first to begin the reformation. The works of Euler, and the French mathematicians Laplace, Lagrange, Lacroix, and numerous others, were introduced and studied with activity. The notation of fluxions and fluents was superseded by the more elegant and powerful algorithm of the Differential and Integral Calculus. Students, who hitherto seldom had courage to labour through more than a few sections of the PRINCIPIA, were now becoming familiar with the pages of Laplace and Lagrange. That the change effected in this great national institution is deep, radical, and permanent, we have public proofs in the works of Herschell, Woodhouse, Babbage, Peacock, and Whewell.

The university of Dublin, though later in adopting these measures of improvement, has not been less vigorous in their prosecution, and will soon accompany her British sister *passibus æquis*. There is something worthy of notice in the circumstances attending the introduction of what is called the "new science" into this university. Great changes in the literary and scientific arrangements of an extensive institution are generally slowly effected, and produced by a combination of the industry and talents of a number of individuals co-operating for the attainment of the same end. In this instance, however, the revolution was great, rapid, and the work of *one man*. About the year 1811, Dr. Bartholomew Lloyd, then a junior fellow, was elected to the professorship of mathematics. The state in which he found the knowledge of that science amongst the students, and, indeed, the state in which it had remained for a century, was nearly as follows.

Students in Dublin must be four years in the university before they become candidates for the degree of bachelor. Of this time, ten months were spent in the acquisition of the first, second, third, and sixth books of Euclid. These constituted the entire mathematical knowledge expected even from the candidates for the highest academical honours. A short selection of mechanics, taken from an old treatise by Helsham, accompanied by a popular introductory pamphlet to Natural Philosophy (both replete with errors), a very few of the first elementary principles of optics, and a selection from Keil's Astronomy, gave the under graduate employment for twelve months. The remainder of the course (two years and two months) was divided between the ancient and modern Logic, and the Ethics of Cicero and Burlemaqui. Such was the state of the undergraduate course. The mathematical and physical knowledge requisite in candidates for fellowships, the situations of highest honour and emolument in the university, consisted of Newton's Arithmetic, the properties of Conic Sections geometrically, Solid Geometry, Keil's Trigonometry, Newton's Optics, and a selection from the PRINCIPIA; Maclaurin's Fluxions were touched upon, but with reserve. Such was actually the state of scientific knowledge in this national academy about the year 1812.

Such a course of study might have been very proper in the university of Dublin in the year 1712; but in the year 1812, with the accumulated discoveries of a century, the various scientific establishments of Great Britain and the continent all actively cultivating physical and mathematical science in their most improved state, the continuance of such a system must have been considered disgraceful. Deeply impressed with this feeling, Dr. Lloyd, singly and unassisted, conceived and executed the most important and rapid revolution ever effected in the details of a great public institution. In order to appreciate the benefits derived from his exertions, it will be only necessary to compare the state of science already described, with its state in the present year 1822. Among the under-graduates, those who now look for high academical honours read the works of Cagnioli and Woodhouse on Trigonometry, Brinkley's Astronomy, a course of Algebraic Geometry, equivalent to the extent of the first part of the present treatise, the Elementary Treatise of Lacroix on the Differential, and part of that on the Integral Calculus; with Peacock's examples as a praxis; a selection from the *Mecanique* of Poisson, including the Statics, the Dynamical principle of D'Alembert, with its various applications; the theory of the moments of inertia, the motion of a body round a fixed axis, and most of the Hydronamics; also the subject of the first seventeen propositions, and the seventh section of the PRINCIPIA, and the theory of projectiles *in vacuo*, all treated analytically.

The course of science read by the candidates for fellowships has also advanced, but not nearly in the same proportion; and it is to be feared, that, until some change takes place in the manner of conducting the examination for fellowships, there can be little hope of improvement. This is a *vivâ voce* examination held in the Latin language. The object being to ascertain the knowledge which the candidates have acquired in the different departments of science and literature, it would appear that the medium of communication between the examiners and candidates ought to be that which would be most readily and clearly apprehended by both, and, therefore, that the English language would be much preferable to any other. For whatever facility may be acquired in speaking a foreign, not to mention a dead language, no one will have the hardihood to assert that it can ever be spoken as freely and fluently as our native tongue. Waving, however, for a moment the objection to the language, concerning which there may possibly exist some difference of opinion, what reason can be given for the exclusion of writing? Will it be credited abroad, that in the university of Dublin, at the election of fellows, there is actually held an *oral* examination in physics and mathematics, without any use whatever of writing? The development of a function by the theorems of Taylor or Lagrange, or the integration of a differential equation effected *vivâ voce*, and in Latin, are probably phenomena new to the learned world! It is unnecessary to extend our observations on this subject further, as its absurdity is so very apparent, that the strongest exposure which can be given to it is a simple statement of the fact.

It has been attempted here to present to the student a very brief sketch of the history of geometry to the present day. That the analytical methods have been almost universally adopted by the moderns in all questions which pass the mere elements of geometry is undeniable. At the same time, however, it is fair to state, that in Great Britain the ancient geometry is not altogether without some remaining partisans, who, in spite of the many proofs of its inefficiency, and in opposition to the judgment of the great mass of scientific talent of Europe, wish to found upon its principles the whole theory of curve lines. To show how vain such an attempt must prove, it is only necessary to examine how far it has succeeded, even when seconded by talents of the first order. Professor Leslie has lately published a work upon the Geometry of Curve Lines, which runs in some measure parallel with the present, and in which he avows himself the champion of "a juster taste in the cultivation of mathematical science." In plainer terms, the object is to produce a counter revolution in geometrical science in Great Britain, and to restore it to the state it had been in before the introduction of the modern analysis.

This work presents the most conclusive proofs how inadequate the method adopted in it is to elucidate most of the subjects to which it is applied. Its failure has betrayed the author in many instances into the use of a phraseology very unsuitable to a mathematical work. Whenever it becomes necessary to explain those properties of curves which demand the higher instruments of analysis, the Professor uses sometimes language which really admits no meaning whatever, and sometimes endeavours to remedy the weakness of the method by the use of an highly metaphorical and figurative style. He states that "the osculating circle may be derived either from the consideration of three approximating points, or from *that of a tangent and a point merging the same contact.*" He describes "points shooting into extreme remoteness, and *vanishing in the distance,*" "lines *thrown off* to indefinite distances," "points *vanishing towards one another,*" "points *absorbing* one another," "curves *migrating* into one another," "tangents *melting* into the curve," &c. &c. If the author had used Taylor's theorem in the investigation of the singular points, and in the determination of the tangents, he would never have been driven to the humiliating necessity of invoking the aid of poetry to establish the theorems of geometry. Had he effected rectification by the use of the integral calculus, his work would never have been encumbered with such a sentence as the following: "The *gradual aggregation* of increments constitute the line to which the *cumulative* amount of the elementary arcs which compose the curve is equal." But these absurdities are not the worst consequences which the imbecility of the geometrical method has produced in this treatise. The Professor has been in many instances led into positive error. The investigation of the osculating circle of the logarithmic, and its point of greatest curvature, presents a remarkable example both of absurdity of style and fallacy of conclusion. After various compositions, conversions, and divisions of ratios, and comparisons of *minute* lines and segments, he concludes, that "the radius of a circle osculating at any point of the logarithmic curve is a fourth proportional to the corresponding ordinate and tangent;" this is immediately followed up by a corollary to discover the point of greatest "incurvation," as the Professor calls it. After spending more than a page in describing the radius of curvature as "occupying a stationary limit," and "suffering a decrement at one end and an equal increment at the other," in the course of a slight mutation, he concludes, that the point so found is the point of greatest incurvation, because the line which represents the radius of curvature is placed in the limit where it has, on the whole, neither increase nor diminution, and has therefore contracted into its minimum. The radius of the osculating circle is, however, not what he professes to prove it, neither is the point assigned by him the point of greatest curvature.

Numerous other objections might be brought against this work, and, indeed, against any other proceeding upon the same principles, such as that by the method of marking the order of a curve by the number of its intersections with a right line, many curves of the fourth order would be reduced to the second, and therefore classed among the conic sections, though having no properties in common with those curves. It may be also observed that there are many singular points, the existence of which are not even recognised; such are conjugate points, points of undulation, &c. Neither is any method given for determining the different degrees of contact and osculation, nor for finding in general the evolutes and involutes of curves. Even those of the lines of the second degree are omitted with the exception of that of the parabola, which is casually thrown among the properties of the semicubical parabola, which the Professor calls the *Paraboloid.* It is unnecessary, however, to pursue these observations farther *.

Professor Leslie is most justly esteemed a man of the highest talents; his works in other departments of science are sufficient to establish his fame, and are so many unanswerable proofs how much the failure of his work on GEOMETRY is to be ascribed to the method, and how little to the author. The Professor engaged in an enterprise which could not have been attended with success had it been supported even by the genius of Newton.

That the preceding observations may not be misconstrued, nor wrested to a sense never con-

* It is a strange circumstance, that in the preface to this work the author states, that "the differential and integral calculus" really derives its main advantage from its algorithm, or that clear and compact form of notation invented by Leibnitz, and improved on the continent by his followers, the Bernouillis, Euler, and Lagrange, and yet at the same time states, that where he has found it necessary to depart from the ancient method, he has substantially applied the principles of the calculus *without its algorithm*, which amounts just to this, that finding the ancient methods, of which he is so enthusiastic an admirer, fail in carrying him even to the limited extent to which he has penetrated into the geometry of curves, he has been driven to the disagreeable necessity of having recourse to the more powerful calculus of the moderns; but that in these cases, he has uniformly taken care not to introduce the use of that from which these methods derive their cardinal excellence.

templated, the student is not to suppose that the following treatise is meant to supersede or replace the ancient geometry. That science must always be viewed with admiration by every person capable of appreciating the clearness, elegance, and variety, which, by the mere exercise of reason, may be drawn from one of the simplest of our ideas. But that admiration can only be co-extensive with the perspicuity and facility it confers on the investigation of the properties of figure. This science then, confined within proper bounds, must always continue to be cultivated and taught; but they are really its greatest enemies who attempt, by straining its powers beyond their natural limit, to apply them to subjects which they can involve in obscurity and difficulty.

As far then as the elements of geometry extend, the ancient methods are used with considerable advantage. Not requiring that abstraction which the more powerful analysis of the moderns demands, and directly addressing the senses as well as the understanding, they are adapted with peculiar fitness for the initiation of a student into the science. But, beyond this point, the young geometer will require engines of greater efficacy; and even though the requisite expertness in the use of these should cost him some labour, the acquisition of the powers with which they will invest him will amply repay him.

The clearness, rigour, and exactitude of the ancient geometry have been much and deservedly extolled, and it is not to be denied that, by great efforts of ingenuity, it may be and has been carried beyond the limits which have been assigned it. The modern methods have been stated to be inferior to them in two respects; in giving less occasion for the exercise of the reasoning faculty, and less rigour to the demonstrations. It may very fairly be answered, that the extent of the knowledge to be acquired is so great, the space allotted by Providence to the life of man so small, and the limits of his intellectual powers so confined, that it is perfect folly to create difficulties for the mere purpose of vanquishing them. Surely the natural obstacles which every where present themselves in the prosecution of scientific speculations are sufficient to exercise our faculties without raising up artificial difficulties. When two methods of arriving at the same truths present themselves, to select the most intricate and difficult, purely for the glory of the conquest, is little short of wilful sacrifice of time and ability.

As to the second objection, that the modern analytical investigations are inferior in rigour to those conducted upon the principles of the ancient methods, it is absolutely unfounded. The truth is, the objectors here confound the terms clearness and rigour, or probably have not a very distinct notion of the true nature of their own objection. Without taking advantage of the obscurity of their ideas, we will first explain the real nature of the objection, and then refute it. Locke very justly observes, that demonstrative truths are less clear, but not less certain than intuitive, and he illustrates his observation by the very appropriate simile of a face seen after many reflections. Owing to the aptitude of the mirrors to absorb part of the light, the brilliancy of the image is deteriorated by every reflection it suffers, but the features continue the same faithful copy of the original. So it is with the certainty of the conclusions to which we are led by the demonstrative process. That certainly admits of no degrees, although the clearness of our perception of it does. As the number of intervening proofs requisite to establish any proposed truth increases, so in proportion does it lose in clearness; but it certainly is in nowise impaired. That equal quantities increased or diminished by equal increments or decrements continue still equal, and that the squares of the lines containing a right angle are together equal to the square of the line joining their extreme points, are propositions equally certain, but by no means equally clear. The reason of this is, that the former is immediately perceived without the intervention of any proof whatever; it carries its own evidence with it, so that it never presents itself before the mind without being accompanied by the reason of its truth; but with the latter it is quite otherwise. Its certainty depends upon a long series of truths antecedently established, which have been registered in the memory, and which themselves must be ultimately capable of a resolution into self-evident elements. Now, if the mind of man were so capacious as to contemplate simultaneously all these, then the clearness of the one proposition would be equal to that of the other. But this is not so. The human mind, circumscribed in its powers of contemplation, can entertain ideas only in succession, and must therefore arrive at demonstrative truths by a succession of proofs. The number and nature of these proofs regulate the clearness of our perception of a truth, but do not affect its certainty.

To apply these reflections to the point in question; if the partisans of the ancient geometry in asserting its superior rigour, mean that it imparts to its demonstrations an higher *degree* of certainty, they speak illogically, and use terms without any distinct import; certainty does not admit *degrees*. If they mean that the conclusions to which the modern analytic method conducts are short of certainty, and must therefore be considered as only probable; the charge can be easily refuted. This method reposes upon the same principles as the ancient geometry. Nothing is assumed in it without proof, but what is also assumed in that science. It is true that much in it is mechanical, and it is in this very circumstance that one of its perfections consists. Regulated by certain rules previously established by proof, the pen of the analyst relieves his mind from many painful details in the demonstrative process, without shaking the validity of his conclusions, and leaves him free and unwearied to pursue new truths. If it be desired, he can always give his demonstrations all that pretended rigour which they are supposed to want by translating the algebraic operations into ordinary language, and which is precisely what Newton has done in his *Principia* *. But most probably what is meant to be imputed to the modern methods is a deficiency in that clearness and perspicuousness with which the use of the ancient method is attended. To this it may be answered, that in elementary questions the excellence of the ancient method is not denied, and that in all geometrical investigations beyond these, this boasted clearness is not to be found; but on the contrary, that the demonstrations are intricate and embarrassed in the extreme, frequently indirect,

* Mais il ne tiendra qu'a l'analyste de donner ensuite à sa demonstration ou à sa solution la rigueur prétendue qu'on croit lui manquer il lui suffira pour cela de traduire cette demonstration dans le langage des anciens, comme Newton a fait la plupart des siennes. D'ALEMBERT.

always tedious, and requiring such a degree of acuteness, that none but an expert geometer is able to follow the thread of the proof; and all this applied to questions that are solved by the analysis of the moderns with perfect facility. On the other hand, the want of clearness in this analysis arises not from any fault in the instrument, but from the very abstruse and general nature of the questions to which it is usually applied; questions which are utterly beyond the most extended powers of the ancient geometry. Those, however, who are skilled in the analytical method feel too sensibly the extent of their powers to undervalue them; and the truth is, they are only decried by those who are ignorant of them, and who, as a learned writer observes, derive a species of consolation from stigmatizing as useless that which they do not understand.

The following treatise is designed to embrace GEOMETRY in its full extent. It is conducted by the modern Analytical Method in its most improved state. It is divided into two parts; the first containing the GEOMETRY OF PLANE CURVES, and the second the GEOMETRY OF CURVED SURFACES. The processes throughout the work have been rendered as elementary as the extensiveness of its object would admit. It is desirable that students who have passed the first elements of plane geometry and the rudiments of algebra should be qualified to commence algebraic geometry. With this view the differential and integral calculus is not introduced into the first part until after a very detailed investigation of the properties of lines of the second degree, and an extensive collection of questions, adapted for exercise, as well in these properties as in the general principles of algebraic investigations. As far as this point the student may proceed without the aid of the calculus, and this part may precede the study of that science with considerable advantage, as it familiarises him with the species of investigations which first led to its invention. Previously to advancing further, it will be necessary to acquire a knowledge of the first principles of the calculus. The elementary work of Lacroix, as far as the section on maxima and minima, with the ordinary methods of integrating algebraic and trancendental functions of one variable will be sufficient for the remainder of the first part. In this part the simplest and most elementary principles of integration are uniformly adopted. Those who are more expert in the use of the calculus will probably, in many instances, find methods more expeditious or elegant than those which have been used. These have in general been chosen, as better suited to the limited knowledge of a junior student, and possibly in some instances from oversight. The general method of drawing rectilinear tangents, rectification and quadrature, the theory of evolutes, the general principles of contact and osculation, and the manner of discovering singular points, are explained by the calculus, and these principles applied to lines of the second degree. Passing to transcendental curves and algebraic curves exceeding the second degree, the properties of all these, which offer any interest to the geometer, whether arising from their intrinsic beauty, or from their utility in physical applications, are very fully discussed. These, besides possessing the student with a large portion of interesting and various geometrical knowledge, serve for exercise in the manner of investigating algebraically curves in general.

The geometry of plane curves is next applied to the illustration of a variety of important theorems relating to the roots of algebraic equations, and the method of determining these roots by the intersection of curves is explained, and examples of its application given. The general properties of algebraic curves are developed as far as they appear to possess any particular interest. To enter further into the discussion of them would have swelled the bulk of the volume without any adequate advantage to the student. Those who may be desirous of further information on this subject are referred to Cramer's *Introduction a l'Analyse des Lignes Courbes*, Euler's *Analysis Infinitorum*, Stirling on Newton's lines of the third order, and De Gua's work entitled *l'Usage de l'Analyse*, &c. The first part is concluded by a very copious collection of questions in geometry and physics, for general exercise in the principles thus far explained, as well as to point out the utility of this science. The questions in physics are adapted to the junior students; this part of the work being altogether superfluous for those who are more advanced.

The second part, which will contain the GEOMETRY OF CURVED SURFACES, will necessarily require a more extensive knowledge of the calculus. The student, however, as he advances, will find little difficulty in gradually extending his knowledge of that science.

Exiguus nascitur, sed opes acquirit eundo.

Hitherto, no treatise whatever on Algebraic Geometry has appeared in Great Britain, and even in France no complete treatise upon the subject has ever been published. The works of the different French mathematicians, entitled "*Geometrie Analytique*" and "*l'Application de l'Algebre à la Geometrie*," do not in general include any curves beyond those of the second degree; and even their discussion of the properties of these is very incomplete. None of them whatever explain the application of the calculus to the geometry of curves; this part of the science being confined to works upon the calculus. One complete system of geometry, proceeding uniformly upon the most improved algebraic and transcendental analysis, seemed a desideratum in science, to supply which has been attempted in the following treatise.

PART THE FIRST.

THE GEOMETRY OF PLANE CURVES.

CORRIGENDA.

Page 6, line 12 and 14, *for* CD, *read* CB

27, 21, *for* $-c$. *read* $-c'$.

29, 3, *for* $by' - x$, *read* $by'x$

—, 4, *for* yx', *read* bx'

30, 15, *for* B', *read* B''

—, 22, et seq., *for* $=$, *read* $=-$

31, 1, *for* $B^{(n)3}$, *read* $B^{(n)2}$

32, 26, *for* $sx' - x''$, *read* $s(x' - x'')$

42, 2 from bottom, *for* 4AE, *read* 4AF

45, 8, *for* R^2, *read* R

47, 3 from bottom, *for* >0, *read* <0

57, 12, *for* (99), *read* (100)

58, 6 from bottom, *for* B, *read* C

60, 14, *for* XX′, *read* YY′

—, last, *for* B, *read* D

77, 11, *for* $\frac{1}{-}$, *read* $\frac{1}{e^2}$

81, last, *for* ex'^2, *read* $e^2x'^2$

82, 21, *for* x, *read* x'

90, 23, *for* $\phi - \omega$, *read* $\omega - \phi$

91, 7, 10, 13, *for* $=-$, *read* $=$

—, 9, *for* P^2, *read* B^2

93, 13, *for* (203), *read* (204)

—, 5 from bottom, *for* Acy', *read* A^2cy'

99, 12, *for* point, *read* part

100, 17, 19, *for* p, *read* p'

107, 3 from bottom, *for* as the, *read* as the squares of the

115, 19, *for* (167), *read* (92)

126, 6 from bottom, *for* $z' - z'$, *read* $z' - z$

141, 5, *for* : R^2, *read* : R

144, 4 from bottom, *for* 4CF, *read* 4AF

—, 3, *for* $-$, *read* $+$

149, 13, *dele* $-$

157, 13, *for* r, *read* z

165, 16, *for* $(y' - y)dx + (x' - x)dy$, *read* $(y' - y)dy + (x' - x)dx$

168, 2, *for* AB have a limit, but AC, *read* AC have a limit, but AB

172, last, *for* $n + 1$, *read* $n - 1$

224, 22, *for* $4ry$, *read* $8ry$

229, 1, 2 from bottom, *for* C, *read* d

230, 5, *for* C, *read* d

231, 6, *for* θ, *read* ϕ

252, 4, *for* AP, *read* AY

254, 5 from bottom, *for* point of contact, *read* origin

266, 1, *for* $Tx + V = 0$, *read* $\frac{Tx}{n} + V$

271, 7, *dele* of *a*

311, 2 from bottom, *for* nth, *read* $\frac{n \cdot n-1}{1 \cdot 2}$th

334, cut, *for* EPB, *read* EP′B

351, 3 from bottom, *for* AD $=-$, *read* AD′ $=-$

355, 13, *for* B′D′, *read* BD′

390, 13, *for* ED, *read* EA

392, 3 from bottom, *for* e^4, *read* e^2

393, 11, *for* 0, *read* θ

408, 15, 21, *for* m, *read* p

417, cut, *for* AB′C and AF′V, *read* ABC and AFV.

A

TREATISE

ON

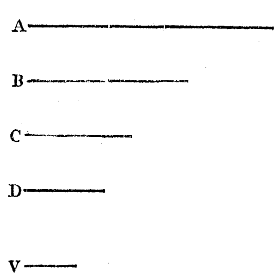
ALGEBRAIC GEOMETRY.

SECTION I.

Of the connection between indeterminate geometrical questions, and algebraical equations between two variables.

(1). The object of *Algebraic Geometry* is the investigation and analysis of the figures and properties of geometrical magnitudes, by means of the symbols and operations of Algebra.

No *necessary connection* subsists between the notation of Algebra and the ideas required to be expressed in geometrical investigation. Some conventional connection must therefore be established between these sciences, in order that the magnitudes and figures contemplated in the one may find corresponding expressions in the symbolical language of the other.



Let several finite right lines, A, B, C, D, be related to any right line V, in the same manner as the algebraical symbols, *a*, *b*, *c*, *d*, are related to unity. The symbols, *a*, *b*, *c*, *d*, are then said to express the right lines A, B, C, D.

The square of the right line V, bears to the rectangle under, B and C, the same relation as unity bears to the product bc.

The rectangle under two lines is therefore expressed by the product of the symbols which express those lines.

In like manner, the square of any symbol represents the square of that line which the symbol expresses.

If A : B :: C : D, and that A, B, and C be expressed by a, b, c, then D will be expressed by $\frac{bc}{a}$.

In like manner, all geometrical relations find representatives in algebraical symbols.

When the expression A $= a$ or B $= b$ is used, the meaning is that a or b is the algebraical expression for the line A or B.

(2.) Having thus established a connection between the language of algebra and the magnitudes, which are contemplated in geometry, either may be conceived to represent the other. That is, a geometrical question can be expressed algebraically, by translating its conditions into algebraical notation, and, *vice versâ*, an algebraical question may be expressed geometrically, by using geometrical magnitudes as representatives of the algebraical symbols.

An example will illustrate this.

A geometrical problem reduced to an algebraical question.

To cut a line (AB), so that the rectangle under the whole line (AB), and one part (BC), shall equal the square of the other part (AC).

Let AB $= a$, AC $= x$, and $\because$ AB $-$ AC $= a - x$.

By the conditions of the question, $a(a - x) = x^2 \because x^2 + ax = a^2$; thus the question becomes an algebraical one, *scil.* to find the roots of $x^2 + ax - a^2 = 0$.

An algebraical question reduced to a geometrical problem.

To find the roots of the equation $x^2 + ax = a^2$.

By transposition $a(a - x) = x^2$. Let $a =$ AB, and AC $= x$ ∴ BC $= a - x$ ∴ rectangle under AB and BC must be equal to the square of AC. Hence the question is reduced to the geometrical problem, *to cut a line so that the rectangle under the whole line, and one part, shall equal the square of the other.*

(3.) It is therefore apparent, that geometrical problems, which relate to mere *magnitude*, without involving the ideas of *figure* or *position*, may with great facility be expressed by the notation of algebra. And that, on the other hand, algebraical questions can with equal facility be represented by geometrical quantities, in which nothing is considered but mere magnitude. But in order to institute a connexion between those sciences, and to bring each under the dominion of the other, much more is necessary. *Figure* and *position* are affections of magnitude, in which the geometer finds objects of investigation much more extensive and interesting than magnitude, considered merely with respect to quantity, could supply. It is, therefore, expedient to establish some principles by which *figure* and *position*, as well as *magnitude*, can be expressed algebraically.

(4.) A method of representing the *figure* of a line by an equation is furnished by a striking analogy, which subsists between indeterminate geometrical problems and equations in which there are two unknown quantities.

In a geometrical problem, whose data are insufficient for its solution, the point which is sought cannot be determined, but yet its position may be considerably restricted; for the conditions which are not sufficient to fix the exact place of the point, may yet be sufficient to confine it, as to position, within certain limits. That is to say, though an indefinite number of positions may be assigned to the sought point, which will all equally fulfil the conditions proposed, yet positions might be assigned which would not fulfil those conditions. The space on which those points are placed, which fulfil the conditions of the question, is called the locus of the sought point. If the conditions require the sought point to be always in a given plane, the locus is usually some line on that plane, the figure and properties of which depend on the conditions of the question. If the point be not restricted to a given plane, the locus is commonly a surface. A very familiar example will illustrate this. Let it be required, *To find a point in a given plane, whose distance from a given point is given.* An indefinite number of points will equally fulfil the conditions of the problem, but yet *all* points will not. The first condition excludes every point of space *except those situate on the given plane.* The second excludes all points on the plane, except *those situate at the intersection of the plane, with a sphere, whose radius equals the given distance, and whose centre is at the given point.* If the first condition were removed, and the second retained, the locus would be the surface of the sphere; and if the second were removed, and the first retained, the locus would be the given plane.

Every line described upon a plane may be considered as the locus of a point, restricted by certain conditions which have a necessary connexion with the nature of the line.

(5.) Analogous to this, in an equation containing two unknown quantities, neither can be absolutely determined. A great diversity of values can be assigned to the symbols representing them, which will all fulfil the conditions of the equation. The symbols, expressive of the unknown quantities, thus capable of receiving different values, are thence called *variables,* in opposition to the other symbols involved in the equation, which are called *constants*, because their values are supposed to remain the same through all the changes which the *variables* undergo. Any value being assigned to either *variable*, a corresponding value of the other will necessarily result, and thus each *variable* is susceptible of such a series of values as render the corresponding values of the other possible. Therefore, though each *variable* cannot be absolutely determined, yet certain limits and restrictions may be assigned to its variation, and those are deducible from the conditions expressed in the equation, just in the same manner as in an indeterminate geometrical problem the position of the sought point restricted, though not absolutely fixed, is deducible from the conditions proposed in the problem.

Thus, for example, in the equation $y = ax$, y and x, the variables are susceptible of an infinite series of values. Their variation is restricted, however, by the condition that x shall vary as y. Again, in the equation $y^2 + x^2 = a^2$, from which results

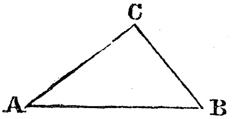
$$y = \sqrt{a^2 - x^2} \qquad x = \sqrt{a^2 - y^2}.$$

The first shows that x is susceptible of all values not exceeding that of a; for any value of x exceeding a would render y impossible. The second equation shows that the values of y are subject to the same restriction.

(6.) The analogy just pointed out originates in this circumstance: *scil.* if an indeterminate geometrical problem be expressed by an algebraical equation, that equation will contain two unknown quantities; and, *vice versâ*, if an equation of two variables be represented geometrically, the result will be an indeterminate problem which will generate a locus.

An indeterminate problem reduced to an equation.

Given the base (AB), and the sum of the sides (AC and BC) of a triangle, to find the vertex (C).



Let AB $= a$, AC $= y$, CB $= x$, and let the excess of the sum of the sides above the base be d.

$$\because y + x = a + d.$$

Any values of y and x, which fulfil the conditions of this equation, represent the sides of a triangle, whose vertex solves the problem.

An equation represented by an indeterminate geometrical problem.

In $y + x = a + d$, to express the values of y and x geometrically. Let $a =$ AB, AC + CB $= a + d$, $\because$ AC and CD represent y and x. That is, describe the locus of the vertex of a triangle, whose base AB $= a$, and the sum of whose sides AC + CD $= a + d$, and then the sides of any triangle on the given base, and whose vertex is placed on the locus described, will be representatives of y and x in the equation $y + x = a + d$.

Since an equation of two variables can be represented by an indeterminate problem, from which a locus may be deduced, the figure of which depends on the conditions of the problem proposed, and therefore on the equation from which the problem results, an equation may, therefore, be conceived to represent the figure of a line, that is, the figure or species of the line is deducible from the equation. By this means *figure*, as well as *magnitude*, is expressed algebraically. The equation from which the species of any line is deduced is said to be the equation of that line, and the line is said to be the locus of the equation.

(7.) In both the preceding examples the process is partly arbitrary, and at the discretion of the analyst. In the first, the sides of the triangle were represented by the variables. These might, however, have been made the representatives of other lines, as the perpendicular and either segment, either side, and the cosine of the angle it forms with the base, or any trigonometrical function of that angle, or trigonometrical functions of the angles at the base, or, in fine, any two quantities, either of which being given would, with the data of the problem, determine the vertex. Hence, in representing an indeterminate problem by an equation, "any quantity, which, being given, would have rendered the problem determinate, may be represented by a variable." Subject to this restriction, the choice of quantities to be represented by variables is arbitrary.

(8.) The form of the equation of a given locus depends on the quantities selected as variables. If, in the example given, the variables represented the perpendicular and either segment, the equation would have been of the second degree; if one of the sides and cosine of the angle, at which it is inclined to the base, had been selected, the equation would also have been of the second degree, but still different from the last.

From these observations it appears that,

1st, "Any line, being considered as the locus of a point, restricted in its position with respect to some fixed points or lines by given conditions, may be expressed by an equation."

2d, "The form of the equation, expressing any given line, depends on the quantities represented by the variables."

(9.) In the second example, the geometrical quantities, selected to represent the algebraical symbols, of which the equation is composed, are arbitrary. Thus, instead of being represented by the sides of the triangle, they might have been represented by the perpendicular and segment, or in any other manner whatever. But on the manner of representing them depends the nature of the line which the equation generates. Thus, had they been represented by the perpendicular and segment, the locus would have been a right line. Hence it appears, 1st, "Any equation between two variables may be conceived to generate a line which is called the locus of the equation."

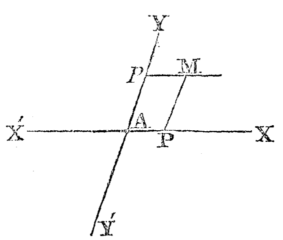
2d, "The species of the line which a given equation generates depends on the manner in which the symbols in that equation are geometrically represented."

SECTION II.

Of the manner of representing equations between two variables by relation to axes of co-ordinates.

(10.) In the investigation of the loci of equations, prosecuted in the following part of this work, the method most usually adopted, of representing geometrically the symbols composing those equations, is as follows:

Let y and x be the variable symbols in any equation.



Two indefinite right lines, (YY′) and (XX′), being assumed in a given plane, intersecting at a given point (A) at a given angle, are called *axes of co-ordinates*. Every system of values of the variables (y and x) resulting from the equation, are represented by portions (Ap and AP) of those axes, measured from the point (A) of their intersection. Through the extremities (P, p) of those values, parallels (PM, pM) to the axes of co-ordinates are drawn, the intersection (M) of which is the point of the locus corresponding to the assumed system of values (Ap, AP) of the variables y and x; and in the same manner all points of the locus are determined.

In order to make a geometrical distinction between the positive and negative values of the variables, they are measured from the point (A) of intersection of the axes in opposite directions. Thus, if the positive values be taken towards Y and X, the negative are taken towards Y' and X'.

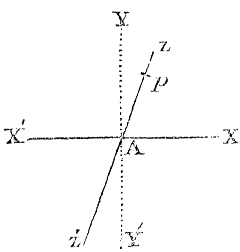
Any system of values of the variables are called *the co-ordinates* of that point whose position they determine.

The point of intersection (A) of the *axes of co-ordinates* is called the *origin of co-ordinates*.

Suppose the positive values of y measured from A towards Y, and those of x from A towards X, then,

$+y$,	$+x$	characterises a point in the angle	XAY
$+y$,	$-x$	- - -	X'AY
$-y$,	$-x$	- - -	X'AY'
$-y$,	$+x$	- - -	XAY'
$y=0$,	$+x$	- - on the line	AX
$y=0$,	$-x$	- - -	AX'
$+y$,	$x=0$	- - -	AY
$-y$,	$x=0$	- - -	AY'
$y=0$,	$x=0$	- - - the origin	A

Particular values of the variables y, x, are distinguished usually by traits, thus, $y'x'$, $y''x''$, &c. and the points distinguished by those values are denominated the points $y'x'$, $y''x''$, &c. A point on YY' is expressed $y'o$, and on XX', $x'o$.



(11.) Another method of representing equations geometrically is also occasionally used. In the preceding method, let the origin (A) and one only (XX') of the axes be given in position. Let the other axis (ZZ') be inclined to it at a variable angle. Let each system of values of the variables be thus represented: let one of the variables (x) be expressed by any trigonometrical function of the angle (ZAX) at which the axes of co-ordinates are inclined, and the other (y) as before, by a portion (Ap) measured along the axis (ZZ$'$), whose position is variable. The extremity (p) of this portion (Ap) is the corresponding point of the locus.

The value of that variable which is represented by the distance of the point in the locus from the origin, is called the *radius vector*. The origin is called the *pole of the equation.*

An equation represented thus is called a *polar equation;* and for distinction the variables represented by the radius vector is called z, and the variable angle by ω. The equation is thus expressed, $z = \text{F}(\omega)$.

Particular values of z and ω are in this case also usually distinguished by traits, thus, $z'\omega'$, $z''\omega''$, &c., and thus characterised are called the points $z'\omega'$, $z''\omega''$, &c.

(12.) As the angles which the axes of co-ordinates form with each other, and with lines which intersect them, and also the angles which lines in general form with each other, become frequently objects of investigation, it is expedient to adopt a concise and clear notation to express them.

The *angle of ordination* is expressed thus, - yx
The angle under radius vector and fixed axis - ω
The angle under any line and an axis of co-ordinates - - - - lx, ly
The angle under two lines - - - ll'
Thus, sin. yx is sine of ordination.
Sin. ll' = sin. of the angle under the lines thus denominated.

All angles are supposed to be measured in the same direction.

(13.) Equations are classed according to their degrees. The degree of an equation is estimated by the number expressing the highest dimension of the variables which enter it. Thus an equation, in which single dimensions only occur, is called an equation of the *first degree.* One, in which the variables enter in dimensions not exceeding two, is called an equation of the *second degree*, &c.

A general equation of any degree is one which embraces within its extension every particular equation of the same degree. Such a formula must necessarily consist of a series of terms, including every dimension and combination of the variables not exceeding the proposed degree, and an absolute term, which for symmetry might be conceived to be involved with dimensions of the variables, whose index is cypher. In this formula each term must include a literal coefficient, expressive, in general, of any value, > 0, < 0, or $= 0$. Thus, the general equation of the *first degree* is,

$$\text{A}y + \text{B}x + \text{C} = 0.$$

That of the *second degree*,

$$\text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0, \text{ \&c. \&c.}$$

And in all such formulæ the symbols A, B, C, &c. are each understood to represent quantities, > 0, < 0, or $= 0$, as the case may be in particular instances.

The loci of equations are investigated according to their degrees, beginning from the first.

The discussion of a general equation is the investigation of its locus, and the effects produced on the locus by the various values and signs which its constant quantities may have in particular cases.

SECTION III.

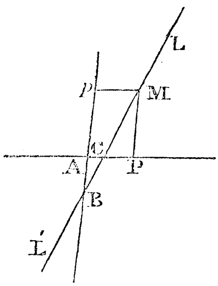
Discussion of the general equation of the first degree.

(14.) Let the fixed axes YY' and XX' be assumed.

In the general equation, $\text{A}y + \text{B}x + \text{C} = 0$, the coefficient of one or other of the variables must be finite; for if $A = 0$, and $B = 0$, $\therefore$ also $C = 0$, and the equation would cease to exist.

Let A represent the finite coefficient, and putting the equation under this form,

$$\frac{y + \frac{C}{A}}{x} = -\frac{B}{A}.$$



Let Ap and AP be any system of values of y and x resulting from this equation. Let $AB = \frac{C}{A}$, $\therefore$ $Bp = y + \frac{C}{A}$ $\therefore$ $\frac{Bp}{AP} = -\frac{B}{A}$. Let the parallels PM and pM be drawn, and since $AP = pM$ $\therefore$ $\frac{Bp}{pM} = -\frac{B}{A}$ $\therefore$ $Bp : pM$ is a constant ratio; and since B is a fixed point, the locus of M must be a right line, L′L.

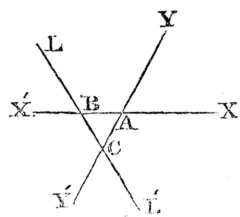
(15.) This right line being designated by the symbol, l, $\frac{\sin. lx}{\sin. ly} = \frac{Bp}{pM} = -\frac{B}{A}$. Hence all equations in which $-\frac{B}{A}$ is the same, represent parallel lines.

(16.) If $B = 0$, $\therefore$ sin. $lx = 0$, $\therefore$ the line LL′ is parallel to XX′, *i. e.* in general, "If the coefficient of either variable $= 0$, the equation is that of a right line parallel to the axis on which the values of that variable would be measured," $\therefore$ $Ay + C = 0$ is the equation of a parallel to the axis XX, and $Bx + C = 0$ a parallel to YY′. In these cases, if $C = 0$, the first, by dividing by A, gives $y = 0$, and the latter, by dividing by B, gives $x = 0$: these are the equations of the axes themselves.

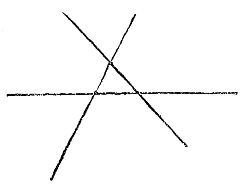
(17.) If neither of the coefficients (A, B) $= 0$, the right line, being parallel to neither axis, meets both. To find the points where it meets YY′, let $x = 0$ in the general equation, $\therefore y = -\frac{C}{A}$, the distance of B from the origin (A). To find where it meets XX′, let $y = 0 \therefore x = -\frac{C}{B}$, the distance AC.

(18.) If $C = 0$, the points C and B coincide with A, $\therefore$ $Ay + Bx = 0$ is the equation of a right line through the origin.

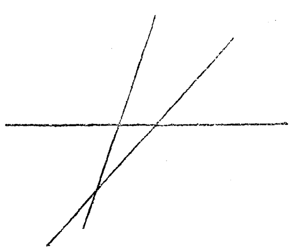
(19.) If A and B have the same sign with C, $\therefore -\frac{C}{A}$ and $-\frac{C}{B}$, are both negative, $\therefore$ the right line intersects both axes of co-ordinates at the negative side of A.

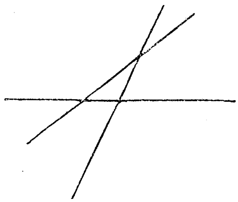


(20.) If A and B have a sign different from C, $\therefore -\frac{C}{A}$ and $-\frac{C}{B}$ are positive, $\therefore$ the right line meets both axes at the positive side of the origin.



(21.) If A and C have a different sign from B, $\therefore -\frac{C}{A}$ is negative, and $-\frac{C}{B}$ is positive, $\therefore$ the right line meets YY′ at the negative and XX′ at the positive side of the origin.





(22.) If B and C have a sign different from A, $\because -\frac{C}{A}$ is positive, and $-\frac{C}{B}$ is negative, $\because$ the right line meets YY' at the positive, and XX' at the negative side of the origin.

SECTION IV.

Of the equations of right lines restricted by certain conditions.

PROP. I.

(23.) *To find the co-ordinates of the point of intersection of two right lines, whose equations are given.*

Let the given equations be

$$Ay + Bx + C = 0,$$
$$A'y + B'x + C' = 0.$$

The point of intersection being that point whose co-ordinates must fulfil the equations of both right lines; let the variables in these equations express them, and the resulting values are,

$$y = -\frac{BC' - B'C}{BA' - B'A}, \quad x = -\frac{AC' - A'C}{AB' - A'B}.$$

(24.) If the numerators of these formulæ be finite, and $AB' - A'B = 0$, the lines are parallel, the point of intersection being supposed infinitely distant. This condition of parallelism was offered before, where it was established that lines are parallel if $\frac{B}{A} = \frac{B'}{A'}$, *i. e.* $BA' - B'A = 0$.

But if at the same time that $AB' - A'B = 0$, also $AC' - A'C = 0$, and $\because BC' - B'C = 0$, the two lines coincide, for then their equations being put under the forms,

$$y + \frac{B}{A}x + \frac{C}{A} = 0,$$

$$y + \frac{B'}{A'}x + \frac{C'}{A'} = 0,$$

are identical, since $\frac{B}{A} = \frac{B'}{A'}$, and $\frac{C}{A} = \frac{C'}{A'}$.

PROP. II.

(25). *To investigate the condition on which three right lines will have a common point of intersection.*

Let the equations of the lines be

$$Ay + Bx + C = 0,$$
$$A'y + B'x + C' = 0,$$
$$A''y + B''x + C'' = 0.$$

By equating either of the co-ordinates of the point of intersection of the first and second, with the corresponding co-ordinate of the intersection of the second and third, there will result, after reduction, the equation,

$$A(B''C' - B'C'') + A'(BC'' - B''C) + A''(B'C - BC') = 0,$$

expressing the required condition.

If any of the lines be parallel to either axis of co-ordinates, the formula must be determined by that variable which is common to the three equations.

PROP. III.

(26). *To find the equation of a right line passing through a given point.*

Let the point be $y'x'$, and the sought equation

$$Ay + Bx + C = 0.$$

Since $y'x'$ is on the right line, $\because$ $Ay' + Bx' + C = 0$, $\because$ by subtraction

$$Ay + Bx - (Ay' + Bx') = 0, \text{ or}$$
$$A(y - y') + B(x - x') = 0.$$

This formula might also be demonstrated thus: the equation must be such as that when $y'x'$ are substituted in it for yx, the whole shall be equal to cypher; hence $\text{C} = -(\text{A}y' + \text{B}x')$.

PROP. IV.

(27.) *To express the equation of a right line passing through two given points.*

Let the co-ordinates of the points be $y'x'$, $y''x''$. By (26) the equation of a line through $y''x''$ is

$$\text{A}(y - y'') + \text{B}(x - x'') = 0.$$

But since this line also passes through $y'x'$, the equation must hold good after substituting $y'x'$ for yx; ∵

$$\text{A}(y' - y'') + \text{B}(x' - x'') = 0.$$

From this and the former, the result is

$$(x' - x'')y - (y' - y'')x + y'x'' - y''x' = 0;$$

$$\text{or, } (x' - x'')(y - y'') - (y' - y'')(x - x'') = 0.$$

PROP. V.

(28.) *To express the equation of a right line making given angles with the axes of co-ordinates.*

Let the given angles be lx, ly. Let the general equation of the right line be divided by A, and it becomes

$$y + \frac{\text{B}}{\text{A}}x + \frac{\text{C}}{\text{A}} = 0.$$

Let $\frac{\text{B}}{\text{A}} = -\frac{\sin. lx}{\sin. ly}$, and $\frac{\text{C}}{\text{A}} \sin. ly = \text{C}'$; ∵ the sought equation is

$$\sin. ly \,.\, y - \sin. lx \,.\, x + \text{C}' = 0.$$

PROP. VI.

(29.) *To express the equation of a right line passing through a given point, and making given angles with the axes of co-ordinates.*

The given point being $y'x'$, and the given angles ly and

lx it follows from (26) and (15) that the sought equation is

$$\sin. ly \, . \, (y - y') - \sin. lx \, (x - x') = 0.$$

PROP. VII.

(30.) *To express the angle under two lines as a function of their equations, and of the angle of ordination.*

Let the equations be

$$Ay + Bx + C = 0,$$
$$A'y + B'x + C' = 0;$$
$$\therefore \frac{\sin. lx}{\sin. ly} = -\frac{B}{A}, \quad \frac{\sin. l'x}{\sin. l'y} = -\frac{B'}{A'}.$$

But, $\quad ly = yx - lx, \quad l'y = yx - l'x;$

$$\therefore \frac{\sin. lx}{\sin. (yx - lx)} = -\frac{B}{A}, \quad \frac{\sin. l'x}{\sin. (yx - l'x)} = -\frac{B'}{A'}.$$

By expanding the denominators, and dividing both numerator and denominator of the first by cos. lx, and of the second by cos. $l'x$, the results solved for tan. lx and tan. $l'x$ are

$$\tan. lx = \frac{B \sin. yx}{B \cos. yx - A}, \quad \tan. l'x = \frac{B' \sin. yx}{B' \cos. yx - A'}.$$

Let the angle under the lines be ll',

$$ll' = (lx - l'x);$$
$$\therefore \tan. ll' = \frac{\tan. lx - \tan. l'x}{1 + \tan. lx \tan. l'x}.$$

Substituting in this formula the values found above,

$$\tan. ll' = \frac{(AB' - A'B) \sin. yx}{AA' + BB' - (AB' + A'B) \cos. yx},$$

which expresses the angle ll' as a function of the two equations, and the angle of ordination.

(31.) *Cor.* 1. If the angle of ordination be right,

$$\cos. yx = 0 \therefore$$
$$\tan. ll' = \frac{AB' - A'B}{AA' + BB'}.$$

(32.) *Cor.* 2. If the angle under the lines be right, tan. ll' is infinite, $\because$

$$AA'+BB'-(AB'+A'B)\cos. yx = 0.$$

(33.) *Cor.* 3. If $ll' = yx$,

$$AA'+BB'-2AB'\cos. yx = 0.$$

(34.) *Cor.* 4. If $ll' = yx = 90°$,

$$AA' + BB' = 0.$$

(35.) *Cor.* 5. If l' coincide with the axis of x, $ll' = lx$; and $B' = 0$, (16.)

$$\tan. lx = \frac{B\sin. yx}{B\cos. yx - A}.$$

(36.) *Cor.* 6. In like manner, if l' coincide with the axis of y, $ll' = ly$; $\because$

$$\tan. ly = \frac{A\sin. yx}{A\cos. yx - B}.$$

PROP. VIII.

(37.) *To find the equation of a right line inclined at a given angle to a given right line.*

Let the given angle be ll', the given right line $Ay + Bx + C = 0$, and the sought line $A'y + B'x + C' = 0$. In the formula

$$\tan. ll' = \frac{(AB'-A'B)\sin. yx}{AA'+BB'-(AB'+A'B)\cos. yx},$$

found in (30), by dividing numerator and denominator by AA', and solving for $\frac{B'}{A'}$, the result is, after reduction,

$$\frac{B'}{A'} = \frac{B\sin.(yx-ll')+A\sin. ll'}{A\sin.(yx+ll')-B\sin. ll'};$$

therefore the sought equation is

$$\{A\sin.(yx+ll')-B\sin. ll'\}y+\{B\sin.(yx-ll')+A\sin. ll'\}x+C'' = 0,$$

where C'' is indeterminate.

(38.) *Cor.* 1. If the angle of ordination be right, sin. $(yx \pm ll') = $ cos. ll'; ∴ in this case the formula becomes
$(\text{A cos. } ll' - \text{B sin. } ll')\, y + (\text{B cos. } ll' + \text{A sin. } ll')\, x + \text{C}'' = 0.$

(39.) *Cor.* 2. *To find the equation of a line perpendicular to a given line.*

In the general formula (37) let $ll' = 90^\circ$, ∴ sin. $(yx \pm ll') = \pm$ cos. yx; ∴ the equation sought is

$$(\text{A cos. } yx - \text{B})\, y - (\text{B cos. } yx - \text{A})\, x + \text{C}'' = 0.$$

(40.) In this case, if the angle of ordination be right, the equation is

$$\text{B}y - \text{A}x - \text{C}'' = 0,$$

which is the equation of a right line perpendicular to a given line, and referred to rectangular co-ordinates.

(41.) *Cor.* 3. *To find the equation of a right line inclined to a given right line at an angle equal to the angle of ordination.*

Let $yx = ll'$, ∴ sin. $(yx - ll') = 0$, and sin. $(yx + ll') = $ sin. $2yx = 2$ sin. yx cos. yx; ∴ the sought equation is

$$(2\text{A cos. } yx - \text{B})\, y + \text{A}x + \text{C}'' = 0.$$

(42.) The formulæ established in the preceding questions may be further modified by subjecting the right lines sought to pass through a given point; the general formula (37) will, in this case, by (26) become

$$\{\text{A sin. } (yx + ll') - \text{B sin. } ll'\}\,(y - y')$$
$$+ \{\text{B sin. } (yx - ll') + \text{A sin. } ll'\}\,(x - x') = 0.$$

(43.) The formula in (40) becomes

$$\text{B}\,(y - y') - \text{A}\,(x - x') = 0.$$

It is clear that (29) is a particular case of (37), and can be deduced from it.

PROP. IX.

(44.) *To express the length of a line joining two points.*

Let the points be yx, $y'x'$, and L the sought length;

$$L = \sqrt{(y-y')^2 + (x-x')^2 + 2(y-y')(x-x')\cos. yx}.$$

If the co-ordinates be rectangular,

$$L = \sqrt{(y-y')^2 + (x-x')^2}.$$

These formulæ are derivable from the common principles of geometry.

PROP. X.

(45.) *To express the intercept of a given right line between two points situate on it.*

Let the right line be $A'y + B'x + C' = 0$, and the points yx, and $y'x'$. By (26) $\frac{y-y'}{x-x'} = -\frac{B'}{A'}$; ∵ since

$$L = (x-x')\sqrt{\frac{(y-y')^2}{(x-x')^2} + 1 + \frac{(2y-y')}{(x-x')}\cos. yx};$$

$$\therefore L = (x-x') \cdot \frac{\sqrt{A'^2 + B'^2 - 2A'B'\cos. yx}}{A'}.$$

PROP. XI.

(46.) *To express the distance between any point on a given line and the point where it intersects another given line.*

Let the lines be $A'y + B'x + C' = 0$, and $Ay + Bx + C = 0$, and let the point given on the first be $y'x'$. In the formula in (45), substituting for x its value for the point of intersection found in (23), the result expresses the sought distance,

$$L = \frac{Ay' + Bx' + C}{AB' - A'B} \sqrt{A'^2 + B'^2 - 2A'B' \cos. yx}.$$

PROP. XII.

(47.) *To express the length of a line drawn from a given point to meet a given right line, and inclined to it in a given angle.*

In the formula of (46) substitute for A' and B' the values for them in the formula found in (37), and the result will be the formula sought; but for brevity, let the substitution be only made in the terms of the denominator, retaining the symbols A', B' under the radical, signifying, however, the values in (37), the result is

$$L = -\frac{Ay' + Bx' + C}{\sin. ll'} \cdot \frac{\sqrt{A'^2 + B'^2 - 2A'B' \cos. yx}}{A^2 + B^2 - 2AB \cos. yx}.$$

(48.) *Cor.* 1. If the co-ordinates be rectangular, the formula is

$$L = -\frac{Ay' + Bx' + C}{\sin. ll' \sqrt{A^2 + B^2}},$$

for $\cos. yx = 0$, and by the values in (38), $A'^2 + B'^2 = A^2 + B^2$.

(49.) *Cor.* 2. *To express the length of a perpendicular drawn from a given point to a given right line.*

In (47) let $\sin. ll' = 1$, ∵

$$L = -(Ay' + Bx' + C) \frac{\sqrt{A'^2 + B'^2 - 2A'B' \cos. yx}}{A^2 + B^2 - 2AB \cos. yx}.$$

(50.) *Cor.* 3. If the co-ordinates be rectangular, also

$$L = -\frac{Ay' + Bx' + C}{\sqrt{A^2 + B^2}}.$$

(51.) *Cor.* 4. *To express the length of a line drawn from a given point to meet a given line, and inclined to it at an angle equal to the angle of ordination.*

In (47) let $ll' = yx$, and let A' and B' have the values in (37) restricted by the condition of $ll' = yx$,

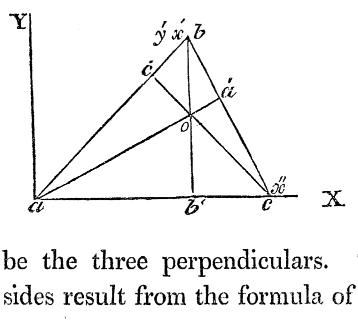
$$L = -\frac{Ay' + Bx' + C}{\sqrt{A^2 + B^2 - 2AB \cos. yx}}.$$

SECTION V.

Propositions calculated for exercise in the application of the equations of the first degree.

PROP. XIII.

(52.) *To investigate the intersection of the three perpendiculars from the angles of a triangle on the opposite sides.*



Let the base, ac, of the triangle be taken as axis of x, and a perpendicular aY through it as axis of y; let the co ordinates of b be $x'y'$, those of c, x'', $y'' = 0$. Let aa', bb', cc', be the three perpendiculars. The equations of the three sides result from the formula of (27). Hence, The equation of ab is - $x'y - y'x = 0.$

The equation of ac is - - $y = 0.$

The equation of bc is $(x' - x'')y - y'(x - x'') = 0.$

Hence, those of the three perpendiculars result from the formula (43):

The equation of aa' is $yy' - (x'' - x')x = 0.$

The equation of bb' is - $x - x' = 0.$

The equation of cc' is $y'y + x'(x - x'') = 0.$

Eliminating y from the first and third, the value of x for the point of intersection is x', and this value being substituted in either, we find the co-ordinates of the point, 0, of intersection of aa' and cc',

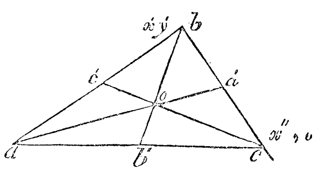
$$\text{Y} = \frac{(x'' - x')x'}{y'}, \quad \text{X} = x'.$$

(53.) *Cor.* Hence, it follows that the three perpendiculars intersect in the same point; for since the values of x for the points b and 0 are the same, the same perpendicular, bb', must pass through both.

PROP. XIV.

(54.) *To investigate the point of intersection of the bisectors of the three sides of a triangle drawn through the several vertices.*

The axes of co-ordinates being as before, and aa', bb', cc', being the bisectors, and the point b being $x'y'$, and c, $x''0$;



The co-ordinates of point a' are $\frac{y'}{2}$, $\frac{x''+x'}{2}$.

The co-ordinates of point b' are 0, $\frac{x''}{2}$.

The co-ordinates of point c' are $\frac{y'}{2}$, $\frac{x'}{2}$.

Hence, by formula (27),

The equation of aa' is $(x' + x'')y - y'x = 0$.

The equation of bb' is $(2x' - x'')y - y'(2x - x'') = 0$.

The equation of cc' is $(x' - 2x'')y - y'(x - x'') = 0$.

The values for y and x found from first and second are;

$$Y' = \frac{y'}{3}, \qquad X' = \frac{x' + x''}{3}.$$

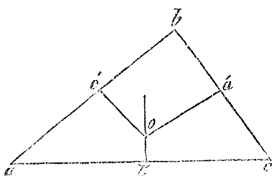
The same values being found from the second and third, proves that the three bisectors meet in this point.

It is obvious from the proportion of Y' to y', that each bisector is divided at their common point of intersection in the ratio of 1 : 2.

(55.) *Cor.* From the principles of *Mechanics*, it is obvious that this point is the *centre of gravity* of the triangle.

PROP. XV.

(56.) *To investigate the point of intersection of perpendiculars to the three sides of a triangle, drawn through their several points of bisection.*



The axes of co-ordinates being as before, the equations of the three perpendiculars result from the equations of the sides expressed in (52), and of the co-ordinates of the points a', b', c', in (54), by the formula (43). Hence,

The equation of $c'o$ is $y'(y - \frac{y'}{2}) + x'(x - \frac{x'}{2}) = 0$.

The equation of $a'o$ is

$$y'(y - \frac{y'}{2}) + (x' - x'')(x - \frac{x' + x''}{2}) = 0.$$

The equation of $b'o$ is $x - \frac{x''}{2} = 0$.

By the first and second equations we find the co-ordinates of the point of intersection of $c'o$ and $a'o$ to be,

$$Y'' = \frac{y'^2 + x'^2 - x'x''}{2y'}, \quad X'' = \frac{x''}{2}.$$

Hence, since the same values result from the second and third, it appears that the three perpendiculars meet in this point.

(57.) *Cor.* The distance R of the point o from each of the angles of the triangle may be hence found,

$$R^2 = Y''^2 + X''^2 = \frac{(y'^2 + x'^2 - x'x'')^2 + y'^2 x''^2}{4y'^2}.$$

Let $ab = c$, $bc = c'$, and $ac = c''$. Hence, $y'^2 + x'^2 = c^2$, $x'' = c''$, also, $c^2 + c''^2 = c'^2 + 2c''x'$, $\because x' = \dfrac{c^2 + c''^2 - c'^2}{2c''}$.

By substituting for $y'^2 + x'^2$, its value, and changing x'' into c'', we have

$$R^2 = \frac{(c^2 - c''x')^2 + y'^2 c''^2}{4y'^2}.$$

And since $2c''x' = c^2 + c''^2 - c'^2$, $\because c^2 - c''x' = \dfrac{c^2 + c'^2 - c''^2}{2}$.

Also, $y'^2 = c^2 - x'^2 = c^2 - \dfrac{(c^2 + c''^2 - c'^2)^2}{4c''^2}$. Making these substitutions, and observing that $(c^2 + c'^2 - c''^2)^2 - (c^2 + c''^2 - c'^2)^2 = 4c^2(c'^2 - c''^2)$, we find that

$$R^2 = \frac{c^2 c'^2}{4y'^2}, \quad \because R = \frac{cc'}{2y'} = \frac{cc'c''}{2y'c''}.$$

Let the area of triangle be A, $\because y'c'' = 2A$, hence

$$R = \frac{cc'c''}{4A}.$$

This expression being symmetrical with respect to the three sides, must be the same for each of the three vertices, and therefore the distances of the point o from the three angles are equal. Hence it appears also, that the point o is the centre of the circumscribed circle, and the value of R is its radius, expressed as a function of the three sides.

The equation $R = \frac{cc'}{2y'}$, which gives $2Ry' = cc'$, proves that the rectangle, under any two sides of a triangle, is equal to the rectangle under the perpendicular on the third side, and the diameter of the circumscribed circle.

If $Y'' = 0$, $c^2 = c''x'$, and hence $c^2 + c'^2 = c''^2$, ∴, therefore, the angle is a right angle; hence the angle in a semicircle is right.

If $Y'' > 0$, $c^2 > c''x'$, ∴ $c^2 + c'^2 > c''^2$, ∴ the angle b is acute, and ∴ the angle in a segment greater than a semicircle is acute.

If $Y'' < 0$, $c^2 < c''x'$, ∴ $c^2 + c'^2 < c''^2$, ∴ the angle b is obtuse; and, therefore, the angle in a segment less than a semicircle is obtuse.

PROP. XVI.

(58.) *The three points of intersection,* 1^0, *of the perpendiculars from the angles of a triangle on the opposite sides;* 2^0, *of the bisectors of the sides;* 3^0, *of the perpendiculars drawn through the points of bisection of the sides, will be in the same right line.*

The equation of a right line through the points YX and Y'X' is,

$$(y - Y')(X - X') - (Y - Y')(x - X') = 0.$$

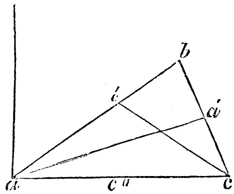
Substituting, in this form, the values already found, it becomes, after reduction,

$$y'(3y - y')(2x' - x'') - \{3(x'' - x')x' - y'^2\}(3x - x' - x'') = 0.$$

The values for Y'' and X'' being substituted for y and x in this equation, will fulfil the conditions, and therefore the right line joining the points YX and Y'X' must pass through Y''X''.

PROP. XVII.

(59.) *To investigate the intersection of the bisectors of the three angles of a triangle.*



The axes of co-ordinates being placed as before, let aa' and cc' bisect the angle a and c respectively.

The equation of aa' is,

$$y - \text{tang.}\ \tfrac{1}{2}\, a \,.\, x = 0.$$

The equation of cc' is, $y + \text{tang.}\ \frac{1}{2}\, c \,.\, (x - c'') = 0.$

But since $\text{tang.}\ \frac{1}{2} = \frac{\sin.\ a}{1 + \cos.\ a}$, and $\text{tang.}\ \frac{1}{2}\, c = \frac{\sin.\ c}{1 + \cos.\ c}$,

and $\sin.\ a = \frac{y'}{c}$, $\cos.\ a = \frac{x'}{c}$, $\sin.\ c = \frac{y'}{c'}$, $\cos. c = \frac{c'' - x'}{c'}$,

the equations, by these substitutions, become,

$$(c + x')y - y'x = 0,$$
$$(c' + c'' - x')y + y'(x - c'') = 0.$$

From the two equations, the co-ordinates of the point of intersection are,

$$\text{Y}''' = \frac{y'c''}{c + c' + c''},\quad \text{X}''' = \frac{c''(c + x')}{c + c' + c''}.$$

But $y'c'' = 2\text{A}$, $\therefore \text{Y}''' = \frac{2\text{A}}{c + c' + c''}$. also

$$2c''x' = c^2 + c''^2 - c'^2 \therefore 2c''(c + x') = (c + c'')^2 - c'^2 =$$
$$(c + c' + c'')\,(c + c'' - c') \therefore \text{X}''' = \frac{c + c'' - c'}{2} = \frac{c + c' + c''}{2} - c.$$

The values of X''' being symmetrical with respect to the sides, will be the same, whichever side is assumed as axis of x; hence it follows, that the three bisectors meet at the same point, and that the perpendicular distances of that point, from the sides of the triangle, are equal. Hence, also, that point is the centre of the inscribed circle; and the value of Y''' expresses the radius of that circle as a function of the sides of the triangle.

PROP. XVIII.

(60.) *To find the locus of a point from which two right lines, drawn at given angles, to two lines given in position, shall have a given ratio.*

Let the equations of the two right lines given, in position referred to rectangular co-ordinates, be,

$$\text{A}y + \text{B}x + \text{C} = 0, \ \text{A}'y + \text{B}'x + \text{C}' = 0.$$

The given angles being ϕ and ϕ', the point, whose locus is sought, being yx; let $\frac{\text{L}}{\text{L}'} = \frac{m}{m'}$. But by (48)

$$\text{L} = -\frac{\text{A}y + \text{B}x + \text{C}}{\sin.\phi\sqrt{\text{A}^2 + \text{B}^2}},$$

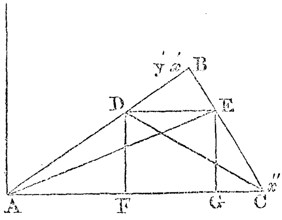
$$\text{L}' = -\frac{\text{A}'y + \text{B}'x + \text{C}'}{\sin.\phi'\sqrt{\text{A}'^2 + \text{B}'^2}},$$

$$\therefore (\text{A}y + \text{B}x + \text{C})\sqrt{\text{A}'^2 + \text{B}'^2} \,.\, m'\sin.\phi' = (\text{A}'y + \text{B}'x + \text{C}')$$
$$\sqrt{\text{A}^2 + \text{B}^2} \,.\, m\sin.\phi n,$$

which being a simple equation, the locus is a right line.

PROP. XIX.

(61.) *A parallel to the base of a triangle being drawn, and its points of intersection being connected with the extremities of the base, to find the locus of the intersection of the connecting lines.*



Let AC be the axis of x, and that of y perpendicular to it; also, let the perpendicular distance of the points D and E, from the base, be b. Since the equation of AB is $yx' - y'x = 0$, and D is a point on it, the value of x for the point D is $\frac{bx'}{y'}$. The equation of BC being

$y(x'-x'')-y'(x-x'')=0$, and the point E being on the line, the value of x for the point E is $\frac{b(x'-x'')+y'x''}{y'}$. Hence,

The equation of AE is, $y=\frac{by'-x}{b(x'-x'')+y'x''}$.

The equation of CD is, $y=\frac{by'}{yx'-y'x''}(x-x'')$.

Eliminating b from these equations, we find the equation of the locus of the point of intersection,

$$y(2x'-x'')-2y'x+y'x''=0, \text{ or}$$

$$y(x'-\frac{x''}{2})-y'x+y'\frac{x''}{2}=0,$$

which by (27) is the equation of a right line passing through the points $y'x'$, and $\frac{x''}{2}$, 0: hence, the locus sought is a right line, bisecting the base, and passing through the vertex.

PROP. XX.

(62.) *A parallel being drawn, as before, to find the locus of the intersection of perpendiculars to the sides through its extremities.*

The co-ordinates of the points D and E, being expressed as above, and the equations of the sides, as in (52) the equation of the perpendicular through D is,

$$y'(y-b)+x'(x-\frac{bx'}{y'})=0.$$

The equation of perpendicular through E is,

$$y'(y-b)+(x'-x'')\left(x-\frac{b(x'-x'')+y'x''}{y'}\right)=0.$$

Eliminating b from these equations, the result is,

$$(y'y+x'x)\{y'^2+(x'-x'')^2\}=\{y'y+(x'-x'')(x-x'')\}(y'^2+x'^2)$$

But,

$$y'^2 + (x' - x'')^2 = c'^2,\ y'^2 + x'^2 = c^2,$$
$$x' = \frac{c^2 + c''^2 - c'^2}{2c''}\ .\ c''y' = 2\text{A},$$

A expressing the area. Making these substitutions, the equation, after reduction, becomes,

$$4\text{A}y(c'^2 - c^2) + x\{c''^2(c'^2 + c^2) - (c'^2 - c^2)^2\} + c''c^2(c^2 - c''^2 - c'^2) = 0,$$

which, being an equation of the first degree, shows the locus sought to be a right line.

PROP. XXI.

(63.) *To find the locus of a point from which the sum of the perpendiculars, drawn to several right lines given in position, may have a given magnitude.*

Let the equations of the right lines given in position be,

$$\text{A}y + \text{B}x + \text{C} = 0,$$
$$\text{A}'y + \text{B}'x + \text{C}' = 0,$$
$$\text{A}''y + \text{B}''x + \text{C}'' = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\text{A}^{(n)}y + \text{B}^{(n)}x + \text{C}^{(n)} = 0.$$

The co-ordinates of the point, whose locus is sought, being expressed by the general symbols yx, the perpendiculars respectively are,

$$\text{P} = \frac{\text{A}y + \text{B}x + \text{C}}{\sqrt{\text{A}^2 + \text{B}^2}},$$
$$\text{P}' = \frac{\text{A}'y + \text{B}'x + \text{C}'}{\sqrt{\text{A}'^2 + \text{B}'^2}},$$
$$\text{P}'' = \frac{\text{A}''y + \text{B}''x + \text{C}''}{\sqrt{\text{A}''^2 + \text{B}''^2}},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\text{P}^{(n)} = \frac{\text{A}^{(n)}y + \text{B}x^{(n)} + \text{C}^{(n)}}{\sqrt{\text{A}^{(n)2} + \text{B}^{(n)2}}};$$

the conditions of the question give the equation,

$$\left\{\frac{A}{\sqrt{A^2+B^2}}+\frac{A'}{\sqrt{A'^2+B'^2}}\cdots+\frac{A^{(n)}}{\sqrt{A^{(n)2}+B^{(n)3}}}\right\}y+$$
$$\left\{\frac{B}{\sqrt{A^2+B^2}}+\frac{B'}{\sqrt{A'^2+B'^2}}\cdots+\frac{B^{(n)}}{\sqrt{A^{(n)2}+B^{(n)2}}}\right\}x+$$
$$\frac{C}{\sqrt{A^2+B^2}}+\frac{C'}{\sqrt{A'^2+B'^2}}+\cdots=M,$$
which being a simple equation, shows the locus to be a right line.

PROP. XXII.

(64.) *To express the area of a polygon as a function of the equations of the sides, and the co-ordinates of a point within it.*

Let the equations of the sides be expressed as in the last prop. By the formula (27) it appears that $y'x'$, $y''x''$, being the co-ordinates of the extremities of the first side, $A=x'-x''$, $B=-(y'-y'')$. Hence, $\sqrt{A^2+B^2}=\sqrt{(y'-y'')^2+(x'-x'')^2}=c$, c being the first side of the polygon; and for the same reason, the several denominators of the values of P, P', P'', &c. are the successive sides c, c', c'', c''', &c.

Let the figure be supposed to be resolved into triangles, by lines drawn from the point within it to the several angles, A being the area

$2A = Pc + P'c' + P''c'' \ldots\ldots P^{(n)}c^{(n)}$. (1) ∵

$2A = (Ay' + Bx' + C) + (A'y' + B'x' + C') \ldots\ldots\ldots$
$(A^{(n)}y' + B^{(n)}x' + C^{(n)})$, (2),

which is the required function, $y'x'$ being the co-ordinates of the point within the polygon. If the figure be a regular polygon, of which c is the side, (by equat. 1), we have $P + P' + P'' \ldots\ldots P^{(n)} = \frac{2A}{c}$. This value being independent of $y'x'$.

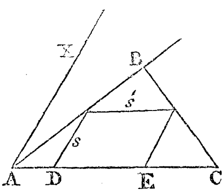
(65.) *Cor.* It follows that, in a regular polygon, the sum of the perpendiculars on the sides from any point within it, is of a certain magnitude. If, at the same time, the perpendiculars are equal to each other,

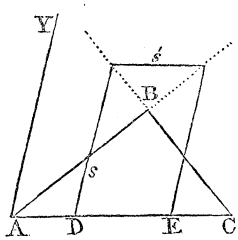
$$\text{P} = \frac{2\text{A}}{nc},$$

which is an expression for the radius of the circle inscribed in a polygon, whose side is c, and whose number of sides is n.

PROP. XXIII.

(66.) *To inscribe in a triangle a parallelogram, whose sides shall have a given ratio.*





Let ABC be any given triangle; let AC be assumed as axis of x, and AY making the angle YAC equal the angle of the proposed parallelogram. The co-ordinates of B being $x'y'$, those of C, $x''o$, the equations of AB and BC are expressed as in (52.) Let s and s' be the sides of the proposed parallelogram; and by the terms of the question, $\frac{s}{s'} = \frac{m}{n}$. Substituting in the equation of AB s for y, we find x, or $\text{AD} = \frac{sx'}{y'}$; and, in like manner, substituting s for y in the equation of BC, we find

$$x, \text{ or AE} = \frac{sx' - x''}{y'} + x''.$$

If the parallelogram be situate as in the first figure, $s' = \text{AE} - \text{AD}$ ∴ $s' = \frac{(y' - s)x''}{y'}$, and this combined with the equation $s' = \frac{sn}{m}$ gives $s = \frac{my'x''}{ny' + mx''}$. But if the parallelogram be inscribed as in the second figure,

$\because s' = \frac{(s - y')x''}{y'}$, and, therefore, $s = \frac{my'x''}{ny' - mx''}$. Hence, in general, $s = \frac{my'x''}{ny' \pm mx''}$, according as the side of the parallelogram parallel to the base lies above or below the vertex.

Hence, there may be two parallelograms inscribed, which will equally fulfil the conditions of the question.

If $m = n$, $s = \frac{y'x''}{y' \pm x''}$.

If $m = n$, and the angle of the parallelogram be right, the formula solves the question, to find the side of a square inscribed in a triangle. In this case y' is the altitude, and $y'x'' = 2\text{A}$, $\therefore s = \frac{2\text{A}}{y' \pm x''}$.

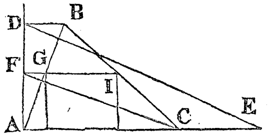
Hence two squares may be inscribed on each side of a triangle, except when the side and perpendicular on it are equal: in that case, the lower sign renders s infinite; and the other value of s, half the side on which the square stands.

(67.) *Cor.* 1. The sides of squares inscribed on the sides of the same triangle, are inversely as the sum of each side, and the perpendicular on it.

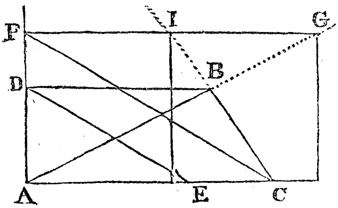
(68.) *Cor.* 2. The formula $\frac{y'x''}{y' \pm x''}$, points out a geometrical construction for the inscription of a square, by the equation being expressed as a proportion,

$$y' \pm x'' : y' :: x'' : s.$$

If the upper sign be taken through B, let BD be drawn parallel to AC, and take CE = AD and join DE, and through C draw CF parallel to ED, and through E let a parallel



to AC be drawn, and GI will be the side of the inscribed square.



If the lower sign be taken, take CE upon CA and equal to AD, and draw ED, and parallel to it draw CF. The parallel to AC through F will determine GI, the side of the square.

If $y' = x''$, ED coincides with AD, and s is infinite.

PROP. XXIV.

(69.) *To find the equation of a right line, such that the perpendiculars drawn from several given points to it shall have a given magnitude* (M.)

The points being $y'x'$, $y''x''$, $y'''x'''$, ... $y^{(n)}x^{(n)}$, let the sought equation be $Ay + Bx + C = 0$.

By the formula (50), the condition of the question is expressed thus:

$$-\frac{Ay' + Bx' + C}{\sqrt{A^2 + B^2}} - \frac{Ay'' + Bx'' + C}{\sqrt{A^2 + B^2}} \cdots \frac{Ay^{(n)} + Bx^{(n)} + C}{\sqrt{A^2 + B^2}} = M,$$

$$\text{or, } - A\,(y' + y'' + \ldots y^{(n)}) - B\,(x' + x'' + \ldots x^{(n)}) - nC - M\sqrt{A^2 + B^2} = 0.$$

By dividing by n, and eliminating C,

$$A\left(y - \frac{y' + y'' \ldots y^{(n)}}{n}\right) + B\left(x - \frac{x' + x'' + \ldots x^{(n)}}{n}\right) - \frac{M}{n}\sqrt{A^2 + B^2} = 0.$$

$$\text{or } \left(y - \frac{y' + y'' \ldots y^{(n)}}{n}\right) + \tan.\, lx \left(x - \frac{x' + x'' + \ldots x^{(n)}}{n}\right) - \frac{M}{n}\sec.\, lx = 0.$$

As the value of the angle lx still remains undetermined, the line sought cannot be absolutely determined; but its position is limited; for let C be a point, whose co-ordinates are

$$Y = \frac{y' + y'' \ldots y^{(n)}}{n} \quad x = \frac{x' + x'' + \ldots x^{(n)}}{n}.$$

A perpendicular, drawn from this point on the sought line, will be (50) $\frac{M}{n}$. Hence it follows, that if with this point as centre, and a radius $= \frac{M}{n}$, a circle be described, any line drawn, touching this circle, will have the required property.

If the question required, that the sum of all the perpendiculars should be $= 0$, *scil.* that the sum of those on each side of the sought line should be equal, then $\frac{M}{n} = 0$, therefore the circle vanishes into the point C, and any right line drawn through this point would have the required property.

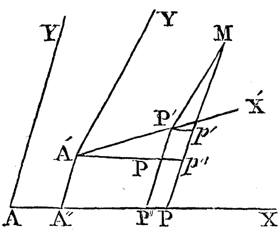
(70.) *Cor.* The point C is manifestly the centre of gravity of a rectilinear figure, formed by joining the given points.

SECTION VI.

Of the transformation of co-ordinates.

(71.) It is frequently desirable to express the equation of the same locus referred to different systems of co-ordinates. This is effected by expressing the values of the co-ordinates of any point related to one system of axes, in terms of the co-ordinates of the same point referred to the other system, and in functions of this position of the two systems of axes with respect to each other. The values thus expressed, being substituted in the equation, related to the one system, give the equation of the same locus referred to the other system. Let yx be the co-ordinates of any point related to one system of axes, and $y'x'$ those of the same point referred to the other system. Let m, n, p, q, a, b, be quantities determined by the mutual position of the axes. Suppose, then,

$y = my' + nx' + a$, and $x = py' + qx' + b$. If these values of y and x be substituted in the equation of any locus related to the axes of y and x, an equation will result between the variables y' and x', *i. e.* one of the same locus related to the other system of axes. The question will, therefore, be resolved when it is shown what functions of the position of the axes the quantities m, n, &c. are.



(72.) Let AY, AX, and A'Y', A'X', be the two systems of axes. Let the co-ordinates of the point M referred to these axes be $y = $ MP, $x = $ AP, $y' = $ MP', $x' = $ A'P'. Draw A'A'' and P'P'' parallel to AY and A'p'', and P'p' parallel to AX. Let AA'' $= x''$, A'A'' $= y''$, $\because y = y'' + p'p'' + p'$M, or $y = y'' + $ P'p $+ p'$M, and $x = x'' + $ A'p $+$ P'p'. Expressing the angles under the respective axis by the notation explained in (12.)

$$\text{P}'p = \frac{\sin. x'x}{\sin. yx} \cdot x', \qquad p'\text{M} = \frac{\sin. y'x}{\sin. yx} \cdot y',$$

$$\text{A}'p = \frac{\sin. yx'}{\sin. yx} x', \qquad \text{P}'p' = \frac{\sin. y'y}{\sin. yx} y'.$$

Hence,

$$y = y'' + \frac{y' \sin. y'x + x' \sin. x'x}{\sin. yx},$$

$$x = x'' + \frac{x' \sin. x'y + y' \sin. y'y}{\sin. yx}.$$

(73.) If the axes A'Y', A'X', be parallel to AY, AX,

$$y = y'' + y', \qquad x = x'' + x'.$$

(74.) If $yx = 90°$, $\because \sin. yx = 1$, $\sin. x'y = \cos. x'x$, and $\sin. y'y = \cos. y'x$, $\because$

$$y = y'' + y' \sin. y'x + x' \sin. x'x,$$

$$x = x'' + x' \cos. x'x + y' \cos. y'x.$$

(75.) If $y'x' = 90^\circ$,

$$y = y'' + \frac{y' \cos. x'x + x' \sin. x'x}{\sin. yx},$$

$$x = x'' + \frac{x' \sin. yx' - y' \cos. yx'}{\sin. yx}.$$

(76.) If $yx = y'x' = 90^\circ$,

$$y = y'' + x' \sin. x'x + y' \cos. x'x,$$
$$x = x'' + x' \cos. x'x - y' \sin. x'x.$$

(77.) If the two systems have the same origin, $y'' = 0$, and $x'' = 0$.

SECTION VII.

The discussion of the general equation of the second decree.

(78.) When an equation is constructed in the manner described in (10), its locus, if it have any, is a line in the plane of the axes of co-ordinates, whose points are determined by supposing each variable susceptible of an unlimited series of values, positive and negative, and the equation thereby furnishing a corresponding unlimited series of values of the other variable, and thus determining the *course of the locus.* Under this view, it might appear that the locus, of every equation whatever, was (like that of the first degree) a line of unlimited extent. This would, in fact, take place did it not frequently happen, that certain values being assigned to either of the variables, the equation furnishes impossible symbols for the values of the other. Such values, since they have no arithmetical, have no geometrical representatives; or, in other words, the *locus* has no point corresponding to such values. In what manner this circumstance affects the figure of the locus, whether by limiting its extent in one or more directions, or by completely circumscribing it, is determined by certain relations between the *constant parts* of the equation. The values of these affect sometimes the form and properties of the line, and sometimes only its position with respect to the axes of co-ordinates. The general equation of the first degree was found to generate a right line, whatever the values of the constant parts might be, and, therefore, in this case they merely affected the position of the line; but its figure and properties were independent of their particular values. This, however, does not happen in other cases. In equations of the higher degrees, it is found that not only the position of the locus, but its nature, form, and properties, depend on the relative values of the constant parts; and that loci of different species, that is, having different forms and properties, will be generated by equations of the same degree, according to the relative values of the constant parts.

(79.) The classification of the different species of lines included under a general equation, and the investigation of the functions of the constant parts, which characterise each of those species, is called the *discussion* of the general equation.

(80.) An equation of the second degree is one which involves the variables in powers or products not exceeding two dimensions. Hence, an equation of the second degree, presented under its most general form, is,

$$\mathrm{A}y^2 + \mathrm{B}xy + \mathrm{C}x^2 + \mathrm{D}y + \mathrm{E}x + \mathrm{F} = 0 \ (a).$$

Where A, B . . . . E represent, generally, the respective coefficients of the dimensions of the variables admissible into an equation of the second degree, and F the sum of all the terms not involved with the variables.

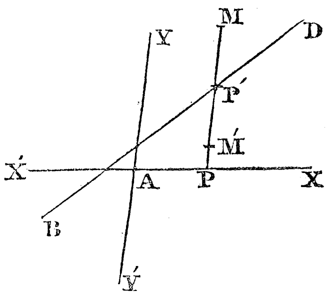
The solution of this equation for the two variables gives

$$y = -\frac{\text{B}x+\text{D}}{2\text{A}} \pm \frac{1}{2\text{A}}\sqrt{(\text{B}^2-4\text{AC})x^2+2(\text{BD}-2\text{AE})x+(\text{D}^2-4\text{AF})} \quad (b)$$

$$x = -\frac{\text{B}y+\text{E}}{2\text{C}} \pm \frac{1}{2\text{C}}\sqrt{(\text{B}^2-4\text{AC})y^2+2(\text{BE}-2\text{CD})y+(\text{D}^2-4\text{CF})} \quad (c).$$

These solutions appear to exclude those equations of the second degree which do not contain the squares of one or both variables. But it will be shown in (86) that these cases can be brought under the above solutions. In what immediately follows the values of A and C will be considered finite.

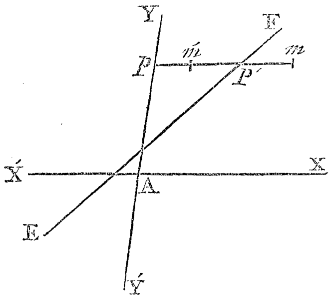
To construct the equation, let any fixed lines, YY' . XX', be assumed as axes of co-ordinates. Let the suffix of the radical in (b) be represented by R^2, and that in (c) by R'^2.



The value of y consists of two parts scil. $-\frac{\text{B}x+\text{D}}{2\text{A}}$ and $\frac{\sqrt{\text{R}^2}}{2\text{A}}$. The first is the value of y in the equation $2\text{A}y + \text{B}x + \text{D} = 0$, therefore, if the line BD be the locus of this equation, and any value, AP, be assigned to x, the corresponding value of $-\frac{\text{B}x+\text{D}}{2\text{A}}$ will be PP' drawn through P parallel to YY' to meet the right line BD. The other part $\frac{\sqrt{\text{R}^2}}{2\text{A}}$ is real, $= 0$, or impossible, ac-according as $\text{R}^2 > 0$, $= 0$, or < 0. If $\text{R}^2 > 0$, let $\text{P}'\text{M} = +\frac{\text{R}}{2\text{A}}$, and $\text{P}'\text{M}' = -\frac{\text{R}}{2\text{A}}$, and the values of y corresponding to $x = \text{AP}$, are PM and PM', and, therefore, M, M', are the points in the locus.

If $\text{R}^2 = 0$, there would be but one value of y, scil. PP', and the corresponding point P' of the locus would be on the line BD. If $R^2 < 0$, y would be impossible, or, in other words, the locus would not meet the parallel PP' in any point whatever.

In like manner the value of x consists of two parts, scil.
$-\frac{By+E}{2c}$ and $\frac{\sqrt{R'^2}}{2c}$.



The first is the value of x in the equation $2cx + By + E = 0$; therefore if the right line, EF, be the locus of this equation, and any value, Ap, be assigned to y, the corresponding value of $-\frac{By+E}{2c}$ will be p, p', a parallel to xx', drawn from p to meet the line EF. The other part $\frac{\sqrt{R'^2}}{2c}$ is real, $= 0$, or impossible, according as $R'^2 > 0 = 0$, or < 0.

If $R'^2 > 0$, let $p'm = +\frac{R'}{2c}$, and $p'm' = -\frac{R'}{2c}$, and pm and $p''m'$ are the values of x, corresponding to $y = Ap$, and m, m', therefore the points are the locus.

If $R'^2 = 0$, there would be but one value of x, scil. pp', and the corresponding point, p', of the locus would be on the line $E'F$.

If $R'^2 < 0$, the locus would not meet the parallel, pp', in any point whatever.

The lines BD and EF have the property of bisecting a system of parallel cords in the locus. Such lines are called *diameters;* and the cords which they bisect are called their *ordinates.*

The course of the locus of the equation of the second degree is limited to that series of values of each variable which give real values of the other. It appears that from that series, all values of x, which fulfil the condition, $R^2 < 0$, and all values of y, which fulfil the condition, $R'^2 < 0$, are excluded. It will therefore be necessary to determine how the sign of R^2 is affected by the values of x, and how that of R'^2 is affected by the values of y. As these circumstances depend on the roots of the equations, $R^2 = 0$, and $R'^2 = 0$, it will be convenient to consider the cases, $B^2 - 4AC > 0$, $B^2 - 4AC = 0$, and $B^2 - 4AC < 0$. (See Notes.)

(81.) If $B^2 - 4AC > 0$, let the roots of the equation, $R^2 = 0$, be x', x''.

If x', x'', be real and unequal, all values of x included between x' and x'', render $\frac{R^2}{B^2 - 4AC} < 0$, and since $B^2 - 4AC > 0$, $\therefore R^2 < 0$; $\therefore$ all values of y corresponding to such a series of values of x are impossible. All values of $x > x''$, or $< x'$, render $R^2 > 0$, and $x = x'$, or $x = x''$, render $R^2 = 0$; $\therefore$ all such give real values of y.

If $x'x''$ be impossible, all values of x give $\frac{R^2}{B^2 - 4AC} > 0$, $\therefore R^2 > 0$; $\therefore$ render all values of y real.

If $x' = x''$, all values of x (except $x = x'$) render $\frac{R^2}{B^2 - 4AC} > 0$, and $\therefore R^2 > 0$, and $x = x'$, gives $R^2 = 0$; $\therefore$ all such values give real values of y.

By the same reasoning, let $y'y''$ be the roots of the equation $R'^2 = 0$.

If $y'y''$ be real and unequal, all values of y between y' and y'' give impossible values of x, and all others real values of x.

If $y'y''$ be impossible, all values of y give real values of x.

If $y' = y''$, all values of y give real values of x.

(82.) If $B^2 - 4AC = 0$, $\therefore R^2 = 2(BD - 2AE)x + (D^2 - 4AF)$.

If $BD - 2AE > 0$, let x' be the root of $R^2 = 0$; all values of $x > x'$ give $\frac{R^2}{2(BD - 2AE)} > 0$, $\therefore R^2 > 0$, and x gives $R^2 = 0$; $\therefore$ all such values of x give real values of y. All values of $x < x'$ give $\frac{R^2}{2(BD - 2AE)} < 0$, $\therefore R^2 < 0$, $\therefore$ give impossible values for y.

If $(BD - 2AE) < 0$, $\therefore$ all values of $x > x'$ give $\frac{R^2}{(BD - 2AE)} > 0$, $\therefore R^2 < 0$, $\therefore$ all values of y impossible. All values of $x < x'$ give $\frac{R^2}{BD - 2AE} < 0$, $\therefore R^2 > 0$, and $x = x'$ gives $R^2 = 0$; all values of y corresponding to such values of x are real.

If $(BD - 2AE) = 0$, $\therefore R^2 = D^2 - 4AF$, $\therefore$ all values of y are real, if $D^2 - 4AF$ be not < 0, and impossible, if $(D^2 - 4AF) < 0$.

In like manner in this case, let $R'^2 = 2(BE - 2CD)y + (E^2 - 4CF)$.

If $BE - 2CD > 0$, all values of $y > y'$, or $y = y'$, give real values of x, and $y < y'$, give impossible values of x.

If $BE - 2CD < 0$, $y > y'$ renders x impossible; but all other values render x real.

If $BE - 2CD = 0$, all values of x are impossible, if $E^2 - 4CF < 0$, real if not.

It is observable that $BD - 2AE = 0$, and $BE - 2CD = 0$, are fulfilled at the same time, for $BD - 2AE = -\frac{B}{2C}(BE - 2CD)$, on condition that $B^2 - 4AC = 0$.

Also, if $B^2 - 4AC = 0$, and $BD - 2AE = 0$, $(D^2 - 4AE)$, and $(E^2 - 4CF)$, will have the same sign, and be at the same time $= 0$, for $E^2 - 4CF = \frac{B^2}{4A^2}(D^2 - 4AF)$.

(83.) If $B^2 - 4AC < 0$, as before, let $x'x''$ be the roots of $R^2 = 0$.

If $x'x''$ be real and unequal, all values of x between x' and x'' give $\frac{R^2}{B^2-4AC} < 0$, $\because R^2 > 0$, and $x = x'$, $x = x''$, give $R^2 = 0$; $\because$ all such values of x give real values of y.

All values of $x > x''$, or $< x'$, give $\frac{R^2}{B^2-4AC} > 0$, $\because R^2 < 0$, $\because$ all corresponding values of y impossible.

If $x'x''$ be impossible, all values of x give $\frac{R^2}{B^2-4AC} > 0$, and $\because R^2 < 0$, $\because$ all values of y impossible.

If $x' = x''$, all values of x (except $x = x'$, or $x = x''$) give $\frac{R^2}{B^2-4AC} > 0$, $\because R^2 > 0$, $\because$ all values of y impossible; but $x = x'$ gives $R^2 = 0$, $\because y$ real.

In like manner, if $y'y''$ be real and unequal, all values of y included between y' and y'', as well as $y = y''$, and $y = y'$, give real values of x, and all other impossible values.

If $y'y''$ be impossible, all values of y give impossible values of x.

If $y' = y''$, all values of x are impossible, except those corresponding to $y = y'$, $y = y''$.

(84.) To determine the conditions by which $x'x''$ and $y'y''$ are real, equal, or impossible, let the equations $R^2 = 0$ and $R'^2 = 0$ be solved; hence the roots are respectively

$$x = \frac{-(BD - 2AE) \pm 2\sqrt{AM}}{B^2 - 4AC},$$

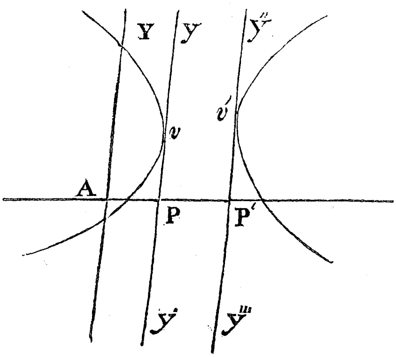
$$y = \frac{-(BE - 2CD) \pm 2\sqrt{CM}}{B^2 - 4AC},$$

where $M = AE^2 + CD^2 + B^2F - BDE - 4ACF$.

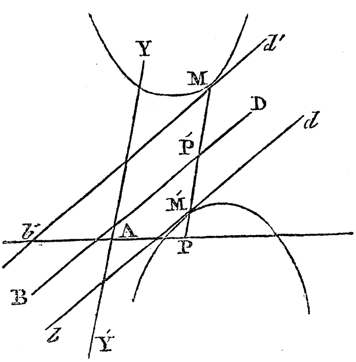
Since A and C are both supposed finite:

If $x'x''$ be real and unequal -	$AM > 0$.
If $y'y''$ be real and unequal -	$CM > 0$.
If $x' = x''$, or $y' = y''$ - -	$M = 0$.
If $x'x''$ be impossible - -	$AM < 0$.
If $y'y''$ be impossible - -	$CM < 0$.

(85.) To investigate the course of the locus under the condition, $B^2 - 4AC > 0$.



1. Let $x'x''$ be real and unequal; let $x' = AP$, $x'' = AP'$, and through P and P' let the indefinite parallels, yy' and $y''y'''$ to YY' be drawn: No point of the locus lies between these parallels (81); but it meets the line yy' at a point v, such, that $Pv = -\dfrac{Bx' + D}{2A}$, and the line $y''y'''$ at the point v', so that $P'v' = -\dfrac{Bx'' + D}{2A}$. Beyond the limits of the parallels, the locus spreads to unlimited extent in two opposite branches (81), touching those lines at v and v'.



2. Let $x'x''$ be impossible, all values of y are in this case real (81). Let AP be that value of x, which renders ${}^2R$ the least possible value; draw PP' parallel to YY' to meet the line BD, whose equation is $2Ay + Bx + D = 0$; take $P'M = +R$, and $P'M' =$

— R. Through the points M, M′, let the indefinite parallels bd, $b'd'$ to BD be drawn. Since P′M, P′M′ is the least value that the radical in (b) can receive, the locus must be excluded from between those lines; but the radical being susceptible of every magnitude, however great it extends in two opposite and unlimited branches beyond them, touching them at the points M, M′.

3. Let $x' = x''$: in this case $R^2 = (x - x')\sqrt{B^2 - 4AC}$, and as all values of y are real, the equation is that of two right lines.

Similar inferences follow with respect to the roots $y'y''$.

1. If they be real and unequal, the curve touches two right lines parallel to XX′, is excluded from between them, and extends indefinitely beyond them.

2. If they be impossible, the curve touches two right lines parallel to the diameter, whose equation is $2Cy + Bx + D = 0$, is excluded from between them, and extends indefinitely beyond them.

3. If $y' = y''$, the equation represents two *right lines.*

Hence, in order that the locus of an equation, fulfilling the character, $B^2 - 4AC > 0$, should be a curve, it must also satisfy the condition, $M >$, or < 0; if not, it will represent right lines.

Curves thus characterised, are called *Hyperbolæ.*

(86.) If the squares of one or both variables be not contained in an equation which does contain their product, it comes within the character $B^2 - 4AC > 0$. But the inferences which have been just made with respect to the locus cannot be immediately applied to this case, because they were made on the supposition, that the equation contained the squares of both variables. However, if the axes of coordinates, to which such an equation is related, be transformed by the general formulæ given in Sect. VI. (72), such a position may be assigned them, that the values of the co-efficients of the squares of the variables shall be finite.

In the equation

$$A'y^2 + B'xy + C'x^2 + D'y + E'x + F' = 0.$$

Suppose A′, or C′, or both = 0, but B′ finite, let the equation resulting from transformation of the axes be

$$Ay^2 + Byx + Cx^2 + Dy + Ex + F = 0.$$

Such values being assigned to the quantities composing the formulæ in Sect. VI. as will render A and C finite.

From the values of A, B, C, in terms of A′, B′, C′, and the angles under the axes of co-ordinates,

$$B^2 - 4AC = (B'^2 - 4A'C')\,\frac{(\sin.y'y\sin.x'x - \sin.x'y\sin.y'x)^2}{\sin.^2 yx} = B'^2.\ \frac{(\sin.yy'\sin.xx' - \sin.x'y\sin.y'x)^2}{\sin.^2 yx}.$$

The quantity (sin. yy' sin. xx' — sin. $x'y$ sin. $y'x$), must be > 0, for being a complete square, it cannot be < 0, neither can it be $= 0$; for if sin. yy' sin. $x'x$ — sin. $x'y$ sin. $y'x = 0$,

and ∵ $\frac{\sin. y'y}{\sin. y'x} = \frac{\sin. x'y}{\sin. x'x}$, ∴ the new axes of co-ordinates would be coincident. Hence, since the quantities B'^2 and $\sin.^2 yx$ are essentially positive, the quantity $B^2 - 4AC > 0$, in which A and C are finite, and which is an equation of the same locus as that in which $A' = 0$ and $C' = 0$, all that has been proved of curves characterised by $B^2 - 4AC > 0$: on the supposition that A and C are finite also, apply to the cases where A or C, or both, are = 0, provided that B is finite.

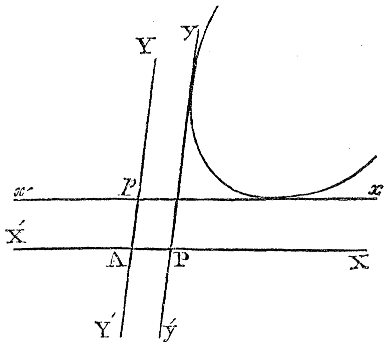
(87.) To investigate the course of the locus when

$$B^2 - 4AC = 0.$$

Let $AP = x'$, $Ap = y'$, and let the indefinite parallels yy' and xx' be drawn.

If $BD - 2AE > 0$, the locus touches yy', and lies entirely at the positive side of it. See (82).

If $BD - 2AE < 0$, the locus touches yy', and lies entirely at the negative side of it.



If $BE - 2CD > 0$, the locus touches xx', and lies entirely at the positive side of it.

If $BE - 2CD < 0$, the locus touches xx', and lies entirely at the negative side of it.

If $BD - 2AE = 0$, and $\because$ also $BE - 2CD = 0$ (82), the equation is that of *right lines*. If $D^2 - 4AF > 0$, or $= 0$, and $\because$ also $E^2 - 4CF > 0$, or $= 0$; but if $D^2 - 4AF < 0$, and $\because$ also $E^2 - 4CF < 0$, there is no locus.

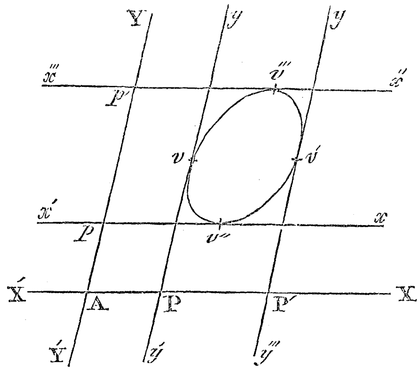
This class of curves characterised by $B^2 - 4AC = 0$, and $BD - 2AE$ finite, and consisting of one unlimited branch, extending in one direction, are called *Parabolæ*.

Equations of the second degree, in which the square of one of the variables and also their product is wanted, come under the character $B^2 - 4AC = 0$; but for the reason before stated, the conclusions preceding cannot be immediately applied to them. However, if a transformation of axes be effected as before, it will follow that since $B' = 0$, and also A' or $C' = 0$, $\because B^2 - 4AC = 0$, for the other factor has been proved finite (86). Hence, since by the transformation, A and C become finite, and at the same time $B^2 - 4AC = 0$; those loci come under the class of *Parabolæ*, and the preceding references apply to them.

(88.) To investigate the course of the locus when $B^2 - 4AC > 0$.

To fulfil this condition, A and C must have the same sign.

1. If $x'x''$ and $\because$ also y', y'' be real and unequal, let



$AP = x'$, $AP' = x''$, $Ap = y'$, $Ap' = y''$. Let the indefinite parallels, yy' and $y''y'''$ to YY', and $x'x'$ and $x''x'''$ to XX' be drawn.

Let $Pv = -\dfrac{Bx'+D}{2A}$,

$Pv' = -\dfrac{B'x''+D}{2A}$,

$pv'' = -\dfrac{By'+E}{2C}$, $p'v''' = -\dfrac{By''+E}{2C}$; from (83) it appears that the locus touches those parallels at v, v', v'', v''', and is included between each system.

2. If $x'x''$, and ∵ also $y'y''$ be impossible, no locus exists (83).

3. If $x' = x''$, and ∵ also $y' = y''$, the variables have each but one real value, scil. $y = -\dfrac{Bx'+D}{2A}$, $x = -\dfrac{Bx''+E}{2C}$; ∵ the locus is in this case a point.

Hence, in order that an equation characterised by $B^2 - 4AC < 0$ may be that of a curve, it must also fulfil the condition $M > 0$.

Curves, thus characterised, are called *Ellipses*.

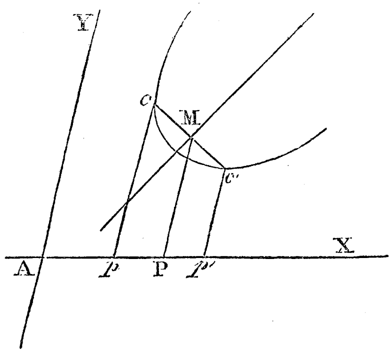
(89.) To recapitulate the preceding results.

If B^2-4AC	>0	and M not $=0$, the equation represents loci,	called *Hyperbolæ*.
- - -	>0	$M=0$: -	*Right lines.*
- - -	$=0$	$BD-2AE$ not $=0$ -	*Parabolæ.*
- - -	$=0$	$BD-2AE=0$, D^2-4AF not <0	*Right lines.*
- - -	$=0$	$BD-2AE=0$, $D^2-4AF<0$	*No locus.*
- - -	<0	$M>0$ - - -	*Ellipses.*
- - -	<0	$M=0$ - - -	*A point.*
- - -	<0	$M<0$ - - -	*No locus.*

SECTION VIII.

Of the diameters, axes, and asymptotes of the lines of the second degree.

(90.) In the discussion of the general equation, it was proved that two right lines bisected systems of chords parallel respectively to the axes of co-ordinates. Hence arose the definition of a diameter. An inquiry naturally presents itself, whether every system of parallel chords has not a corresponding diameter. To determine this, let $ay + bx + c = 0$ be a line meeting the curve at C, C′. To consider this as one of a system of parallel chords, let $\frac{b}{a}$ be considered as given,



and $\frac{c}{a}$ as indeterminate. By eliminating y by this and the general equation, the roots Ap, Ap' of

$$x^2 + \frac{2\text{A}bc - \text{B}ca - \text{D}ba + \text{E}a^2}{\text{A}b^2 - \text{B}ba + \text{C}a^2} \cdot x + \frac{\text{F}a^2 + \text{A}c^2 - \text{D}ca}{\text{A}b^2 - \text{B}ba + \text{C}a^2} = 0,$$

which is the resulting equation, will be the values of x for the points C, C′. Let CC′ be bisected at M, MP be drawn parallel to AY,

$$\because \text{AP} = \frac{\text{A}p + \text{A}p'}{2}, \because \text{AP} = -\frac{2\text{A}bc - \text{B}ca - \text{D}ba + \text{E}a^2}{2(\text{A}b^2 - \text{B}ba + \text{C}a^2)}.$$

By substituting for c its value in $ay + bx + c = 0$, and denominating AP by x, and PM by y, the equation of the locus of M is found to be,

$$(\text{B}a - 2\text{A}b)\, y + (2\text{C}a - \text{B}b)\, x + \text{E}a - \text{D}b = 0.$$

This being an equation of the first degree, the locus sought is a right line, and consequently a *diameter* to which the parallel chords are *ordinates*. If the curve be a parabola, the condition $B^2 - 4AC = 0$, gives

$$2Ay + Bx + \frac{2A(Ea - Db)}{Ba - 2Ab} = 0,$$

by eliminating C. The co-efficients of the variables in this equation being constant, prove that all diameters of a parabola are parallel to the line $2Ay + Bx = 0$.

As $\frac{B^2}{2C} = 2A$, the equation may also be expressed

$$By + 2Cx + \frac{2C(Ea - Db)}{2Ca - Bb} = 0.$$

PROP. XXV.

(91.) *Given a diameter, to determine its ordinates.*

1°. If $B^2 - 4AC$ be not $= 0$, let the given diameter be $a'y + b'x + c' = 0$, and its ordinates $ay + bx + c = 0$.

$$\frac{b'}{a'} = \frac{2Ca - Bb}{Ba - 2Ab}, \quad \therefore \quad \frac{b}{a} = \frac{2Ca' - Bb'}{Ba' - 2Ab'}$$

Which equations determine either the diameter or its ordinates when the other is given.

2°. If $B^2 - 4AC = 0$, let the diameter be $2Ay + Bx + c' = 0$,

$$c' = \frac{2A(Ea - Db)}{Ba - 2Ab}, \quad \therefore \quad \frac{b}{a} = \frac{2AE - Bc'}{2A(D - c')}.$$

PROP. XXVI.

(92.) *To find the equation of a diameter through a given point.*

The equation of any line through the given point $y'x'$ is

$$a'y + b'x - (a'y' + b'x') = 0.$$

This being a diameter, let its ordinates be $ay + bx + c = 0$, $\therefore$ $a' = Ba - 2Ab$, $b' = 2Ca - Bb$, $a'y' + b'x' = Db - Ea$.

$$\frac{b'}{a'} = -\frac{(B^2 - 4AC)y' + (BE - 2CD)}{(B^2 - 4AC)x' + (BD - 2AE)}.$$

Therefore the equation of the diameter is

$$\{(B^2 - 4AC)x' + (BD - 2AE)\}(y - y') -$$
$$\{(B^2 - 4AC)y' + (BE - 2CD)\}(x - x') = 0.$$

The equation of its ordinates is,

$$(2Ay' + Bx + D)y + (2Cx' + By' + E)x + c = 0,$$

where c is indeterminate.

(93.) *Cor.* 1. If $B^2 - 4AC = 0$, and therefore $\frac{BE - 2CD}{BD - 2AE} = -\frac{B}{2A} = -\frac{2C}{B}$, the *diameter* becomes either of

$$2A(y - y') + B(x - x') = 0,$$
$$B(y - y') + 2C(x - x') = 0.$$

(94.) *Cor.* 2. If $B^2 - 4AC$ be not $= 0$, the equation of the *diameter* being divided by $(B^2 - 4AC)$ becomes

$$\left(x' + \frac{BD - 2AE}{B^2 - 4AC}\right)(y - y') - \left(y' + \frac{BE - 2CD}{B^2 - 4AC}\right)(x - x') = 0,$$

which is a right line through the point,

$$y'' = -\frac{BE - 2CD}{B^2 - 4AC}, \qquad x'' = -\frac{BD - 2AE}{B^2 - 4AC};$$

therefore all diameters of an *ellipse*, or *hyperbola*, intersect each other at this point, and, *vice versâ*, all right lines passing through this point are diameters.

(95.) *Def.* The point $y''x''$ is called the *centre*, and the *ellipse* and *hyperbola* are thence called by the common name of *central curves*. Since $B^2 - 4AC = 0$ renders the co-ordinates of this point infinite, the *parabola* may be conceived to have a centre at an infinite distance.

PROP. XXVII.

(96.) *In central curves, if any diameter be parallel to the ordinates of another diameter, the latter will be also parallel to the ordinates of the former.*

For, in (91), if $\frac{b}{a}$ determine the position of a diameter, $\frac{b'}{a'}$ determines that of its ordinates, and *vice versâ*.

Def. Such diameters are called *conjugate diameters*.

PROP. XXVIII.

(97.) *To discover whether any and what diameters intersect their ordinates perpendicularly.*

1°. If $B^2 - 4AC$ be not $= 0$, let the sought diameter be

$$a'(y-y'') + b'(x-x'') = 0,$$

y'' x'' being the co-ordinates of the centre; and its ordinates $ay + bx + c = 0$, by (91), $\frac{b'}{a'} = \frac{2Ca - Bb}{Ba - 2Ab}$, and by (32)

$$aa' + bb' - (a'b + ab')\cos. yx = 0; \text{ hence,}$$

$$(B - 2A\cos. yx)b'^2 + 2(A-C)b'a' + (2C\cos. yx - B)a'^2 = 0,$$

$$\frac{b'}{a'} = \frac{C - A \pm \sqrt{(C-A)^2 + B^2 + 2\cos. yx(2AC\cos. yx - BA - BC)}}{B - 2A.\cos. yx}.$$

These values of $\frac{b'}{a'}$ are always real. For if the quantity under the radical be arranged by the dimensions of B, and equated with zero, we shall find

$$B^2 + 2(A+C)\cos. yx.\ B + 4AC\cos. yx + (C-A)^2 = 0,$$

which, solved for B, gives after reduction

$$B = (C+A)\cos. yx \pm (C-A)\sin. yx\sqrt{-1};$$

which being impossible, the suffix of the radical in the values of $\frac{b'}{a'}$ is always positive. The equations sought are ∵

$$\frac{(B - 2A\cos. yx)(y-y'') + (C-A)}{\pm\sqrt{(c-A)^2 + B^2 + 2\cos. yx(2AC\cos. yx - BA - BC)}} \times (x - x'') = 0.$$

2°. If $B^2 - 4AC' = 0$, let the sought diameter be

$$2Ay + Bx + \frac{2A(Ea - Db)}{Ba - 2Ab} = 0.$$

Since it is perpendicular to the line $ay + bx + c = 0$,

$$2Aa + Bb - (2Ab + Ba)\cos. yx = 0, \because \frac{b}{a} = \frac{B\cos. yx - 2A}{B - 2A\cos. yx}.$$

Making this substitution

$$2Ay + Bx + \frac{2A\{BE + 2AD - (BD + 2AE)\cos. yx\}}{B^2 - 4AB\cos. yx + 4A^2} = 0.$$

Such diameters are called axes, and it appears that ellipses and hyperbolæ have two, and parabolæ but one.

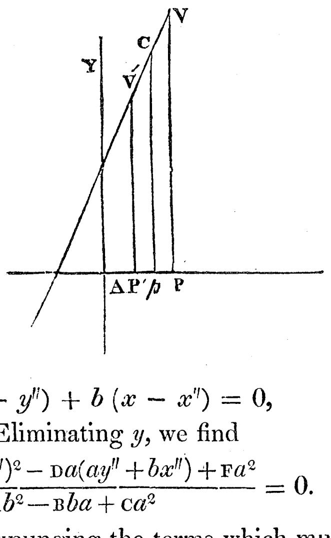
(98.) The two values of $\frac{b'}{a'}$ fulfil the condition (32), therefore the axes of central curves are at right angles.

(99.) Hence also the axes are conjugate diameters.

PROP. XXIX.

(100.) *To find the intersection of a curve with its diameter.*

1°. If $B^2 - 4AC = 0$. Let the equation of the diameter be $2Ay + Bx + c' = 0$.



The elimination of y between this and the general equation gives

$$\frac{4AF + c'^2 - 2Dc'}{2(BD - 2AE)}.$$

Therefore every diameter of a parabola meets the curve in one, and but one point.

2°. If $B^2 - 4AC$ be not $= 0$, the diameter is $a(y - y'') + b(x - x'') = 0$, where $y''x''$ is the centre. Eliminating y, we find

$$x^2 - 2x''x + \frac{A(ay'' + bx'')^2 - Da(ay'' + bx'') + Fa^2}{Ab^2 - Bba + Ca^2} = 0.$$

The roots of which, after expunging the terms which mutually destroy each other, and dividing both terms of the fraction under the radical by $B^2 - 4AC$, are

$$x = x'' \pm \sqrt{-\frac{M}{B^2 - 4AC} \cdot \frac{a^2}{Ab^2 - Bba + Ca^2}}.$$

Where M retains its signification in (84).

Supposing these values of x to be real, let C be the centre, and V, V′ the points of intersection. Since Pp and P′p are the two values of the radical, they are equal $\because$ CV = CV′; therefore every diameter which meets an *ellipse* or *hyperbola* is bisected at the *centre*. It is from this property that the *centre* has received its name.

(101.) *Def.* The points where a diameter meets the curve are called the *vertices* of that diameter.

(102.) *Def.* The vertex of an axis is called a *vertex* of the curve.

(103.) When a diameter of central curves is spoken of as a finite line, that portion of the diameter intercepted between its vertices is meant.

PROP. XXX.

(104.) *To find what diameters of central curves meet them.*

It will be necessary to determine how the values of a and b affect the suffix of the radical in (100) negative, and what not.

The sign of the factor $\frac{a^2}{\mathrm{A}b^2 - \mathrm{B}ba + \mathrm{C}a^2}$ depends on the relation of the values of a and b to the roots of

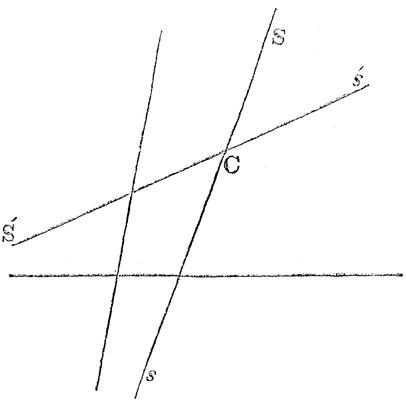
$$\mathrm{A}\frac{b^2}{a^2} - \mathrm{B}\frac{b}{a} + \mathrm{C} = 0, \; i.\,e. \; \frac{b}{a} = \frac{\mathrm{B} \pm \sqrt{\mathrm{B}^2 - 4\mathrm{AC}}}{2\mathrm{A}}.$$

Let these values of $\frac{b}{a}$ be r, r'. If they be real and unequal, *scil.* if $\mathrm{B}^2 - 4\mathrm{AC} > 0$, all values of $\frac{b}{a}$ between r and r' render $\frac{a^2}{\mathrm{A}b^2 - \mathrm{B}ba + \mathrm{C}a^2} < 0$. If $\frac{b}{a} = r$, or $\frac{b}{a} = r'$, $\because$ $\frac{a^2}{\mathrm{A}b^2 - \mathrm{B}ba + \mathrm{C}a^2}$ is infinite; and if $\frac{b}{a}$ have any value $> r$, or $< r'$, $\frac{a^2}{\mathrm{A}b^2 - \mathrm{B}ba + \mathrm{C}a^2} > 0$.

If r and r' be impossible, *scil.* if $\mathrm{B}^2 - 4\mathrm{AC} < 0$, all values of $\frac{b}{a}$ render $\frac{a^2}{\mathrm{A}b^2 - \mathrm{B}ba + \mathrm{C}a^2} > 0$.

The roots r and r' cannot be equal, for then $\text{B}^2-4\text{AC}=0$, which is contrary to the hypothesis.

If $\text{B}^2-4\text{AC}>0$, and $\text{M}>0$, the factor $-\frac{\text{M}}{\text{B}^2-4\text{AC}}<0$, therefore the real values of x are those corresponding to $\frac{a^2}{\text{A}b^2-\text{B}\,ba+\text{C}a^2}<0$, or to those values of $\frac{b}{a}$ intermediate between r and r', and the impossible values are those which correspond to values of $\frac{b}{a}$, $>r$, or $<r'$. Let the diameter be called d, $\because \frac{b}{a}=\frac{\sin. dx}{\sin. dy}$. Through the centre C, let the lines Ss and S$'s'$ be drawn, so that, calling Ss, l, and S$'s'$, l', $r=\frac{\sin. lx}{\sin. ly}$, $r'=\frac{\sin. l'x}{\sin. l'y}$.



In order, therefore, that a diameter d should meet the curve, $\frac{\sin. dx}{\sin. dy}$ must be $>\frac{\sin. l'x}{\sin. l'y}$, and $<\frac{\sin. lx}{\sin. ly}$. The lines S$s$ and S$'s'$ extend *ad infinitum* without meeting the curve.

Those diameters fulfilling the condition $\frac{\sin. dx}{\sin. dy}$, $>\frac{\sin. lx}{\sin. ly}$, or $<\frac{\sin. l'x}{\sin. l'y}$, do not meet the curve. Hence the angles SCs' and S$'$Cs include between their sides all those diameters which meet the curve, and consequently include the curve itself; and the angles S$'$CS$'$ and sCs' include all those diameters which do not meet the curve, and consequently exclude the curve itself.

(105.) The values of r, r', being

$$\frac{b}{a}=\frac{\text{B}\pm\sqrt{\text{B}^2-4\text{AC}}}{2\text{A}}.$$

Since $\frac{\text{B}-\sqrt{\text{B}^2-4\text{AC}}}{2\text{A}}=\frac{2\text{C}}{\text{B}+\sqrt{\text{B}^2-4\text{AC}}}$, the values of r, r', may be expressed thus,

$$r=\frac{\text{B}+\sqrt{\text{B}^2-4\text{AC}}}{2\text{A}},$$

$$r'=\frac{2\text{C}}{\text{B}+\sqrt{\text{B}^2-4\text{AC}}}.$$

The equations of the lines ss and $s's'$ are therefore,

$$2\text{A}(y-y'')+(\text{B}+\sqrt{\text{B}^2-4\text{AC}})\,(x-x'')=0,$$
$$2\text{C}(x-x'')+(\text{B}+\sqrt{\text{B}^2-4\text{AC}})\,(y-y'')=0.$$

Though these right lines pass through the centre, yet they are not diameters, for if they were, the equation of their ordinates would be (92) respectively,

$$2\text{A}y+(\text{B}^2+\sqrt{\text{B}^2-4\text{AC}})x+c=0,$$
$$2\text{C}x+(\text{B}+\sqrt{\text{B}^2-4\text{AC}})y+c'=0.$$

That is, the ordinates would be coincident with the diameters themselves, which is contrary to the definition of ordinates.

(106.) These lines, therefore, are not themselves diameters, but may be considered as the limits of diameters. They separate those diameters which meet the curve, called *transverse diameters*, from those which do not meet it, called *second diameters*. As the diameters, both transverse and second, approach to coincidence with these lines, they also approach to coincidence with their ordinates; and the lines ss and $s's'$ are the limits at which that coincidence actually takes place: these lines are called *asymptots*.

(107.) From the position of *transverse* and *second dia-*

meters, it is plain that the ordinates of the former intersect the same branch of the curve, but those of the latter opposite branches.

(108.) If $B^2 - 4AC > 0$, and $M < 0$, inferences similar to those already made will follow, with this difference, that the angle SCS', scs', will then include the curve; and the diameters which meet it and the angles SCs', $s'Cs$, include the second diameters.

(109.) If $B^2 - 4AC < 0$. In this case, if the equation represent an *ellipse*, $M > 0$, therefore $-\frac{M}{B^2 - 4AC} > 0$; but the values of r, r', are impossible, and therefore $\frac{a^2}{Ab^2 + Bba + Ca^2} > 0$, hence the values of x, in (99), are always real and unequal, therefore every diameter of an ellipse intersects it in two points.

(110.) If the axes of an ellipse be unequal, the greater is generally called the transverse, and the lesser the conjugate axis. In an hyperbola, the axis which meets the curve is called the transverse, and the other the conjugate axis.

SECTION IX.

Of the different forms of the equations of lines of the second degree, related to different axes of co-ordinates.

(111.) That an equation of the second degree should include under it any or all of the three classes of curves which have been investigated in the discussion, it is not necessary that every dimension of the variables, consistent with its general character, should be found among its terms. A term wanted does not necessarily render the equation less general, if its generality be estimated only by the curves included under it. But, in another sense, the generality is always impaired by such deficiency, which, though it may not exclude from the extension any of these classes of curves, yet it may restrict the curve in its position with respect to the axes of co-ordinates by which the equation is constructed. As this circumstance gives great facility to the development of the properties of lines of the second degree, it will be useful to ascertain the form of the equation, (that is, the terms of which it consists,) corresponding to certain particular positions which the curve may assume with respect to the axes of co-ordinates.

PROP. XXXI.

(112.) *To find the form of the equation when the curve passes through the origin of co-ordinates.*

In order that this should happen, the conditions $y = 0$ and $x = 0$, should be co-existent, $\therefore$ F $= 0$, $\therefore$ the form is

$$\text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x = 0.$$

PROP. XXXII.

(113.) *To find the form of the equation when a diameter and its ordinates are parallel to the axes of co-ordinates.*

The diameter, whose ordinates are parallel to YY′, is

$$2\text{A}y + \text{B}x + \text{D} = 0.$$

In order that this should be parallel to XX′, the condition B $= 0$ is necessary; therefore the form sought is

$$\text{A}y^2 + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0.$$

In this case, also, provided that A and B are both finite, the diameter

$$2\text{C}x + \text{B}y + \text{E} = 0,$$

has its ordinates parallel to XX′, and therefore the curve is central, and the axes of co-ordinates parallel to a system of conjugate diameters.

PROP. XXXIII.

(114.) *To find the form of the equation when either axis of co-ordinates is coincident with a diameter whose ordinates are parallel to the other.*

In addition to the condition $B = 0$ in (113) let $D = 0$, then the diameter $2Ay + Bx + D = 0$ will be coincident with XX′, in this case the form is

$$Ay^2 + Cx^2 + Ex + F = 0.$$

(115.) But if in addition to $B = 0$, also $E = 0$, then the diameter $2Cx + By + E = 0$ will be coincident with YY′, and the form will be

$$Ay^2 + Cx^2 + Dy + F = 0.$$

In this case, if $F = 0$, the origin is at the vertex of the diameter, and the equation becomes

$$Ay^2 + Cx^2 + Ex = 0.$$

(116.) If all these conditions, $B = 0$, $E = 0$, $D = 0$, be fulfilled together, the axes of co-ordinates coincide with a system of conjugate diameters, and the form is

$$Ay^2 + Cx^2 + F = 0.$$

(117.) In any of these cases, if the origin be on the curve, the form is had by omitting F.

(118.) In case $B = 0$, if the curve be a parabola, A or C must also $= 0$.

PROP. XXXIV.

(119.) *To find the form of the equation when the centre of the curve is at the origin.*

The co-ordinates of the centre in (94) must each $= 0$, in order that the centre should be at the origin; ∵

$$BD - 2AE = 0, \qquad BE - 2CD = 0.$$

If D and E were finite, these equations would give $B^2 - 4AC = 0$, which united with either of the above conditions, would render the equation either impossible, or that of right lines; therefore, in order that the equation should be that of a curve, the conditions must be satisfied by $\text{D} = 0$ and $\text{E} = 0$, which shows that when the centre is on the origin, the form is

$$\text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{F} = 0.$$

PROP. XXXV.

(120.) *To find the form of the equation of the hyperbola when the axes of co-ordinates are, one or both, parallel to the asymptots.*

In order that Ss (105) should be parallel to YY′, $\text{A} = 0$, and in order that S′s' should be parallel to XX′, $\text{C} = 0$, and in order that both should take place together, $\text{A} = 0$, $\text{C} = 0$; hence,

(121.) If an asymptot be parallel to XX′, the form is

$$\text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0.$$

The equations of the asymptots are in this case

$$(x - x'') = 0, \text{ or } x + \frac{\text{D}}{\text{B}} = 0,$$

$$\text{C}(x - x'') + \text{B}(y - y'') = 0.$$

(122.) If an asymptot be parallel to XX′, the form is

$$\text{A}y^2 + \text{B}xy + \text{D}y + \text{E}x + \text{F} = 0;$$

and the equations of the asymptots are

$$y + \frac{\text{E}}{\text{B}} = 0,$$

$$\text{A}(y - y'') + \text{B}(x - x'') = 0.$$

(123.) If both axes be parallel to the asymptots, the form is

$$\text{B}xy + \text{D}y + \text{E}x + \text{F} = 0;$$

and the equations of the asymptots are

$$x + \frac{\text{D}}{\text{B}} = 0, \quad y + \frac{\text{E}}{\text{B}} = 0.$$

(124.) If an asymptot be coincident with YY′, $\because \text{A} = 0$, $\text{B} = 0$; therefore the form is

$$Cx^2 + Bxy + Ex + F = 0;$$

and the equations of the asymptots are

$$x = 0, \quad Cx + B(y - y'') = 0.$$

(125.) If one asymptot be coincident with XX′, C = 0, E = 0, the form is

$$Ay^2 + Bxy + Ey + D = 0;$$

and the equations of the asymptots are

$$y = 0, \quad Ay + B(x - x'') = 0.$$

(126.) If both asymptots be coincident with the axes of co-ordinates,

$$A = 0,\ C = 0,\ D = 0,\ E = 0; \because$$
$$Bxy + F = 0.$$

PROP. XXXVI.

(127.) *To find the form of the equation of the parabola when one axis is a diameter and the other parallel to its ordinates, the origin being at its vertex.*

If the equation be that of the parabola, C = 0, and the origin being on the curve, F = 0, therefore the form is

$$Ay^2 + Ex = 0.$$

PROP. XXXVII.

(128.) *To express the equation of a central curve related to a system of conjugate diameters as axes of co-ordinates, and in terms of those parts of the diameters which are intercepted within the curve.*

In (116) $y = 0$ gives $x^2 = -\frac{F}{C}$, and $x = 0$ gives $y^2 = -\frac{F}{A}$, let $-\frac{F}{C} = A'^2$, and $-\frac{F}{A} = B'^2$. If the curve intersects the axes of co-ordinates, 2A′ and 2B′ will be the parts intercepted, and the equation sought is

$$A'^2y^2 + B'^2x^2 = A'^2B'^2;$$

A'^2 being positive, or made so by changing the signs, the curve will be an ellipse, if $B'^2 > 0$; an hyperbola, if $B'^2 < 0$, for $B^2 - 4AC = - 4A'^2B'^2$.

(129.) If $A'^2 = R'^2 = B'^2$ and $yx = 90°$, the equation is

$$y^2 + x^2 = R'^2.$$

In this case the curve is a circle, since all points are a given distance R' from the origin.

(130.) To express the equation of the circle in its most general form, the origin and inclination of the axis should not be limited. A circle being defined to be a curve, every point of which is equidistant from a fixed point $y'x'$, its equation must be (44)

$$(y - y')^2 + (x - x')^2 + 2(y - y')(x - x') \cos. yx = R'^2,$$

$$\text{or } y^2 + 2 \cos. yx \,.\, yx + x^2 - 2(y' + x' \cos. yx)y -$$

$$2(x' + y' \cos. yx)x + y'^2 + x'^2 + 2y'x' \cos. yx - R'^2 = 0.$$

Hence the general equation represents a circle, if $A = C$ and the axes of co-ordinates are assumed at an angle, whose cosine is $\frac{B}{2A}$.

(131.) To express the equation,

$$Ay^2 + Cx^2 + Ex = 0.$$

In terms of the conjugate diameters; if $y=0$, the value of x being $2A'$,

$$2A' = -\frac{E}{C}, \because A' = -\frac{E}{2C};$$

if $x = A'$, the value of y will be B', $\because$

$$B'^2 = \frac{E^2}{4AC}:$$

hence $\frac{A}{C} = -\frac{A'^2}{B'^2}$, and the equation becomes

$$A'^2y^2 + B'^2x^2 - 2A'B'^2x = 0.$$

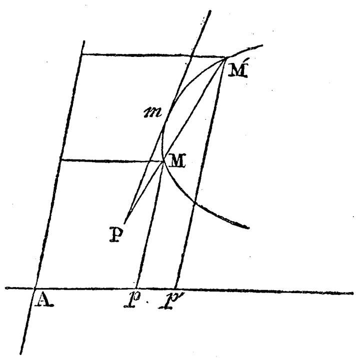
SECTION X.

Of the equations of tangents, normals, subtangents, and subnormals.

Previous to an investigation of the properties of those curves already defined, it will be necessary to determine the equations of certain lines related to the curve, and on which those properties depend.

PROP. XXXVIII.

(132.) *To express the equation of a line passing through a given point and touching a curve of the second degree.*



Let the value of y in the equation $a(y - y') + b(x - x') = 0$ of a right line (PM) passing through the point (P) $y'x'$ be substituted in the general equation of the second degree, and the result solved for x, gives an equation of the form

$$x = \frac{m \pm a\sqrt{\mathrm{R}'^2 a^2 - 2\mathrm{P}ab + \mathrm{R}^2 b^2}}{2n},$$

in which R^2 and R'^2 represent the quantities under the radicals in (80) (b) and (c), $y'x'$ being substituted for yx, and

$$-\mathrm{P} = (\mathrm{B}^2 - 4\mathrm{AC})y'x' + (\mathrm{BD} - 2\mathrm{AE})y' + (\mathrm{BE} - 2\mathrm{CD})x' - (\mathrm{DE} - 2\mathrm{BF});$$

the values of m and n being of no importance to the present inquiry.

In like manner the value of x in the equation of the line PM being substituted in the general equation, and the result solved, for y gives

$$y = \frac{m' \pm b\sqrt{R'^2a^2 - 2Pab + R^2b^2}}{2n}.$$

The line is a tangent when the points M and M′ unite as in Pm, therefore they must have the same co-ordinates; hence,

$$R'^2a^2 - 2Pab + R^2b^2 = 0,$$

which gives

$$\frac{b}{a} = \frac{P \pm 2\sqrt{MF'}}{R^2},$$

$$\text{or } \frac{a}{b} = \frac{P \pm 2\sqrt{MF'}}{R'^2};$$

$$\text{and therefore } \frac{b}{a} = \frac{R'^2}{P \pm 2\sqrt{MF'}}.$$

Where M represents the formula (84), and

$$F' = Ay'^2 + Bx'y' + Cx'^2 + Dy' + Ex' + F;$$

hence, the equation sought is

$$R^2(y - y') + (P \pm 2\sqrt{MF'})(x - x') = 0.$$

Since the radical is susceptible of two signs, there may be two right lines from the same point touching the curve; their equations may separately be represented thus,

$$R^2(y - y') + (P + 2\sqrt{MF'})(x - x') = 0,$$

$$R'^2(x - x') + (P + 2\sqrt{MF'})(y - y') = 0.$$

(133.) If the point $y'x'$ be on the curve $F' = 0$, and

$$\frac{P}{R^2} = \frac{R'^2}{P} = \frac{R'}{R} = \frac{2Cx' + By' + E}{2Ay' + Bx' + D};$$

therefore the equation of a tangent to a point $y'x'$ on the curve is

$$(2Ay' + Bx' + D)(y - y') + (2Cx' + By' + E)(x - x') = 0.$$

Hence, and by (92), it follows that the ordinates to a diameter are parallel to tangents through its vertices, and that, therefore, these tangents are parallel to each other. It also follows, that the tangents through the vertices of a diameter are parallel to its conjugate.

(134.) *Def.* A right line passing through the point of contact, and perpendicular to the tangent, is called a normal.

PROP. XXXIX.

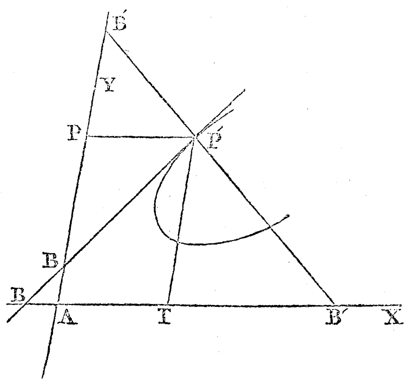
(135.) *To find the equation of the normal.*

From the equation of the tangent and the formula in (39), it may be inferred that the equation of the normal is

$$[(2\text{C}x' + \text{B}y' + \text{E}) - (2\text{A}y' + \text{B}x' + \text{D})\cos. yx](y - y')$$
$$-[(2\text{A}y' + \text{B}x' + \text{D}) - (2\text{C}x' + \text{B}y' + \text{E})\cos. yx](x - x') = 0.$$

PROP. XL.

To find the subtangent.



(136.) The portion of either axis of co-ordinates intercepted between the points B, P, where the tangent and a parallel P′P to the other axis through the point P′ of contact intersect it, is called a subtangent. In (133) the value of $(x' - x)$ corresponding to $y = 0$ is the value of the subtangent S on the axis of x, and the value of $(y' - y)$ corresponding to $x = 0$ is the subtangent S′ on the axis of y; therefore

$$\text{S} = -y' \cdot \frac{2\text{A}y' + \text{B}x' + \text{D}}{2\text{C}x' + \text{B}y' + \text{E}},$$

$$\text{S}' = -x' \cdot \frac{2\text{C}x' + \text{B}y' + \text{E}}{2\text{A}y' + \text{B}x' + \text{D}}.$$

PROP. XLI.

(137.) *To find the subnormal.*

A portion, PB′, of each axis of co-ordinates similarly situated with respect to the normal and the parallel PP′ is called a subnormal, and its value is found in the same manner from (135),

$$s = -y' \cdot \frac{(2Cx'+By'+E)-(2Ay'+Bx'+D)\cos. yx}{(2Ay'+Bx'+D)-(2Cx'+By'+E)\cos. yx},$$

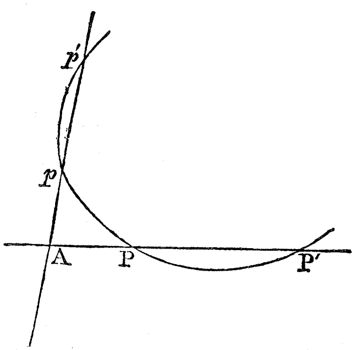
$$s' = -x' \cdot \frac{(2Ay'+Bx'+D)-(2Cx'+By'+E)\cos. yx}{(2Cx'+By'+E)-(2Ay'+Bx'+D)\cos. yx}.$$

SECTION XI.

Of the general properties of lines of the second degree.

PROP. XLII.

(138.) *If several pairs of intersecting right lines parallel to two right lines given in position meet a curve of the second degree, the rectangles under their segments intercepted between the several points of intersection and the corresponding points of occurse with the curve, will be in a constant ratio.*



Let the axis of co-ordinates be those lines which meet the curve, the points where they intersect it are found by supposing successively $y = 0$ and $x = 0$ in the general equation, and are therefore determined by the roots of

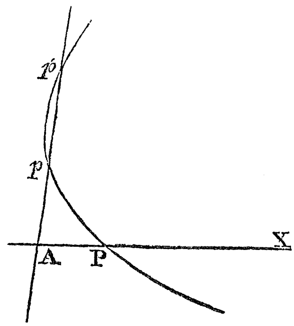
$$Cx^2 + Ex + F = 0, (2)$$
$$Ay^2 + Dy + F = 0, (3).$$

Hence, $AP \times AP' = +\frac{F}{C}$, and $Ap \times Ap' = +\frac{F}{A}$, therefore

$$\frac{\text{AP} \times \text{AP}'}{\text{A}p \times \text{A}p'} = \frac{\text{A}}{\text{C}}.$$

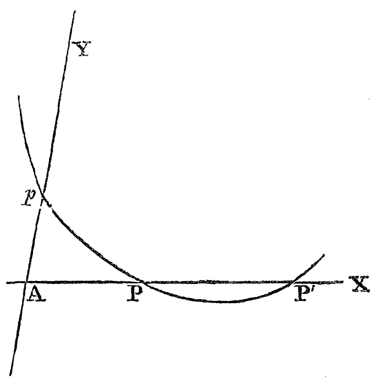
The values of A and C are not affected by a transformation of origin without a change of direction, and therefore, since the axes of co-ordinates are supposed parallel to right lines given in position, $\frac{\text{A}}{\text{C}}$ is constant.

(139.) *Cor.* 1. If the roots of (2) or (3) or both are equal, the lines AX or AY or both will be tangents, and the rectangle under the roots is the square of the tangent; hence the proposition (138) is extended to the squares of tangents intersecting secants or intersecting each other.



(140.) *Cor.* 2. If A or C = 0 in (2) or (3), the equation in which this takes place has but one root, and the secant intersects the curve in but one point.

(141.) *Cor.* 3. If C = 0, the right line AX intersects the curve but once, in this case $\text{A}p \times \text{A}p' \propto \text{AP}$.

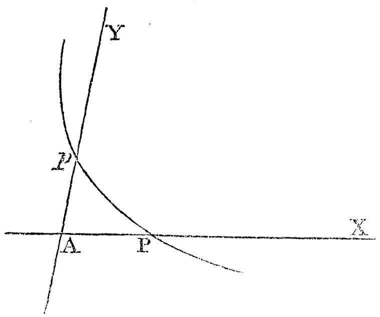


(142.) *Cor.* 4. If A = 0, in like manner Ap meets the curve but once, and

$$\text{AP} \times \text{AP}' \propto \text{A}p.$$

(143.) *Cor.* 5. If A = 0 and C = 0, each of the lines AX and AY meets the curve but once, and $\text{AP} \propto \text{A}p$.

(144.) *Cor.* 6. If when A = 0, or C = 0, B be finite, the curve must be an



hyperbola, and AY or AX is parallel to an asymptot. Hence, in an hyperbola, if AP be parallel to an asymptot, and App' be a secant parallel to a line given in position,

$$Ap \times Ap' \propto AP.$$

(145.) *Cor.* 7. But if $A = 0$ or $C = 0$, and also $B = 0$, the curve must be a parabola, and AY or AX a diameter. Hence a similar inference follows with respect to the diameter of a parabola, as the parallel to the asymptot of an hyperbola scil., if AX be the diameter, $Ap \times Ap' \propto AP$.

(146.) *Cor.* 8. If $A = 0$ and $C = 0$, the curve is an hyperbola, and the lines AX and AY are parallel to the asymptots. Hence, in this case $AP \propto Ap$.

(147.) *Cor.* 9. By (141) and (144), it appears that a parallel to the asymptot of an hyperbola and a diameter of a parabola intersect the curve but once.

(148.) *Cor.* 10. In central curves the rectangles under the segments of secants are as the squares of the diameters to which they are parallel.

(149.) *Cor.* 11. In central curves the squares of the ordinates are as the rectangles under the segments of the diameter to which they are applied.

(150.) *Cor.* 12. In a parabola the squares of the ordinates to any diameter are as the intercepts between them and the vertex of the diameter to which they are applied.

(151.) *Cor.* 13. In a circle the rectangle under the segments of secants and the squares of tangents drawn through the same point are equal.

(152.) *Cor.* 14. In central curves intersecting tangents are as the parallel diameters.

(153.) *Cor.* 15. If AP, AP′ (fig. to Art. 157) be tangents, and DCE be parallel to AP, then DC : DF : DE. For

$$DC \times DE : DP^2 :: AP'^2 : AP^2 :: DF^2 : DP^2.$$

PROP. XLIII.

(154.) *To express the equation of a line joining the points of contact of two tangents drawn from a given point.*

In the equation found in (133), let yx be considered constant, and the co-ordinates $y'x'$ of the point of contact variable, and their denominations consequently changed, the equation becomes

$$(2Ay' + Bx' + D)y + (2Cx' + By' + E)x + Dy' + Ex' + 2F = 0,$$

by considering that the point yx must fulfil the conditions of the general equation of the curve.

PROP. XLIV.

(155.) *The line joining the points of contact is an ordinate to the diameter passing through the point of intersection of the tangents.*

For the equation found in (156) is that of a line parallel to the line whose equation is found in (92), as that of the ordinates of a diameter through $y'x'$.

PROP. XLV.

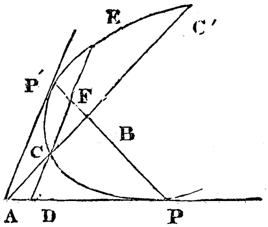
(156.) *The locus of the intersection of tangents through the extremities of a chord parallel to a line given in position is the diameter to which that chord is an ordinate.*

For tangents through the extremities of any ordinate intersect on the diameter to which it is an ordinate.

PROP. XLVI.

(157.) *Every secant drawn from the point of intersection of two tangents, and meeting the curve in two points, is cut harmonically by the curve and the line joining the points of contact.*

Suppose the intersection A of the tangents, the origin, and



the secant (ACC'), the axis of x; hence,

$$AC \times AC' = +\frac{F}{C},$$

$$\text{and } AC + AC' = -\frac{E}{C}.$$

In the equation (154) of the line joining the points of contact, let $y=0$, which gives $AB = -\frac{2F}{E}$, therefore $AB = \frac{2AC \times AC'}{AC + AC'}$, and hence, AC, AB, AC$'$ are in harmonical progression.

(158.) *Cor.* If ACB intersect the curve in but one point C, AB will be bisected at C, since in that case AC$'$ is infinite, and therefore the ratio of AC to AB is 1 : 2. This takes place when AC is the diameter of a parabola or parallel to the asymptote of an hyperbola.

PROP. XLVII.

(159.) *To find the locus of the intersection of tangents through the extremities of a chord passing through a given point.*

In the equation found in (154), let the variables yx be changed into constant co-ordinates ($y'x'$) of the given point, and let the co-ordinates $y'x'$ of the point of intersection of the tangents be changed into variables yx, and the equation becomes

$$(2Ay' + Bx' + D)y + (2Cx' + By' + E)x + Dy' + Ex' + 2F = 0.$$

Hence, the locus sought is a right line parallel to the ordinates of the diameter passing through the given point, and intersects that diameter when the tangents through the extremities of the ordinate through $y'x'$ intersect it.

(160.) *Cor.* Hence, if the given point be upon the axis the locus will be a right line perpendicular to the axis.

PROP. XLVIII.

(161.) *The lines joining the points of contact of every pair of tangents drawn from points in any right line intersect each other at the same point.*

For if the diameter be drawn whose ordinates are parallel to this right line, and from their point of intersection two tangents be drawn, the point at which the line joining the points of contact of these tangents intersect the diameter, is that through which the line joining the points of contact of every such system of tangents pass.

(162.) *Def.* Any diameter being axis of x, and a tangent through its vertex axis of y, the equation is

$$\text{A}y^2 + \text{C}x^2 + \text{E}x = 0.$$

The line representing $-\frac{\text{E}}{\text{A}}$ is called the *parameter* of the diameter, which coincides with the axis of x.

To express the equation of the curve in terms of the parameter p we have

$$p = -\frac{\text{E}}{\text{A}} = \frac{2\text{B}'^2}{\text{A}'}$$

by which substitution the equation becomes

$$y^2 + \frac{p}{2\text{A}}x^2 - px = 0.$$

It appears that the *parameter* of any diameter of an *ellipse* or *hyperbola* is a third proportional to the diameter itself, and the diameter conjugate to it.

(163.) *Def.* The *parameter* of the *axis* is called the *principal parameter.*

(164.) *Def.* A point of the *axis*, whose ordinate is equal to half the *principal parameter*, is called the *focus*.

PROP. XLIX.

To find the distance of the focus from the vertex.

Let the equation be

$$\text{A}y^2 + \text{C}x^2 + \text{E}x = 0.$$

In this, $-\frac{E}{2A}$ being substituted for y, the result is

$$4ACx^2 + 4AEx + E^2 = 0,$$

the roots of which are

$$x = -\frac{E}{2C}\left\{1 \mp \sqrt{\frac{A-C}{A}}\right\}.$$

The value of x expressed as a function of the semiaxes $A'B'$ is

$$x = A'(1 \mp \sqrt{1 - \frac{B'^2}{A'^2}}) = A' \mp \sqrt{A'^2 - B'^2}.$$

If the curve be an *ellipse*, A'^2 and B'^2 have the same sign, and therefore the value of x is real only where $A' > B'$. Hence, there are no *foci* on the lesser axis of an ellipse, and there are two on the greater axis, equally distant from the centre, and the square of their distance (c) from the centre is equal to the difference of the squares of the semiaxes; *i. e.*

$$c^2 = A'^2 - B'^2.$$

If $A' = B'$, the distance between the foci vanishes, and they both coincide with the centre, which takes place when the *ellipse* is a *circle*.

The quantity $\frac{c}{A'}$ is called the *eccentricity* of the ellipse, and therefore a circle is an ellipse whose *eccentricity* $= 0$.

If the curve be an hyperbola, A'^2 and B'^2 have different signs. In this case, if $A'^2 > 0$ and $B'^2 < 0$, the value of x is real, and $c = \sqrt{A'^2 + B'^2}$; but if $A'^2 < 0$ and $B'^2 > 0$, the value of x is impossible. Hence, in an *hyperbola* there are no *foci* on the *one axis*, but two on the other equally distant from the centre; and the square of their distance from the centre is equal to the sum of the squares of the semi-axes; *i. e.*

$$c^2 = A'^2 - B'^2.$$

If the curve be a *parabola*, $C = 0$; therefore one value of x becomes infinite, and the other is $-\frac{E}{4A} = \frac{1}{4}p$, where p expresses the the *principal parameter*.

Hence, in a *parabola* there is but *one focus* on the axis at a distance from the vertex equal to a fourth of the principal parameter.

The axis passing through the foci of an *ellipse* or *hyperbola* is the *transverse axis*, and the other the *conjugate axis*.

(165.) *Def.* The right line, which is the locus of the intersections of tangents drawn through the extremities of any chord passing through the focus, is called the *directrix*.

PROP. L.

(166.) *To determine the position of the directrix.*

The equation related to an axis and a tangent through its vertex being

$$\text{A}y^2 + \text{C}x^2 + \text{E}x = 0;$$

and the co-ordinates of the foci being

$$y' = 0, x' = -\frac{\text{E}}{2\text{C}}\left\{1 \pm \sqrt{\frac{\text{A} - \text{C}}{\text{A}}}\right\}$$

the equation of the *directrix* must be (159)

$$x - \frac{\text{E}}{2\text{C}}\left\{\sqrt{\frac{\text{A}}{\text{A}-\text{C}}} - 1\right\} = 0;$$

but if $\text{C} = 0$, the equation of the locus is

$$x - x' = 0, \text{ or } x - \frac{p}{4} = 0.$$

If the curve be the *ellipse* or *hyperbola*, the equation of the *directrix* expressed as a function of the *axes*, is

$$x = \text{A} \pm \frac{\text{A}^2}{c}.$$

Hence the distance of the *directrix* from the centre is $\frac{\text{A}^2}{c}$.

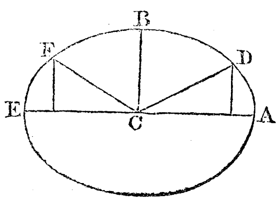
An *ellipse* or *hyperbola* has therefore two directrices equally distant from the centre, and perpendicular to the transverse axis, and a parabola but one, which is also perpendicular to the axis.

SECTION XII.

The properties of the ellipse and hyperbola.

PROP. LI.

(167.) *An ellipse or hyperbola being expressed by an equation related to its axes as axes of co-ordinates, to express the lengths of any semidiameter, and its semiconjugate, in terms of the co-ordinates of its vertex.*



Let $y'x'$ be the vertex of the given semidiameter $CD = A'$.

$$A'^2 = y'^2 + x'^2;$$

but by equation of curve

$$A^2y'^2 + B^2x'^2 = A^2B^2;$$

hence,

$$A'^2 = \frac{A^2B^2 + c^2x'^2}{A^2} = B^2 + e^2x'^2.$$

Where $c = \sqrt{A^2 - B^2}$, and $e = \frac{c}{A}$, the distance of the focus from the centre.

The equation of CD being $yx' - y'x = 0$, that of CF its conjugate must be, (92),

$$A^2y'y + B^2x'x = 0.$$

By this equation, and that of the curve, the co-ordinates of their intersection are,

$$y = \frac{Bx'}{A}, x = \frac{Ay'}{B}.$$

Therefore, if $CF = B'$,

$$B'^2 = \frac{B^2x'^2}{A^2} + \frac{A^2y'^2}{B^2} = \frac{A^4 - c^2x'^2}{A^2},$$

$$\text{or } B'^2 = A^2 - e^2x'^2.$$

In the ellipse $A > ex'$, and in the hyperbola $A < ex'$;

hence, in ellipse A'^2 and B'^2 are both positive; but in the hyperbola B'^2 is negative, ∵ B' is impossible, ∵ in each system of conjugate diameters of an hyperbola, one is a transverse diameter, and the other a second diameter.

PROP. LII.

(168.) *In an ellipse, the sum of the squares of any system of conjugate diameters is equal to the sum of the squares of the axes; and, in an hyperbola, the difference of the squares is equal to the difference of the squares of the axes.*

For, by adding the values of A'^2 and B'^2, in (167), the result for ellipse is,

$$A'^2 + B'^2 = A^2 + B^2.$$

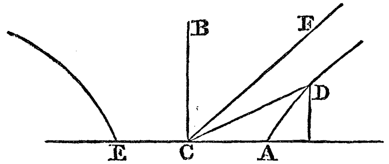
And, since in hyperbola B'^2 and B^2 are both negative,

$$A'^2 - B'^2 = A^2 - B^2.$$

(169.) *Cor.* Hence, if the axes of an hyperbola be equal, every system of conjugate diameters will be equal: such a curve is called an *equilateral hyperbola.*

PROP. LIII.

(170.) *To find the relation between the angles, at which any two conjugate diameters are inclined to the transverse axis.*



By (167) the equations of CD and CF are,

$$yx' - y'x = 0,$$

$$A^2y'y + B^2x'x = 0,$$

hence,

$$\text{tang. DCA} = \frac{y'}{x'}, \text{ and tang. FCA} = -\frac{B^2x'}{A^2y'}; \text{ therefore,}$$

$$\text{tang. DCA} \times \text{tang. FCA} = -\frac{B^2}{A^2}.$$

In the *ellipse*, therefore, the product of the tangents is negative, and therefore they must always have different signs, ∵ the angles must have different affections. Hence if CD lies in the angle BCA, CF must lie in the angle BCE; and if CD lie in BCE, CF must lie in BCA.

In the *hyperbola* the same product is positive, since B'^2 is negative, and therefore CD and CF lie both in the same angle.

(171.) *Cor.* 1. In an *ellipse*, if a second system of conjugate diameters were at right angles, it would be a circle; for in this case tang. DCA. tang. FCA $+ 1 = 0$, therefore $B^2 = A^2$, therefore the curve would be a circle.

(172.) *Cor.* 2. In an *hyperbola*, if $B^2 = A^2$, tang. DCA. tang. FCA $= 1$, hence in an *equilateral hyperbola*, the conjugate diameters make complemental angles with the transverse axis.

PROP. LIV.

(173.) *To express the polar equation of an ellipse, or hyperbola, the centre being the pole, and the angle being measured from the transverse axis.*

By (167) $A'^2 = B^2 + e^2x^2$, for A' substitute z, and for x z cos. ω, and the equation will become $z^2 = \dfrac{B^2}{1 - e^2 \cos.^2 \omega}$, which is the equation required.

PROP. LV.

(174.) *Diameters which make equal angles with the transverse axis are equal.*

For z in the last Prop. is a function of cos.^{2} ω, and if two diameters make equal angles, the angles which they form, when measured in the same direction, are supplemental: the squares of their cosines are equal.

PROP. LVI.

(175.) *To find the greatest and least diameters.*

The value of z in the polar equation is a maximum, when e cos. ω is a maximum. In the ellipse $e < 1$, $\because e$ cos. $\omega < 1$, $\because 1 - e$ cos. ω is a minimum, when cos. ω a maximum, *i. e.* when $\omega = 0$. Also z is a minimum, when cos. ω a minimum, *i. e.* when $\omega = \frac{\pi}{2}$. Hence, in an *ellipse*, the greatest diameter is the transverse axis, and the least the conjugate axis.

In the *hyperbola*, z will be infinite when $\cos.^2 \omega = \frac{1}{e^2} = \frac{A^2}{A^2 + B^2}$. Between this value of cos. ω, and cos. $\omega = 0$, the values of z are impossible, and between it and unity they are continually diminishing.

Hence, if a line be drawn through the centre, represented by the equation $y -$ tang. $\omega \, . \, x = 0$, or $Ay - Bx = 0$, all the diameters between this line and the transverse axis meet the curve, and all between it and the conjugate are second diameters. Hence the least transverse diameter of an hyperbola is the *transverse axis*.

(176.) *Cor.* 1. The line represented by the equation,

$$Ay - Bx = 0,$$

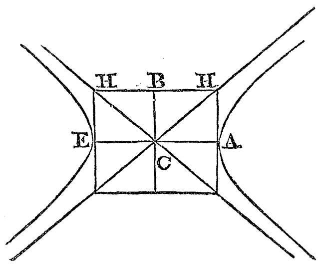
is an *asymptote* (105) for similar reasons: the other is,

$$Ay + Bx = 0.$$

PROP. LVII.

(177.) *The asymptots of an hyperbola make equal angles with the transverse axis, and are the diagonals of a rectangle, formed by lines drawn through the vertices of each parallel to the other.*

For the tangents of the angles which they make with the



transverse axis measured in the same direction, are $+\frac{B}{A}$, and $-\frac{B}{A}$, which are therefore supplemental, and therefore the angles HCA, H′CE are equal, ∴ CH, CH′ are the asymptots.

PROP. LVIII.

(178.) *To find whether any and what system of conjugate diameters in an ellipse are equal.*

In order to be equal they must, by (174), make equal angles with the axis, ∴ tang. DCA = tang. FCA, but

tang. DCA . tang. FCA $= -\frac{B^2}{A^2}$, ∴ tang. DCA $= \frac{B}{A}$, and

tang. FCA $= -\frac{B}{A}$, ∴ the equation of the equal conjugate diameters are,

$$Ay - Bx = 0, \quad Ay + Bx = 0.$$

(179.) *Cor.* 1. The equal conjugate diameters are the diagonals of the rectangle, formed by tangents through the extremities of the axes, and are in that respect analogous to the asymptots of an hyperbola.

(180.) *Cor.* 2. If an ellipse and hyperbola have the same axes, the equal conjugate diameters of the ellipse are the asymptots of the hyperbola.

(181.) *Cor.* 3. The equation of the ellipse, referred to equal conjugate diameters as axes of co-ordinates, is $y^2 + x^2 = A'^2$, being analogous to that of the circle.

(182.) *Cor.* 4. The co-ordinates of the vertices of equal conjugate diameters are found from the equations

$$Ay - Bx = 0, \text{ and } A^2y^2 + B^2x^2 = A^2B^2.$$

They are, therefore,

$$y = \pm \frac{\text{B}}{\sqrt{2}}, \qquad x = \pm \frac{\text{A}}{\sqrt{2}}.$$

(183.) *Cor.* 5. If A' be one of the equal conjugate diameters,

$$\text{A}'^2 = \frac{\text{A}^2 + \text{B}^2}{2}.$$

(184.) *Cor.* 6. The value of x in the cor. 4. being independent of B, and that of y independent of A, shows, that if one axis of an ellipse is given the locus of the extremities of equal conjugate diameters are parallel lines.

PROP. LIX.

(185.) *To find when the rectangle, under a system of conjugate diameters, is a maximum and minimum.*

By (167), $\text{A}'^2 = \text{B}^2 + e^2x^2$, $\text{B}'^2 = \text{A}^2 - e^2x^2$, $\therefore \text{A}'^2\text{B}'^2 = (\text{A}^2 - e^2x^2)(\text{B}^2 + e^2x^2)$.

For the *ellipse*, the factors of this product have the same sign, $\because$ their sum is constant, $\therefore$ the product is a *maximum* when they are equal; hence, the *major limit* is the equal conjugate diameters.

It is evident, also, the product is a *minimum* when they are most unequal, *i. e.* when x is a maximum, $\therefore x = \text{A}$; hence the *minor limit* is the axes.

For the *hyperbola*, the factors have different signs, therefore their difference is given, consequently there is no *major limit*. The *minor* limit is found by taking x a minimum, *i. e.* $x = \text{A}$, $\therefore$ the minor limit is the axes.

PROP. LX.

(186.) *To find the limits of the sum and difference of a system of conjugate diameters.*

Let $\text{S}^2 = \text{A}'^2 + \text{B}'^2 + 2\text{A}'\text{B}'$, and $\text{D}^2 = \text{A}'^2 + \text{B}'^2 - 2\text{A}'\text{B}'$.

In an ellipse, $\text{A}'^2 + \text{B}'^2$ is a given magnitude, $\therefore$ S is a maximum or minimum at the same time with $A'B'$; hence, the major limit is the equal conjugate diameters, and the minor the axes.

For the same reason the major limit of D is the difference of the axes, and it has no minor limit.

In the *hyperbola*, $A'^2 - B'^2$ is constant, and ∵ since A' increases without limit, B' must also increase without limit, and ∵ S must increase without limit.

Also, since $SD = A'^2 - B'^2$, and S increases without limit, D must diminish without limit.

Also S is a minimum where A' and B' are so; *i. e.* where they are the axes. It is evident that D is at the same time a maximum.

PROP. LXI.

(187.) *A system of conjugate diameters being axes of co-ordinates, to find the equation of a tangent through a given point.*

The given point being $y'x'$, the equation sought by substituting for $A'^2y'^2 + B^2x'^2$, its value $A'^2B'^2$ is,

$$A'^2y'y + B'^2x'x = A'^2B'^2.$$

PROP. LXII.

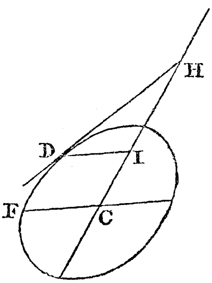
(188.) *To express the subtangent and subnormal of an ellipse and hyperbola related to a system of conjugate diameters as axes of co-ordinates.*

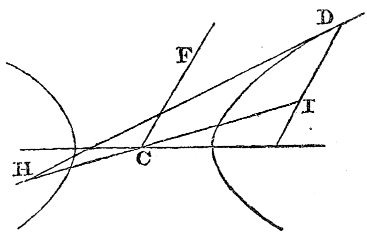
Let S = subtangent, s = the subnormal, and θ = the angle under the conjugate diameters.

By the formula, in (136), $S = \frac{A'^2y'^2}{B'^2x'}$, $s = \frac{A'^2y'^2 \cos.\theta - B'^2y'x'}{A'^2y' - B'^2x' \cos.\theta}$.

For the ellipse $A'^2y'^2 = B'^2(A'^2 - x'^2)$ ∵ $S = \frac{A'^2}{x'} - x'$. And if A′, B′, be the axes of the curve $s = -\frac{B'^2 x'}{A'^2}$. For the hyperbola $A'^2 y'^2 = B'^2(x'^2 - A'^2)$, $\because$ $s = x' - \frac{A'^2}{x'}$; and if A′, B′ be the axes of the curve $s = \frac{B'^2 x'}{A'}$.

(189.) *Cor.* 1. Since in ellipse $CH = x' + s = \frac{A'^2}{x'}$, and in hyperbola $CH = x' - s = \frac{A'^2}{x'}$. It follows that, in an ellipse or hyperbola, if a tangent and ordinate be drawn from any point D to meet the same diameter, the semidiameter is a mean proportional between the parts of the diameter intercepted between them and the centre.





(190.) *Cor.* 2. The value of s, being independent of B′, is the same for any number of ellipses or hyperbolas described on 2A′ as diameter, and having the conjugate diameters coincident with CF.

PROP. LXIII.

(191.) *To express the magnitude of the normal related to the axes, as axes of co-ordinates.*

If s be the subnormal, and $y'x'$ the point on the curve, N being the normal, $N^2 = s^2 + y'^2$; but by the last Prop.

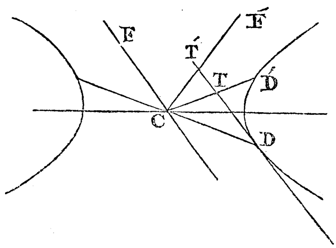
$$s^2 = \frac{B^4 x'^2}{A^4}, \quad \because \quad N^2 = \frac{A^4 y'^2 + B^4 x'^2}{A^4}.$$

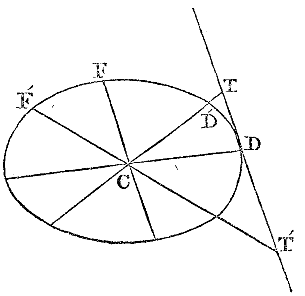
But $A^2 y'^2 = A^2 B^2 - B^2 x'^2$, $\because$ $A^4 y'^2 + B^4 x'^2 = B^2(A^4 - c^2 x'^2)$.

$$N^2 = \frac{B^2}{A^2}(A^2 - ex'^2).$$

PROP. LXIV.

(192.) *Any semidiameter is a mean proportional between the parts of the tangent which is parallel to it, intercepted between the point of contact, and any system of conjugate diameters.*





Let the semidiameter CD through the point of contact, and its conjugate CF be the axes of co-ordinates, and let CD$'$, and CF$'$ be any other system of conjugate diameters.

The point D$'$ being $x'y'$, the equation of CD$'$ is

$$yx' - y'x = 0,$$

and that of CF$'$ is

$$A'^2 y'y + B'^2 x'x = 0.$$

In each of these, let A$'$ be substituted for x, and the corresponding values of y are,

$$DT = \frac{A'y'}{x}, \text{ and } DT' = -\frac{B'^2 x'}{A'y'}.$$

Hence, $$DT \times DT' = - B'^2.$$

The sign being negative for the ellipse, and positive for the hyperbola, shows that they are at different sides of CD in the one, and on the same in the other.

PROP. LXV.

(193.) *The triangles formed by ordinates to any diameter* CD *from the extremities of a system of conjugate diameters, and the intercepts between them and the centre are equal.*

For if the point D$'$ be $y'x'$, the co-ordinates yx of the point

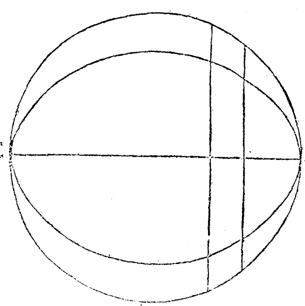
F' are by (167), $y = \frac{B'x'}{A'}$, $x = \frac{A'y'}{B'}$.

Since the angles $y'x'$, and yx are supplemental, their sines are equal, therefore the area of the one triangle is $\frac{y'x' \sin. yx}{2}$, and of the other is

$$\frac{yx \sin. yx}{2} = \frac{B'x'}{A'} \times \frac{A'y'}{B'} \cdot \frac{\sin. yx}{2} = \frac{y'x' \sin. yx}{2}.$$

PROP. LXVI.

(194.) *If on the axes of an ellipse as diameters circles be described, that on the transverse axis will be entirely outside the ellipse, touching it at the extremities of this axis; that on the conjugate will be entirely within the ellipse, touching at the extremities of its conjugate axis.*



Let A be the semitransverse axis, and B the semiconjugate; let Y be the ordinate of the large circle on A,

$$Y^2 = A^2 - x^2;$$

but in the ellipse

$$y^2 = \frac{B^2}{A^2}(A^2 - x^2) = \frac{B^2 Y^2}{A^2},$$

and since $A^2 > B^2$, $Y > y$, therefore every part of the circle must be outside the ellipse.

In like manner, let X be the ordinate of the diameter $2B$ of the other circle.

$$X^2 = B^2 - y^2;$$

but in the ellipse $x^2 = \frac{A^2}{B^2}(B^2 - y^2) = \frac{A^2}{B^2} X^2$. And since $B < A$, $X < x$, therefore every part of the circle lies within the ellipse.

It is obvious that they touch as stated above.

(195.) *Cor.* $\frac{Y}{y} = \frac{A}{B}$, all ordinates to the diameter AE of the circle are cut in the same ratio by the ellipse.

PROP. LXVII.

(196.) *If a circle be orthographically projected on a plane, to which it is inclined at an angle* (θ), *its projection will be an ellipse.*

The projection of the diameter of the circle, which is parallel to the plane, is a line on the plane equal and parallel to it. Ordinates being supposed to be drawn to this diameter of the circle, their projections will be perpendicular to the projection of the diameter, and have to the ordinates themselves the ratio of the cosine of angle of projection to radius, which being a constant ratio, the locus of their extremities must be an ellipse, by (195.)

PROP. LXVIII.

(197.) *The angles in the semiellipse, whose base is the transverse axis, are obtuse; those in the semiellipse, whose base is its conjugate, are acute.*

The proof is obvious from (194), and the angle in the semicircle being right.

PROP. LXIX.

(198.) *To find the limits of the angle inscribed in a semiellipse on either axis.*

Let any point on the ellipse be $y'x'$, the equation of two lines passing through the extremities of the axis and that point are,

$$y(x'-A) - y'(x-A) = 0, \qquad y(x'+A) - y'(x+A) = 0.$$

By the formula, in (31),

$$\text{tang. } ll' = -\frac{2Ay'}{y'^2 + x'^2 - A^2}.$$

But by the equation of curve

$$x'^2 = \frac{A^2}{B^2}(B^2 - y'^2), \because \text{tang. } ll' = -\frac{2AB^2}{(A^2 - B^2)y'}.$$

If $A > B$ this is negative, therefore the angle being obtuse, is a maximum when its tangent is a minimum, which is when $y = B$, since AB^2 is invariable. But if $A < B$, the angle being acute must be a minimum in the same case.

Hence in a semiellipse, whose base is a transverse axis, the greatest angle which can be inscribed is that whose vertex is at the extremity of the conjugate axis. And in a semiellipse, whose base is the conjugate axis, the least angle which can be inscribed is that whose vertex is at the extremity of the transverse axis.

PROP. LXX.

(199.) *If two right lines be drawn from the extremities of a diameter of an ellipse or hyperbola to any point on the curve, the diameters parallel to these are conjugate.*

In order that the two lines through the centre,

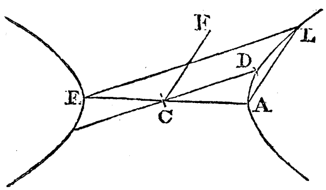
$$y - ax = 0, y - a'x = 0,$$

should be conjugate diameters, the conditions $aa' = -\frac{B^2}{A^2}$ must be fulfilled. But if the two right lines connect a point in the curve with the extremities of a diameter, their equations related to that diameter, and its conjugate are,

$$y(x' - A) - y'(x - A) = 0, \ y(x' + A) - y'(x + A) = 0,$$

and, in this case, $aa' = \frac{y'^2}{x'^2 - A^2}$, and by the equation of the curve $y'^2 = \frac{B^2}{A^2}(A^2 - x'^2)$, $\because \frac{y'^2}{x'^2 - A^2} = -\frac{B^2}{A^2}$, $\because aa' = -\frac{B^2}{A^2}$; hence lines parallel to these must be conjugate diameters.

(200.) *Cor.* 1. Hence is obvious a geometrical method of drawing a diameter conjugate to a given one.



Let CD be the given diameter, and AE any other; from E draw a line EL parallel to CD, join AL, and through C draw a diameter, CF, parallel to AL; CF is the semidiameter conjugate to CD.

(201.) *Cor.* 2. To find a system of conjugate diameters, which shall contain a given angle.

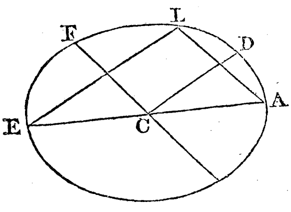
On the transverse axis describe a segment of a circle, which shall contain the given angle, and join the extremities of the axis with the point where this segment intersects the ellipse, diameters parallel to these lines will be conjugate, and contain the given angle.

(202.) *Cor.* 3. The equal conjugate diameters are parallel to the lines joining the extremities of the axes.

(203.) *Cor.* 4. The property expressed in the proposition furnishes a geometrical method of drawing a tangent at a given point. Find, as in Cor. 1, the diameter conjugate to that through the point, and a line through the point parallel to this is the tangent.

PROP. LXXI.

(204.) *To find the most oblique conjugate diameters.*



Let a perpendicular P be drawn from the extremity D of any diameter A′ on its conjugate, sin. DCF$(\theta) = \frac{P}{A'}$. But the equation of CF being

$$A^2 y' y + B^2 x' x = 0.$$

By formula (50), $P = -\frac{A^2 y'^2 + B^2 x'^2}{\sqrt{A^4 y'^2 + B^4 x'^2}}$. But by the equation of the curve,

$$A^2y'^2 = A^2B^2 - B^2x'^2, \because A^4y'^2 + B^4x'^2 = B^2(A^4 - c^2x'^2) = A^2B^2(A^2 - e^2x'^2), \because$$

$$P = -\frac{AB}{\sqrt{A^2 - e^2x^2}}.$$

But, by (167),

$$B'^2 = A^2 - e^2x'^2, \because \sin.\,\theta = \frac{AB}{A'B'}.$$

Hence the sine θ is a minimum, when $A'B'$ is a maximum, *i. e.* when $A' = B'$, hence the most oblique conjugate diameters are those which are equal.

(205.) *Cor.* Since a tangent through the vertex of any diameter is parallel to its conjugate, the value of the sine of the angle under any diameter and the tangent through its vertex is, $\frac{AB}{A'B'}$.

PROP. LXXII.

(206.) *The rectangle under the normal to any point, and the transverse axis is equal to the rectangle under the conjugate axis, and the semi-diameter conjugate to that passing through the point.*

For, by (191), $N = \frac{B}{A}\sqrt{A^2 - e^2x^2}$, and by (167),

$$B' = \sqrt{A^2 - e'^2x^2} \because 2AN = 2BB'.$$

PROP. LXXIII.

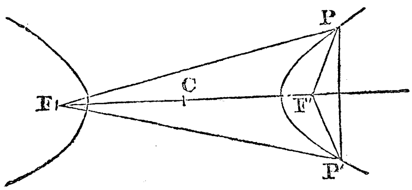
(207.) *To find the magnitude of a parallelogram formed by tangents through the vertices of a system of conjugate diameters.*

Since the sides of the parallelogram are parallel respectively to the conjugate diameters $2A'$, $2B'$, they must be equal to them, and inclined at the same angle θ. Hence the area of the parallelogram is $4A'B'\sin.\,\theta$, which, by (204), is equal

4AB. Hence, all parallelograms formed by tangents through the vertices of a system of conjugate diameters are equal the rectangle under the axes.

PROP. LXXIV.

(208.) *To find the distance of any point in an ellipse or hyperbola from the focus.*

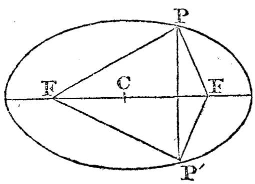


The axes of the curve being assumed as axes of co-ordinates, the equation is,

$$A^2y^2 + B^2x^2 = A^2B^2.$$

Let D be the distance sought, $D^2 = y^2 + (x - c)^2$,

$\because A^2D^2 = A^2y^2 + A^2(x - c)^2 = A^2B^2 - B^2x^2 + A^2(x - c)^2$;



in which substituting c^2 its value, and taking the square root of the result, $AD = \pm (A^2 - cx)$,

$$\therefore D = \pm (A - \frac{c}{A}x) = \pm (A - ex),$$

where e expresses the eccentricity.

For the same value of x, there are, therefore, two equal values of D, which is what should be expected; y^2 was eliminated, which has two equal roots $\pm y$, and the two values of D correspond to these. The two values of D are represented in the figures by FP and FP'.

If c be taken negatively, the distance D' will be that of the point from the focus F' on the negative side of the centre. Hence,

$$D' = \pm (A + \frac{c}{A}x) = \pm (A + ex).$$

In an ellipse D and D' must have the same sign, for c and x being both less than A, $\frac{c}{A}x$ must be less than it also.

But in a hyperbola, since c and x are both greater than A, $\frac{cx}{A}$ must also be greater than A; ∵ in an hyperbola D and D′ have different signs. By the solution of the equation,

$$A^2y^2 + B^2x^2 = A^2B^2;$$

and from the consideration that $B^2 > 0$ in the ellipse, and $B^2 < 0$ in the hyperbola, it is obvious that any value of $x > A$ in the ellipse, and $< A$ in the hyperbola, would render y impossible.

PROP. LXXV.

(209.) *In an ellipse the sum of the distances of any point from the foci, and in an hyperbola the difference of those distances, are respectively equal to the transverse axis.*

For adding the values of D and D′ in the last Prop.

$$D \pm D' = 2A,$$

D′ being positive for the ellipse, and negative for the hyperbola.

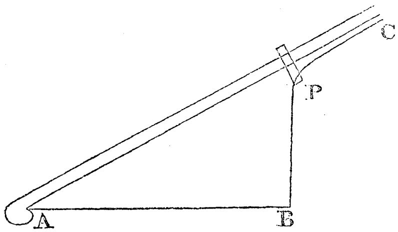
(210.) *Cor.* 1. Hence, an ellipse is the locus of the vertex of a triangle, of which the base and sum of the sides are given; and an hyperbola is the locus, when the base and difference of the sides is given.

PROP. LXXVI.

(211.) *To describe an ellipse and hyperbola mechanically.*

1°. Let the extremities of a cord be fixed to two points, a pencil looped in the cord, moved so as continually to keep the cord stretched, will describe an ellipse of which the points are the foci, and the length of the cord the transverse axis.

2°. Let one extremity A of a straight ruler be fixed, so that the ruler can move round it; in the same plane, to another point B let the extremity of a cord be fixed. The



ruler being turned so as to pass through the point B, let the cord pass through a ring attached to the ruler at P, and capable of sliding upon it, and be fastened to it at any distant point C. The ruler being moved in the same plane round the point A, a pencil attached to the sliding ring at the point P will describe an hyperbola.

PROP. LXXVII.

(212.) *To express the polar equation, the focus being the pole, and the transverse axis the axis from which the angles are measured.*

For the value of D found in (208), let z be substituted, and $z \cos.\omega + c$ for x; the result after reduction is

$$z = \frac{A(1-e^2)}{1+e\cos.\omega};$$

or since $(1-e^2) = \frac{B^2}{A^2}$, $\because A(1-e^2) = \frac{B^2}{A} = \frac{p}{2}$; $\because$

$$z = \frac{p}{2(1+e\cos.\omega)}.$$

If the angle ω be measured from any right line making with the transverse axis an angle ϕ,

$$z = \frac{A(1-e^2)}{2\{1+e\cos.(\phi-\omega)\}} = \frac{p}{2\{1+e\cos.(\phi-\omega)\}}.$$

PROP. LXXVIII.

(213.) *The rectangle under the distances of any point from the foci is equal to the square of the semidiameter conjugate to that passing through the point.*

For from (208) $DD' = A^2 - e^2x^2$, and by (167),

$$B'^2 = A^2 - e^2x^2, \because DD' = B'^2.$$

PROP. LXXIX.

(214.) *To find the length of a perpendicular* P' *from the focus on a tangent through any point* $(y'x')$.

The equation of a tangent being

$$A^2y'y + B^2x'x - A^2B^2 = 0,$$

the value of P' is found by (50),

$$P' = -B^2 \cdot \frac{A^2 - cx'}{(A^4y'^2 + B^4x'^2)^{\frac{1}{2}}}.$$

But by the equation of the curve, it appears that

$$A^4y'^2 + B^4x'^2 = P^2A^2(A^2 - e^2x'^2). \quad \text{Hence,}$$

$$P' = -B\left(\frac{A - ex'}{A + ex'}\right)^{\frac{1}{2}}.$$

If c be taken negatively, the length of the perpendicular P'' from the other focus on the tangent is,

$$P'' = -B\left(\frac{A + ex'}{A - ex'}\right)^{\frac{1}{2}}.$$

PROP. LXXX.

(215.) *The rectangle under the perpendiculars from the foci on a tangent through any point is equal to the square of the semiconjugate axis.*

For by the last Prop.

$$P'P'' = B^2.$$

PROP. LXXXI.

(216.) *The perpendiculars from the foci on a tangent through any point are as the distances of that point from the focus.*

For, from (214),

$$\frac{P'}{P''} = \frac{A - ex'}{A + ex'} = \frac{D}{D'}.$$

PROP. LXXXII.

(217.) *The lines connecting any point with the foci are equally inclined to the tangent.*

For the angles at which D and D' are inclined to the tangent being θ and θ',

$$\sin. \theta = \frac{P'}{D} \text{ and } \sin. \theta' = \frac{P''}{D'};$$

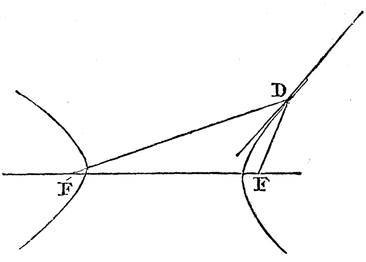
but by last Prop.

$$\frac{P'}{P''} = \frac{D}{D'}, \therefore \frac{P'}{D} = \frac{P''}{D'}, \therefore \sin. \theta = \sin. \theta'.$$

(218.) *Cor.* 1. $\text{Sin.}^2 \theta = \frac{P'P''}{DD'}$; but by (215), $P'P'' = B^2$, and $DD' = B'^2$ by (213);

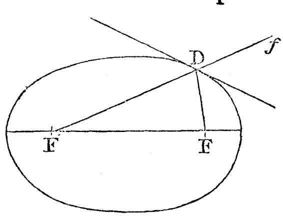
$$\therefore \sin. \theta = \frac{B}{B'}.$$

(219.) *Cor.* 2. The normal bisects the angle under the focal distances.



(220.) *Cor.* 3. The property expressed in the Prop. points out a geometrical method of drawing a tangent to a point on the curve.

For let lines DF, DF′, be drawn from the point to the foci, and if the curve be an ellipse DF′ produced, the line which bisects the angle FD*f* is a tangent. If the curve be an hyperbola, the line which bisects FDF′ is the tangent.



(221.) *Cor.* 4. If one of the foci be a point from which rays emerge, which obey the same law of reflection as those of light, and that the curve be a reflecting substance, the reflected rays, if the curve be an ellipse, will converge to the other focus; and if it be an hyperbola, they will diverge from it. It is from this property that the foci have received their name.

PROP. LXXXIII.

(222.) *A line being drawn from the focus to the point of contact of a tangent, and a line from the centre parallel to it, to find the length of the latter intercepted between the centre and the tangent.*

Let the line sought be z, and the angle it makes with the tangent $= \theta$, and the perpendicular from centre on the tangent $=$ P.

Hence, $z = \frac{\text{P}}{\sin.\theta}$; but $\text{P} = \frac{\text{AB}}{\text{R}'}$ by (203), and by (218),

$$\sin.\theta = \frac{\text{B}}{\text{B}'}, \quad \therefore z = \text{A}.$$

Hence, the locus of the intersection of this line with the tangent is the periphery of a circle described on the transverse axis as diameter.

PROP. LXXXIV.

(223.) *To find the locus of the intersection of a tangent and a right line perpendicular to it passing through the focus.*

The equation of the tangent is

$$\text{A}^2y'y + \text{B}^2x'x = \text{A}^2\text{B}^2.$$

The equation of the perpendicular is

$$\text{A}^2y'x - \text{B}^2x'y = \text{A}cy'.$$

Eliminating $y'x'$, observing the condition,

$$\text{A}^2y'^2 + \text{B}^2x'^2 = \text{A}^2\text{B}^2,$$

and arranging the terms according to the dimensions of y, we have

$$y^4+(2x^2-2cx-\text{B}^2)y^2+(x^4-2cx^3-\text{B}^2x^2+2\text{A}^2cx-\text{A}^2c^2)=0,$$

which resolved, gives

$$y^2 = \frac{-2x^2+2cx+\text{B}^2 \pm (2cx+\text{B}^2-2\text{A}^2)}{2},$$

which gives the two equations,

$$y^2 + (x - c)^2 = 0,$$
$$y^2 + x^2 = \text{A}^2.$$

The first is satisfied only by $y = 0$, $x = c$, which are the co-ordinates of the one extremity of the perpendicular; the latter is the equation of the circle described on the transverse axis as diameter, which is therefore the locus sought.

PROP. LXXXV.

(224.) *In an ellipse or hyperbola the semitransverse axis is a mean proportional between the distances of the focus and directrix from the centre.*

For the distance of the directrix from the centre is $\frac{\text{A}^2}{\text{C}}$ by (166.)

(225.) *Cor.* 1. Hence, in an ellipse the vertex lies between the centre and directrix; but in the hyperbola, the directrix lies between the centre and the vertex; for

$$\text{C} > \text{A}, \therefore \text{A} > \frac{\text{A}^2}{\text{C}}.$$

(226.) *Cor.* 2. The perpendicular distance of any point $y'x'$ in the curve from the directrix is

$$\frac{\text{A}^2}{\text{C}} - x' = \frac{\text{A}^2 - \text{C}x'}{\text{C}}.$$

PROP. LXXXVI.

(227.) *The distance of any point in an ellipse or hyperbola from the focus has a constant ratio to the perpendicular distance of the same point from the directrix.*

For by (208) $\text{D} = \frac{\text{A}^2 - \text{C}x}{\text{A}}$, and by the last Prop., the distance from directrix is $\frac{\text{A}^2-\text{C}x}{\text{C}}$, the ratio of which C : A, is independent of the co-ordinates of the point.

This is a ratio of minor inequality for the ellipse, and of major inequality for the hyperbola.

PROP. LXXXVII.

(228.) *A line being drawn from the focus to any point in the curve, to find the locus of the intersection of a perpendicular to this line drawn through the focus with the tangent.*

The equation of the line drawn from the focus to the point $y'x'$ being

$$(x' - \text{C})y - y'x + y'\text{C} = 0.$$

The equation of perpendicular to it is

$$y'y + (x' - \text{C})x + \text{C}(\text{C} - x') = 0.$$

If y' be eliminated by means of this equation, and that of the tangent through $y'x$, the result after reduction is

$$x = \frac{\text{A}^2}{\text{C}}.$$

Hence the locus sought is the *directrix*.

PROP. LXXXVIII.

(229.) *The asymptot of the hyperbola is the limit of the position of the tangent, the distance of the point of contact from the centre being indefinitely increased.*

Let the point of contact be $y'x'$: the equation of the tangent solved for y, and the value of y' being substituted in it, gives

$$y = \mp \frac{\text{B}x'}{\text{A}\sqrt{x'^2-\text{A}^2}}x + \frac{\text{AB}}{\sqrt{x'^2-\text{A}^2}} = \frac{-\text{B}}{\text{A}\sqrt{1-\frac{\text{A}^2}{x'^2}}}x + \frac{\text{AB}}{\sqrt{x'^2-\text{A}^2}}.$$

As x' is indefinitely increased, the value of y approaches $-\frac{\text{B}}{\text{A}}x$ as a limit; but $y = \mp \frac{\text{B}x}{\text{A}}$ is the equation of the asymptots.

(230.) *Cor.* 1. Hence, it appears that the asymptots are the diagonals of a parallelogram formed by tangents through the vertices of every system of conjugate diameters.

(231.) *Cor.* 2. If a line be drawn connecting the extremities of any pair of conjugate diameters, it will be bisected by one asymptot, and parallel to the other: for these extremities are the points of bisection of the sides of the parallelograms, of which the asymptots are the diagonals.

PROP. LXXXIX.

(232.) *To find the equation of the asymptots related to any system of conjugate diameters.*

The equation of the tangent related to any system of conjugate diameters $\text{A}'\text{B}'$ is

$$\text{A}'^2y'y + \text{B}'^2x'x = \text{A}'^2\text{B}'^2,$$

$$\text{or } \text{A}'y\sqrt{1 - \frac{\text{A}'^2}{x'^2}} + \text{B}'x = \frac{\text{A}'^2\text{B}'}{x'}.$$

The limit of this when x' is indefinitely increased is the equation of the asymptots, $\text{A}'y \pm \text{B}'x = 0$, which is the same form as when related to the axes.

PROP. XC.

(233.) *The intercept of a tangent to an hyperbola between the two asymptots is equal to the diameter to which it is parallel, and is bisected at the point of contact.*

The diameter through the point of contact and its conjugate being axes of co-ordinates, the equation of the asymptots is

$$\text{A}'y \mp \text{B}'x = 0.$$

If in this $x = A'$, $y = \mp B'$, hence the proposition is manifest.

PROP. XCI.

(234.) *If any right line intersect an hyperbola, and be produced to meet both asymptots, the two intercepts between the curve and asymptots are equal.*

The diameter parallel to the right line and that to which it is an ordinate being taken as axes of co-ordinates, the equation of the hyperbola is

$$A'^2y^2 - B'^2x^2 = - A'^2B'^2;$$

the equation of asymptots,

$$A'y \mp B'x = 0.$$

From the form of these equations, it is evident that the axis of x bisects the part of the line intercepted between the two asymptots, as the two values of y are equal with opposite signs. It also bisects the part intercepted within the curve; and hence it follows that the two intercepts between the curve and the asymptots are equal.

PROP. XCII.

(235.) *A right line being intercepted between the asymptots, the rectangle under the segments of it made by the curve is equal to the square of the parallel semidiameter.*

The axes of co-ordinates being as in the last Proposition, the segments are

$$\frac{B'}{A'}x' - \frac{B'}{A'}\sqrt{x'^2 - A'^2},$$

$$\frac{B'}{A'}x' + \frac{B'}{A'}\sqrt{x'^2 - A'^2},$$

being the sum and difference of the values of y for the curve and asymptot, which, when multiplied, give B'^2.

PROP. XCIII.

(236.) *To find the intercept of a parallel to an asymptot between the curve and the directrix.*

Let the point where the parallel meets the curve be $y'x'$; the perpendicular distance of the point from the directrix is by (226), $\frac{\text{A}'^2 - \text{C}x'}{\text{C}}$; and the sine of the angle at which the parallel to the asymptot is inclined to the directrix is $\frac{\text{C}}{\text{A}}$. Hence, the intercept of the parallel required is

$$\frac{\text{A}'^2 - \text{C}x'}{\text{A}},$$

and therefore the distance of any point in the curve from the focus (208), is equal to a parallel to the asymptot drawn from the same point to the directrix.

PROP. XCIV.

(237.) *The asymptots of an equilateral hyperbola intersect at right angles.*

For their equation is $y \mp x = 0$, $\because$ each is inclined to the transverse axis at half a right angle, and therefore they make with each other a right angle.

PROP. XCV.

(238.) *If from any point in an hyperbola, parallels to each asymptot be drawn to meet the other, the parallelogram under these is of a constant magnitude, and equal to a fourth part of the parallelogram formed by lines joining the extremities of the axes.*

The equation of hyperbola related to its asymptots is

$$yx = -\frac{\text{F}}{\text{B}}.$$

The line joining the vertices of the axes is equal to c; and since it is parallel to one asymptot, and bisected by the other, when $y=\frac{1}{2}c$, $x=\frac{1}{2}$ of other diagonal of rectangle under axes, $\therefore x=\frac{1}{2}c$; hence $-\frac{F}{B}=\frac{c^2}{4}$, $\therefore yx=\frac{c^2}{4}$. And as the parallelogram under yx is equiangular with that whose side is c, they are equal.

PROP. XCVI.

(239.) *The subtangent of an hyperbola related to its asymptots, as axes of co-ordinates, is equal to the intercept of the asymptot between the ordinate of the point and the centre.*

Since the point of the tangent intercepted between the asymptots is bisected at the point of contact, the ordinate parallel to each asymptot from the point of contact, must bisect the parts of the other intercepted between the tangent and the centre.

SECTION XIII.

Of the parabola.

PROP. XCVII.

(240.) *A parabola is the limit of an ellipse or hyperbola, the parameter of which being given, the transverse axis is increased without limit.*

For the equation of an ellipse or hyperbola, the origin being at the vertex, is (131),

$$A^2y^2 + B^2x^2 - 2AB^2x = 0;$$

the parameter being p,

$$\frac{Ap}{2} = B^2,$$

making this substitution, and dividing by A^2, the equation becomes

$$y^2 + \frac{p}{2A}x^2 - px = 0.$$

If A be supposed to be increased without limit, p remaining unvaried, the second term disappears, and the equation becomes

$$y^2 - px,$$

which is that of a *parabola.*

PROP. XCVIII.

(241.) *To find the equations of a tangent and normal of a parabola.*

The equation of the parabola related to a diameter and a tangent, through its vertex as axes of co-ordinates, is

$$y^2 - p'x = 0,$$

p' being the parameter of that diameter (162). The equation of the tangent is, therefore, (133),

$$2y'(y - y') - p(x - x') = 0;$$

or since $y'^2 - p'x' = 0$,

$$2y'y - p(x + x') = 0.$$

The equation of the normal is therefore

$$(p' + 2y' \cos. yx)(y - y') + (2y' + p' \cos. yx)(x - x') = 0;$$

and if the axis of x be that of the curve, it becomes

$$p'(y - y') + 2y'(x - x') = 0.$$

PROP. XCIX.

(242.) *To find the subtangent and subnormal of the parabola.*

The subtangent being S, and subnormal s, their values by the formulæ (136), (137), become

$$S = 2x',$$

$$s = y', \frac{p' + 2y' \cos. yx}{2y' + p' \cos. yx}.$$

If the axis of x be that of the curve, the value of the subnormal becomes

$$s = \frac{p}{2}.$$

Hence, in a parabola the subtangent is bisected by the curve, and the subnormal relative to the axis is constant, and equal to half the principal parameter.

PROP. C.

(243.) *To find the distance of a point in a parabola from the focus.*

Let the sought distance be z, and $y'x'$ the point. The co-ordinates of the focus being $y = 0$, $x = \frac{p}{4}$, $\because$

$$z^2 = y'^2 + \left(x' - \frac{p}{4}\right)^2 = \left(x' + \frac{p}{4}\right)^2,$$

$$\therefore z = x' + \frac{p}{4}.$$

PROP. CI.

(244.) *To find the polar equation of a parabola, the focus being the pole.*

Let φ be the angle which the axis of the parabola makes with the fixed axis from which the values of ω are measured. If yx be any point on the curve, by (243) $z = x + \frac{p}{4}$; but $\left(x - \frac{p}{4}\right) = z \cos. (\omega - \varphi)$. Hence,

$$z = z \cos. (\omega - \varphi) + \frac{p}{2},$$

$$\therefore z = \frac{p}{2\{1 - \cos. (\omega - \varphi)\}};$$

or, since $4 \sin.^2 \frac{1}{2} (\omega - \varphi) = 2\{1 - \cos. (\omega - \varphi)\}$,

$$z = \frac{p}{4 \sin.^2 \frac{1}{2}(\omega - \phi)};$$

and if the angle ω be measured from the axis $\phi = 0$, therefore the polar equation is

$$z = \frac{p}{4 \sin.^2 \frac{1}{2}\omega}.$$

If ω be measured on the negative side of the focus, these equations become

$$z = \frac{p}{2(1 + \cos. \omega)},$$

$$z = \frac{p}{4 \cos.^2 \frac{1}{2}\omega}.$$

(245.) *Cor.* Hence the equation

$$z = \frac{p}{2(1 + e \cos. \omega)},$$

includes all three species of lines of the second degree. It represents an *ellipse* if $e < 1$, a *parabola* if $e = 1$, and an *hyperbola* if $e > 1$.

PROP. CII.

(246.) *A right line being drawn through the focus of any line of the second degree, and terminated in the curve, to find the relation between the parts intercepted between the focus and the curve.*

By the polar equation the intercepts z, z' are,

$$z = \frac{p}{2(1 + e \cos. \omega)},$$

$$z' = \frac{p}{2\{1 + e \cos. (\pi + \omega)\}} = \frac{p}{2(1 - e \cos. \omega)}.$$

Hence follow, by multiplication and addition,

$$zz' = \frac{p^2}{4(1 - e^2 \cos.^2 \omega)},$$

$$z + z' = \frac{p}{1 - e^2 \cos.^2 \omega};$$

and therefore $4zz' = p(z + z')$.
That is, the rectangle under the sum of the segments and the principal parameter, is equal to four times the rectangle under the segments.

(247.) *Cor.* 1. The rectangle under the segments varies as the whole intercept.

(248.) *Cor.* 2. Half the principal parameter is an harmonical mean between the segments.

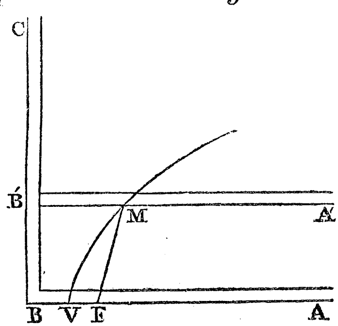
PROP. CIII.

(249.) *The distance of any point in a parabola from the focus is equal to the perpendicular distance of the same point from the directrix.*

By (166), the perpendicular distance of the directrix from a tangent through the vertex is $\frac{p}{4}$, therefore the perpendicular distance of a point in the curve from the directrix is $x + \frac{p}{4}$; but this by (243) is the distance of the same point from the focus.

PROP. CIV.

(250.) *To describe a parabola mechanically.*



Let F be the focus, and V the vertex of the proposed parabola. Take BV = FV, and BC perpendicular to AB will be the directrix. Let a square ABC be applied to the right angle under the axis and the directrix. The extremity of a cord being fastened to any distant point on the side BA of the square, and being passed round a sliding pin at V, let it be fixed to the point F. If the square thus adjusted be moved in the direction BC parallel to itself, the point M will describe a parabola, since B'M always equals MF.

PROP. CV.

(251.) *To find the length of a perpendicular from the focus on a tangent.*

The equation of the tangent being

$$2y'y - p(x' + x) = 0;$$

and the co-ordinates of the focus,

$$y = 0, x = \frac{p}{4}.$$

The perpendicular required is

$$P = \frac{p(x' + \frac{1}{4}p)}{(4y'^2 + p^2)^{\frac{1}{2}}} = \left\{ \tfrac{1}{4}p(x + \tfrac{1}{4}p) \right\}^{\frac{1}{2}}.$$

PROP. CVI.

(252.) *The perpendicular on the tangent through any point, is a mean proportional between the distances of that point and the vertex from the focus.*

For, by (243), the distance from the focus is $x + \frac{p}{4}$, and the distance of the vertex from the focus is $\frac{p}{4}$, therefore, by (251), the perpendicular is a mean proportional between these.

PROP. CVII.

(253.) *To find the locus of the point of intersection of the perpendicular from the focus of a parabola, with the tangent.*

The equations of the tangent and the perpendicular are,

$$2y'y - p(x' + x) = 0,$$
$$2py + 4y'x - py' = 0.$$

Eliminating $y'x'$ by means of these equations, and the equation of the parabola, the result, after reduction, is

$$x\left\{16y^2 + (p - 4x)^2\right\} = 0,$$

which gives

$$x = 0,$$
$$16y^2 + (p - 4x)^2 = 0.$$

The locus of the first is the axis of y, and the latter can only be fulfilled by the conditions,

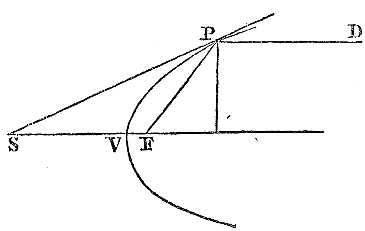
$$y = 0,\ x = \frac{p}{4},$$

which are the co-ordinates of the focus.

Thus, one of the results gives the co-ordinates of the point from which the perpendicular is let fall, and the other shows that the locus of the extremity which meets the tangent, is the tangent to the curve which passes through the vertex.

PROP. CVIII.

(254.) *The part of the axis of a parabola intercepted between a tangent and the focus, is equal to the distance of the point of contact from the focus.*



For, by (243), the distance FP from the focus is,

$$z = x + \frac{p}{4};$$

and since the *subtangent* is bisected by the vertex (242), the intercept of the axis between the tangent and vertex is x, and therefore the intercept between the tangent and focus is $x + \frac{p}{4}$, $\therefore$ FP = FS.

PROP. CIX.

(255.) *A tangent to any point of a parabola being drawn, a diameter, and a line through the focus from the same point, are equally inclined to the tangent.*

For, by the last proposition, the line PF, from the focus, being equal to the intercept FS of the axis between the focus and tangent, the tangent must be equally inclined to PF and the axis; but, since all diameters of a parabola are parallel to the axis (93), the diameter PD and PF are also equally inclined to the tangent.

(256.) *Cor.* If any rays, which obey the law of equality of *incidence* and *reflection*, move in right lines parallel to the axis of a reflecting surface, generated by the revolution of a parabola round its axis, the reflected rays will all converge to the focus; or if they diverge from a lucid point placed at the focus, they will be reflected parallel.

PROP. CX.

(257.) *The distance of any point in a parabola from the focus, is equal to a perpendicular to the axis passing through the same point, intercepted between the axis and the focal tangent.*

In the general equation of a tangent through any point $y'x'$, substitute $\frac{p}{4}$ for x', and $\frac{p}{2}$ for y', and the equation becomes

$$y = x + \frac{p}{4},$$

that of the focal tangent; but this value of y is the same as that of z in (243).

PROP. CXI.

(258.) *To find the relation between the principal parameter, and the parameter of a diameter passing through any given point.*

The equation of the parabola related to the axis, and a tangent through the vertex, as axes of co-ordinates, is

$$y^2 = px,$$

p being the principal parameter. Let the co-ordinates of the point through which the diameter passes be $y'x'$. The axes of co-ordinates being removed to this diameter as axis of x, and a tangent through its vertex as axis of y. The transformed equation, by the formulæ (74), becomes

$$y^2 \sin.^2 tx + (2y' \sin. tx - p \cos. tx)y - px + y'^2 - px' = 0.$$

Since the sin. $x'x = 0$, and cos. $x'x = 1$, the new axis of a being parallel to the former, and expressing by tx the angle under the tangent and diameter, which is the same with $y'x$ in the formula.

Also, since the point $y'x'$ is on the curve,

$$y'^2 - px' = 0.$$

And since tang. $tx = \frac{p}{2y'}$, (241), $\because$

$$2y' \sin. tx - p \cos. tx = 0.$$

Hence the transformed equation becomes

$$y^2 = (p + 4x')x = p'x,$$

observing that $\sin.^2 tx = \frac{p}{p+4x'}$.

Hence the parameter p' of the diameter, through the point $y'x'$, is equal to four times the distance of the point from the focus, since the distance from the focus is $x + \frac{1}{4}p$.

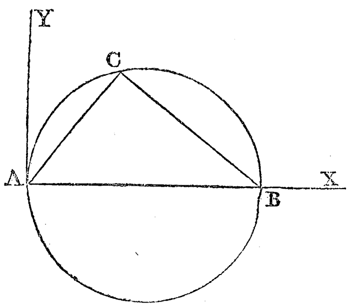
(259.) *Cor.* $\text{Sin.}^2 tx = \frac{p}{p'}$. Hence the parameters of diameters of a parabola are inversely as the sines of the angles at which these diameters are inclined to their ordinates.

SECTION XIV.

Problems relative to lines of the second degree, illustrative of the application of the preceding principles.

PROP. CXII.

(260.) *Given the base* (AB), *and vertical angle of a triangle, to find the locus of a vertex.*



The base (AB) being assumed as axes of x, and a perpendicular (AY) through its extremity (A), as axis of y, let the co-ordinates of C be yx, let AB $= b$, and ACB $= \theta$, ∴ tang.A$=\frac{y}{x}$, tang.B$=\frac{y}{b-x}$,

$$\because \text{tang. C} = -\frac{\text{tang. A} + \text{tang. B}}{1 - \text{tang.A tang.B}} = \frac{yb}{y^2 + x^2 - bx}.$$

Hence the equation sought is,

$$y^2 + x^2 - \cot.\ \theta\ .\ b\ .\ y - bx = 0,$$

which (130) is the equation of a circle, the co-ordinates of whose centre are,

$$y' = \tfrac{1}{2} \cot.\ \theta\ .\ b, \qquad x' = \tfrac{1}{2} b.$$

If $\theta = \frac{\pi}{2}$, cot. $\theta = 0$, ∴ $y' = 0$, which shows that in this case the centre is at the point of bisection of the base.

Cot. θ is positive or negative, according as C is acute or obtuse; ∴ the centre is above the base in the former case, and below it in the latter. From these results may be inferred,

1st, That all angles in the same segment of a circle are equal.

2d, That the angle contained in a semicircle is right.

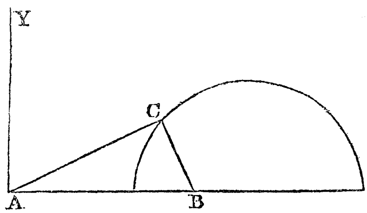
3d, That the angle contained in a segment, greater than a semicircle, is acute.

4th, That the angle contained in a segment, less than a semicircle, is obtuse.

PROP. CXIII.

(261.) *Given the base* (AB) *of a triangle, and the ratio of the sides, to find the locus of the vertex* (C).

The axis of co-ordinates being placed as before, let AC $= a$, and CB $= c$; and let $a = nc$, $\because a^2 = n^2c^2$, but



$$a^2 = y^2 + x^2, \qquad c^2 = y^2 + (b - x)^2.$$

The equation of the locus sought is, $\because$

$$y^2 + x^2 + \frac{2n^2}{1 - n^2} \cdot bx - \frac{n^2b^2}{1 - n^2} = 0,$$

which is the equation of a circle, the co-ordinates of whose centre are

$$y' = 0, \quad x' = \frac{n^2b}{n^2 - 1}.$$

The points where the circle intersects the base are found by supposing $y = 0$, which gives

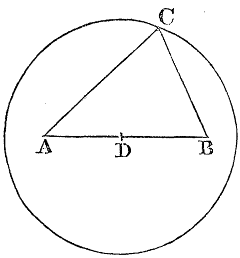
$$x = \frac{nb}{n \mp 1},$$

which values show that the circle cuts the base internally and externally in the given ratio, and the part intercepted between these points is the diameter of the circle.

PROP. CXIV.

(262.) *Given the base* (AB) *of a triangle, and the sum of the squares of the sides, to find the locus of the vertex.*

Let the point of bisection (D) of the base be taken as



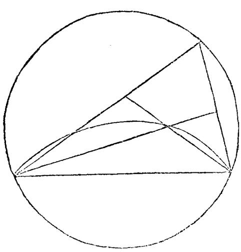
origin, the base as axis of x, and the perpendicular through D as axis of y. Let AD $= b$, and let the given sum of the squares of the sides be s^2.

$$AC^2 = y^2 + (x + b)^2,$$
$$BC^2 = y^2 + (x - b)^2,$$
$$\therefore y^2 + x^2 = \tfrac{1}{2}s^2 - b^2,$$

which is the equation of a circle, whose centre is at the origin, and whose radius is $\sqrt{\tfrac{1}{2}s^2 - b^2}$.

PROP. CXV.

(263.) *Given the base and vertical angle of a triangle, to find the locus of the intersection of the perpendiculars from the angles on the opposite sides.*



The axis of co-ordinates being placed as in Prop. CXII., and the significations of the symbols being retained, the co-ordinates of the intersection of the perpendiculars are

$$Y = \frac{(b - x)x}{y}, \quad X = x;$$

and from these the values of y and x being found, and substituted in the equation for the locus of the vertex found in (52), the result is,

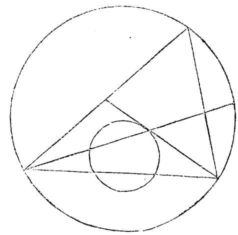
$$Y^2 + X^2 + \cot.\,\theta\,.\,bY - bX = 0.$$

This is the equation of a circle; and since it differs from the equation in (260), which gives the locus of the vertex, only by the sign of cot. θ, the locus sought is a segment of a circle, containing an angle supplemental to θ.

PROP. CXVI.

(264.) *Given the base* (AB), *and vertical angle* (θ), *to find the locus of the intersection of the bisectors of the sides.*

By (54) the co-ordinates of the intersection of the bi-



sectors are

$$\text{Y} = \frac{y}{3}, \text{X} = \frac{x+b}{3},$$

substituting the values of y and x in these equations in that of the locus of the vertex in (260), the result is

$$\text{Y}^2 + \text{X}^2 - \tfrac{1}{3}\cot.\theta\, b\text{Y} - b\text{X} + \tfrac{2}{9}b^2 = 0,$$

which is the equation of a circle, the co-ordinates of whose centre are

$$x' = \frac{b}{2}, y' = \frac{\cot.\theta\, .\, b}{6}.$$

To find the points where this circle meets the base, let $\text{Y} = 0$ in the above equation, and the corresponding values of x are,

$$x = \tfrac{1}{3}b, x = \tfrac{2}{3}b,$$

which shows that the circle intersects the base at the points of trisection.

Let φ be the angle contained in the segment of this circle, whose chord is one-third of the base.

$$\text{Tang.}\,\varphi = \frac{x' - \frac{1}{3}b}{y'} = \text{tang.}\,\theta.$$

Hence this segment contains an angle equal to the vertical angle of the triangle.

PROP. CXVII.

(265.) *Given the base and vertical angle of a triangle to find the locus of the centre of the inscribed circle.*

The lines bisecting the base angles intersect at the centre of the inscribed circle (59), ∵ the sum of the angles which they form with the base is $\frac{\pi - \theta}{2}$, θ being the vertical angle; and ∵ the angle formed by the two bisectors is $\frac{\pi + \theta}{2}$. This being a given angle, the locus sought is the segment of the circle which contains it.

PROP. CXVIII.

(266.) *To express the circle by a polar equation.*

The general equation of the circle related to rectangular co-ordinates, is

$$(y - y')^2 + (x - x')^2 = r^2.$$

Let the distance of any point in the circle from the pole be z, and the angle it makes with the axis of x be ϕ', and the distance of the centre be z', and the angle it makes with the axis of x, ϕ'. By substituting $z \sin. \phi$, $z' \sin. \phi'$, $z \cos. \phi$, $z' \cos. \phi'$, for y, y', x, x', respectively, and observing the conditions,

$$\sin.^2 \phi + \cos.^2 \phi = 1, \cos. \phi \cos. \phi' + \sin. \phi \sin. \phi' = \cos. (\phi - \phi'),$$

the equation becomes

$$z^2 + z'^2 - 2zz' \cos. (\phi - \phi') = r^2.$$

If the pole be on the curve $z' = r$, and the equation becomes

$$z - 2r \cos. (\phi - \phi') = 0,$$

and if, at the same time, the axis from which ϕ is measured passes through the centre, $\phi' = 0$, and the equation is

$$z - 2r \cos. \phi = 0.$$

PROP. CXIX.

(267.) *A right line being drawn from a given point* (P) *to a given circle, to find the locus of the point at which it is divided in a given ratio.*

Let the intercept between the given point, and the point whose locus is sought, be z'', and let $nz'' = z$. By this substitution in the polar equation of the circle, we find

$$z''^2 + \frac{z'^2}{n^2} - 2z'' \frac{z'}{n} \cos. (\phi - \phi') = \frac{r^2}{n^2}.$$

Hence the locus sought is a circle, whose centre is found by dividing the line connecting the given point with the centre of the given circle in the given ratio, and whose radius is to that of the given circle as $1 : n$.

PROP. CXX.

(268.) *To find the locus of a point, from which lines being drawn to several given points, the sum of their squares shall be of a given magnitude.*

Let the given points be $y'x'$, $y''x''$, $y'''x'''$ .... $y^{(n)}x^{(n)}$, and the point whose locus is sought yx. The squares of the lines respectively are

$$(y-y')^2+(x-x')^2,$$
$$(y-y'')^2+(x-x'')^2,$$
$$(y-y''')^2+(x-x''')^2,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$(y-y^{(n)})^2+(x-x^{(n)})^2,$$

which being added, and their sum expressed by s^2, and the result divided by n, give

$$y^2+x^2-2\frac{y'+y''+y'''\dots y^{(n)}}{n}.y-2\frac{x'+x''+x'''\dots x^{(n)}}{n}.x$$
$$+\frac{y'^2+x'^2+y''^2+x''^2\dots-s^2}{n}=0,$$

which is the equation of a circle, whose centre is the Centre of Gravity of the figure formed by lines joining the given points (70).

PROP. CXXI.

(269.) *To find the locus of a point, from which lines being drawn at given angles to the sides of a given rectilinear figure, the sum of their squares shall have a given magnitude.*

The equations of the sides of the figure being respectively

$$\begin{aligned}
&Ay + Bx + C = 0,\\
&A'y + B'x + C = 0,\\
&A''y + B''x + C'' = 0,\\
&\ldots\\
&\ldots\\
&A^{(n)}y + B^{(n)}x + C^{(n)} = 0.
\end{aligned}$$

Let the angles the lines make with the sides be ϕ, ϕ', ..., the squares of the lines are respectively,

$$\frac{(Ay + Bx + C)^2}{(A^2 + B^2)\sin.^2\phi},$$

$$\frac{(A'y + B'x + C')^2}{(A'^2 + B'^2)^2\sin.^2\phi'},$$

$$\ldots$$

$$\ldots$$

$$\frac{(A^{(n)}y + B^{(n)}x + C^{(n)})^2}{(A^{(n)2} + B^{(n)2})\sin.^2\phi^{(n)}},$$

which being added together, and their sum equated with a constant quantity, give a complete equation of the second degree, which is that of the locus sought.

PROP. CXXII.

(270.) *To find the locus of a point, from which two right lines being drawn at given angles to two given right lines, the rectangle under them shall have a given magnitude.*

Let the equations of the two lines be

$$Ay + Bx + C = 0,$$

$$A'y + B'x + C' = 0.$$

The lines making given angles with these from the point yx, are

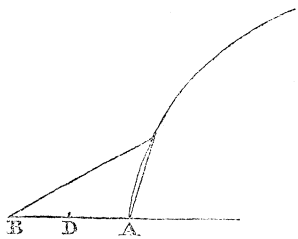
$$-\frac{Ay + Bx + C}{\sqrt{A^2 + B^2}\sin.\phi},$$

$$-\frac{A'y + B'x + C'}{\sqrt{A'^2 + B'^2}\sin.\phi'}.$$

These being multiplied, and their product equated with a constant magnitude, the result is a complete equation of the second degree, which is that of the locus sought.

PROP. CXXIII.

(271.) *Given the base of a triangle, and the difference between the base angles, to find the locus of the vertex.*



The middle point D of the base being assumed as origin, and the base as axis of x, let the co-ordinates of the vertex be yx, and $AD = b$, and the difference of the angles $= \theta$.

$$\text{Tang. } \theta = \frac{\text{tang. A} - \text{tang. B}}{1 + \text{tang. A tang. B}} = \frac{2yx}{y^2 - x^2 + b^2}.$$

Hence the equation of the sought locus is

$$y^2 - 2 \cot. \theta\, yx - x^2 + b^2 = 0.$$

This is manifestly the equation of an hyperbola, since $B^2 - 4AC = 4(\cot.^2 \theta + 1) = 4 \operatorname{cosec.}^2 \theta > 0$; its centre is the origin. The position of the diameter conjugate to AB may be determined by the equation found in (167), which becomes in this case

$$y = - \text{tang. } \theta \, . \, x,$$

∵ the diameter conjugate to AB is inclined to it at an angle $= \theta$.

The axis of y being transformed from its present position to coincidence with the conjugate diameter through the point D, by substituting $y \, . \sin. \theta$ for y, and $x - y \cos. \theta$ for x, which are what the formulæ (74) become in this case, the transformed equation is

$$y^2 - x^2 = - b^2,$$

which shows that the hyperbola is equilateral, and that its semiaxis squared is $b^2 \sin. \theta$ (169).

(272.) *Cor.* 1. Hence it follows, that in an equilateral hyperbola, if lines be drawn from the vertices of any diameter to any point in the curve, the difference of the angles which they form with the diameter is equal to the angle under that diameter and its conjugate.

(273.) *Cor.* 2. If the difference of the angles be a right angle, the base is the transverse axis, and *vice versâ.*

PROP. CXXIV.

(274.) *Given the base of a triangle, and the product of the tangents of the base angles, to find the locus of the vertex.*

The axes of co-ordinates being placed as in the last Proposition, let the product of the tangents be m, ∵

$$y^2 + mx^2 - mb^2 = 0,$$

which is the equation of the locus sought. This locus is therefore an ellipse if $m > 0$, and an hyperbola if $m < 0$, the base being the transverse axis.

PROP. CXXV.

(275.) *Given the base of a triangle, and the sum of the tangents of the base angles, to find the locus of the vertex.*

The axes of co-ordinates being placed as before, let the given sum be m, ∵

$$m = \frac{2by}{b^2 - x^2},$$

hence the equation of the locus is

$$mx^2 + 2by - mb^2 = 0.$$

This equation being put under the form

$$x^2 = -\frac{2b}{m}\left(y - \frac{mb}{2}\right),$$

shows that if the origin be removed to a point in the axis of y, whose distance from the present origin is $\frac{mb}{2}$, the equation becomes

$$x^2 = -\frac{2b}{m}y.$$

Hence the locus sought is a parabola, whose axis is a perpendicular through the middle point of the base, and whose vertex is at a perpendicular distance from the base equal to $\frac{mb}{2}$, and whose principal parameter is $-\frac{2b}{m}$.

PROP. CXXVI.

(276.) *Given the base of a triangle, and the difference of the tangents of the angles at the base, to find the locus of the vertex.*

The axes of co-ordinates being placed as before, let the given difference be m, ∵

$$m = \frac{+2yx}{b^2 - x^2}:$$

the equation sought is ∵

$$2yx + mx^2 - mb^2 = 0.$$

This is the equation of an hyperbola (124), the axis of y being an asymptote, and the origin at the centre (119); the base of the triangle is therefore a diameter, the equation of the diameter conjugate to which is

$$y + mx = 0.$$

Hence the tangent of the angle (θ), at which this conjugate diameter is inclined to the base, is equal to the difference of the tangents of the angles at the base.

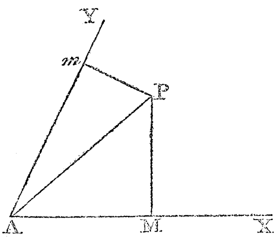
The axis of y being transformed to coincidence with the conjugate diameter by substituting y sin. θ for y, and $x + y$ cos. θ for x, and $-$ tan. θ for m, the equation becomes

$$\cos.^2 \theta \,.\, y^2 - x^2 + b^2 = 0.$$

Hence, the square of the semi-second diameter conjugate to the base is $-b^2 \sec.^2 \theta$.

PROP. CXXVII.

(277.) *To find the locus of a point* (P), *from which perpendiculars drawn to the sides of a given angle* (XAY), *shall contain a quadrilateral of a given area.*



The sides of the given angle (θ) being assumed as axes of co-ordinates, and the co-ordinates of the point P being yx, the area of PMA is

$$\tfrac{1}{2}y \sin. \theta(x + y \cos. \theta),$$

and that of PmA is

$$\tfrac{1}{2}x \sin. \theta(y + x \cos. \theta).$$

Let the magnitude of the quadrilateral be m^2, the equation of the locus sought is

$$y^2 + 2 \sec. \theta \,.\, yx + x^2 - 2m^2 \sec. \theta \operatorname{cosec.} \theta = 0.$$

Since $B^2 - 4AC = 4(\sec.^2 \theta - 1) = 4\tan.^2 \theta > 0$, the locus is an hyperbola, of which the vertex of the angle is the centre (119).

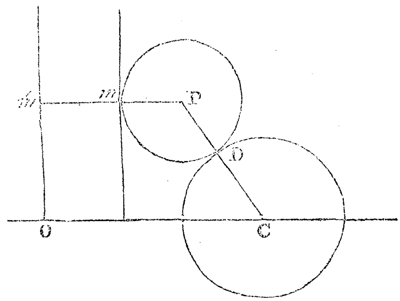
PROP. CXXVIII.

(278.) *To find the locus of the centre of a circle touching a given right line, and passing through a given point.*

The locus must be a parabola, of which the given point is the focus, and the given line the directrix, as is evident from (249).

PROP. CXXIX.

(279.) *To find the locus of the centre of a circle touching a given right line and a given circle.*



Let P be the centre whose locus is sought, and C the centre of the given circle; PD = Pm. Let Pm be produced, so that mm' = CD, and through m' parallel to the given line let another right line be drawn, ∵ Pm' = PC; ∴ the locus is a parabola whose focus is C, and whose directrix is m'O.

PROP. CXXX.

(280.) *To find the locus of the centre of a circle which touches two given circles.*

This is equivalent to being given the base and difference of the sides of a triangle to find the locus of the vertex. The locus is therefore an hyperbola whose foci are the centres of the two given circles, and whose transverse axis is the difference of their radii.

PROP. CXXXI.

(281.) *To find the locus of the intersection of tangents to a given parabola which intersect at a given angle.*

Let the points of contact be $y'x'$, $y''x''$, the point of intersection yx, the given angle θ, and the equation of the given parabola $y^2 = px$. The equations of the tangents through the given points are

$$2y'y - px - y'^2 = 0 \quad (1),$$
$$2y''y - px - y''^2 = 0 \quad (2).$$

The tangents of the angles which these make with the axis of the parabola being $\frac{p}{2y'}$ and $\frac{p}{2y''}$,

$$\tan.\ \theta = \frac{2p(y''-y')}{p^2 + 4y'y''} \quad (3).$$

Subtracting (2) from (1), and dividing the result by $(y''-y')$, we find $2y - y' = y''$. Substituting this value in (3), and multiplying by the denominator,

$$8\tan.\ \theta \, . \, y'y - 4py - 4\tan.\ \theta \, . \, y'^2 + \tan.\ \theta \, . \, p^2 + 4py' = 0.$$

Multiplying (1) by 4tan. θ, and subtracting it from this, the result divided by p is

$$4\tan.\ \theta \, . \, x + \tan.\ \theta \, . \, p + 4y' - 4y = 0;$$

and hence,

$$y' = y - \tfrac{1}{4}\tan.\ \theta \, . \, p - \tan.\ \theta \, . \, x,$$

which substituted in (1) gives

$$y^2 - \tan.^2\ \theta \, . \, x^2 - p(1 + \tfrac{1}{2}\tan.^2\ \theta)\ x - \tfrac{1}{16}\tan.^2\ \theta \, . \, p^2 = 0,$$

which is the equation of the locus sought. This must be an hyperbola, since $(\text{B}^2 - 4\text{AC}) = 4\tan.^2\ \theta > 0$.

The co-ordinates of the centre are

$$y = 0,\ x = -\tfrac{1}{2}p\ (\cot.^2 . \ \theta + \tfrac{1}{2});$$

the origin being removed to this point, the equation becomes

$$y^2 - \tan.^2\ \theta x^2 = -\frac{p^2}{4}\operatorname{cosec.}^2\ \theta,$$

which shows that the squares of the semiaxes are

$$\text{A}^2 = \frac{p^2}{4} \cdot \frac{\cos.^2\ \theta}{\sin.^4\ \theta},\ \text{B}^2 = -\tfrac{1}{4}p^2 \operatorname{cosec.}^2\ \theta.$$

In this investigation the $\tan.^2\ \theta$ includes $+\tan.\ \theta$ and $-\tan.\ \theta$, *i.e.* tan. θ and tan. $(\pi - \theta)$, which shows that the process includes the locus of the intersection of tangents, which contain an angle supplemental to θ. Of the opposite branches of the hyperbola, one is the locus of the intersection of tangents containing the angle θ, and the other of tangents containing its supplement.

If $\theta = \frac{\pi}{2}$, the equation after division by $\tan.^2 \theta$, becomes

$$x + \frac{p}{4} = 0,$$

which is the equation of the directrix of the given parabola. Hence, if tangents to a parabola intersect at right angles, the locus of their intersection is the directrix.

PROP. CXXXII.

(282.) *To find the locus of the intersection of tangents to an ellipse or hyperbola, which shall be inclined to the transverse axis at angles, the product of whose tangents is given.*

Let the equations of two right lines meeting the curve be

$$y + ax + b = 0 \quad (1),$$
$$y + a'x + b' = 0 \quad (2);$$

the equation of the curve being $A^2y^2 + B^2x^2 = A^2B^2$. Eliminating y by this, and each of the equations of the right lines, and finding the value of x in the resulting equation, and equating the radical in each with 0, we find

$$A^2a^2 + B^2 - b^2 = 0,$$
$$A^2a'^2 + B^2 - b'^2 = 0.$$

The values of b and b' in (1) and (2) being substituted, and the equations arranged by the dimensions of a and a',

$$a^2 - \frac{2xy}{A^2 - x^2}a + \frac{B^2 - y^2}{A^2 - x^2} = 0,$$

$$a'^2 - \frac{2xy}{A^2 - x^2}a' + \frac{B^2 - y^2}{A^2 - x^2} = 0.$$

The values of a and a' being the roots of either of these equations, let $aa' = m$, $\therefore$

$$m = \frac{B^2 - y^2}{A^2 - x^2}.$$

Hence, the equation of the locus sought is

$$y^2 - mx^2 + mA^2 - B^2 = 0;$$

the locus is therefore an ellipse or hyperbola, according as

$$m < 0 \text{ or } > 0.$$

Let the semiaxes of this curve be A', B',

$$A'^2 = \frac{mA^2 - B^2}{m}, \qquad B'^2 = -mA^2 + B^2;$$

hence $A'^2 : B'^2 :: 1 : m$.

If the curve be an hyperbola, and $m < 0$, the locus is impossible when $mA^2 < B^2$, or $m < \frac{B^2}{A^2}$, which shows that in this curve the product of the tangents of the angles, which two tangents make with the axis, cannot be less than the product of the tangents of the angles, which the asymptotes make with it (176).

PROP. CXXXIII.

(283.) *To find the locus of the intersection of two tangents to an ellipse which intersect at a right angle.*

In the last Proposition, if $m = -1$, the tangents will intersect perpendicularly, the equation of the locus is therefore

$$y^2 + x^2 = A^2 + B^2,$$

which is the equation of a circle concentrical with the ellipse, and whose radius equals the line joining the extremities of the axes.

PROP. CXXXIV.

(284.) *To find the locus of the intersection of two tangents to an hyperbola, which intersect at a right angle.*

In this case, in (282), $m = -1$ and $B^2 < 0$, $\because$ the equation of the locus is

$$y^2 + x^2 = A^2 - B^2,$$

which is the equation of a circle concentrical with the hyperbola, and whose radius equals $\sqrt{A^2 - B^2}$. This equation is impossible if $B > A$, which shows that, in an hyperbola of this kind, two tangents cannot intersect at a right angle.

PROP. CXXXV.

(285.) *To find the locus of the intersection of two tangents to an ellipse or hyperbola, which make angles with the transverse axis, which, measured in the same direction, are together equal to a right angle.*

In this case, in (282), $m = 1$, $\because$ the equation of the locus is

$$y^2 - x^2 = -(A^2 \mp B^2),$$

which is the equation of an equilateral hyperbola, whose axis is the distance between the foci of the given ellipse or hyperbola.

PROP. CXXXVI.

(286.) *To find the locus of the intersection of tangents to an ellipse, which are parallel to conjugate diameters.*

In this case, $m = -\frac{B^2}{A^2}$ (170), $\because$ the equation of the locus sought is

$$A^2y^2 + B^2x^2 = 2A^2B^2,$$

which is the equation of an ellipse, whose semiaxes are $\sqrt{2} . A$, $\sqrt{2} . B$, and which is therefore similar to the given ellipse.

This is obviously equivalent to finding the locus of the vertices of the angles of parallelograms circumscribed round systems of conjugate diameters.

PROP. CXXXVII.

(287.) *To find the locus of the intersection of tangents to an hyperbola, which are parallel to conjugate diameters.*

In this case, $m = \frac{B^2}{A^2}$ (170), $\because$ the equation of the locus is

$$A^2y^2 - B^2x^2 = 0,$$

which is resolved into $Ay + Bx = 0$ and $Ay - Bx = 0$, which are the equations of the asymptotes, which are the locus sought.

PROP. CXXXVIII.

(288.) *To find the locus of the intersection of tangents to an ellipse, which make angles with the transverse axis, the product of whose tangents is* $\frac{B^2}{A^2}$.

In (282), if $m = \frac{B^2}{A^2}$, the equation of the locus is

$$A^2y^2 - B^2x^2 = 0,$$

which is resolved, as before, into $Ay + Bx = 0$, $Ay - Bx = 0$, which are the diagonals of the rectangle formed by tangents through the vertices of the axes, and which are therefore the locus sought.

PROP. CXXXIX.

(289.) *To find the locus of the intersection of two tangents to a parabola, which are inclined to its axis at angles, the product of whose tangents is constant.*

Let the equations of two right lines meeting the parabola be

$$y - ax - b = 0 \quad (1),$$
$$y - a'x - b' = 0 \quad (2).$$

By these and the equation of the curve finding values for

x, and equating the radical in each with 0, we find

$$4ab = p, \quad 4a'b' = p;$$

eliminating b and b' by means of these and (1), (2), the results are

$$a^2 - a \cdot \frac{y}{x} + \frac{p}{4x} = 0,$$

$$a'^2 - a' \cdot \frac{y}{x} + \frac{p}{4x} = 0;$$

the values aa' being the roots of either of these equations, and the given product being expressed by m, we have

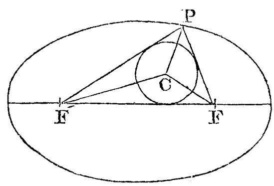
$$m = + \frac{p}{4x}, \text{ or } 4mx - p = 0,$$

which is the equation of a right line perpendicular to the axis, and meeting it at a point whose distance from the vertex is $+ \frac{p}{4m}$.

If the tangents intersect at right angles, $m = -1$, and the locus is the directrix.

PROP. CXL.

(290.) *Two lines being drawn from the foci of an ellipse to any point in the curve, to find the locus of the centre of the circle which touches these and the transverse axis.*



1. Let the circle touch the three lines as in the figure. Let the co-ordinates of the point (P) on the ellipse be $y'x'$, and those of the centre (C) yx. The area of the triangle $\text{FPF}' = y'c$, where c is the distance of the focus from the centre. Also the area of the same triangle $= y(\text{A} + c)$, since $\text{A} + c$ is the semiperimeter of the triangle;

$$\because y'c = y(\text{A} + c);$$

also, since the line FC bisects the angle PFF',

$$\tan.\ \text{CFF}' = \frac{\sin.\ \text{PFF}'}{1+\cos.\ \text{PFF}'};$$

but $\sin.\ \text{PFF}' = \frac{y'}{z}$, and $\cos.\ \text{PFF}' = \frac{c-x'}{z}$, and also

$$\tan.\ \text{CFF}' = \frac{y}{c-x},\ \text{hence}\ \frac{y}{c-x} = \frac{y'}{z+c-x'}.$$

Now by (208) $z = \text{A} - \frac{cx'}{\text{A}}$, which being substituted, the values of $y'x'$ resulting from this and the first equation are

$$y' = y\,\frac{\text{A}+c}{c},$$

$$x' = \frac{\text{A}x}{c}.$$

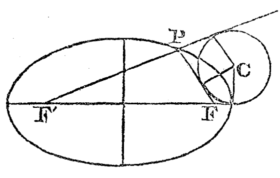
Substituting these values in the equation $\text{A}^2y'^2+\text{B}^2x'^2=\text{A}^2\text{B}^2$, the result after reduction is

$$(\text{A}+c)^2y^2+\text{B}^2x^2 = \text{B}^2c^2,$$

this is the equation of the sought locus, which is therefore an ellipse whose axes coincide with those of the given ellipse. Let the semiaxes be A', B', ∴

$$\text{A}' = c,$$

$$\text{B}' = \frac{\text{B}c}{\text{A}+c}.$$



2. Let the circle touch the three lines as in this figure. In this case, if $\text{FP} = z$, $\text{F}'\text{P} = z'$, we have

$$y'c = y(c+\tfrac{1}{2}z'-\tfrac{1}{2}z);$$

but by (208), $z'-z' = \frac{2cx'}{\text{A}}$, hence $\text{A}y' = (\text{A}+x')y$; also

$$\tan.\ \text{CFP} = \tan.\ \tfrac{1}{2}(\pi-\text{PFF}'),\ ∴\ \tan.\ \text{CFP} = \frac{y'}{z-c+x'};$$

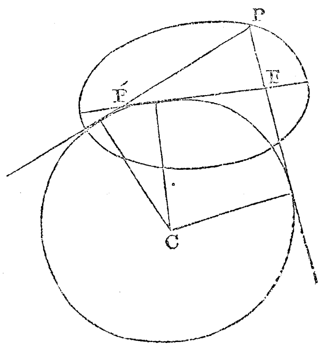
$$\text{but also}\ \tan.\ \text{CFP} = \frac{y}{x-c},\ \text{hence}\ \frac{y}{x-c} = \frac{y'}{z-c+x'}.$$

Eliminating y' by this and the former equation, the result after reduction is

$$x-\text{A} = 0,$$

which shows that the locus in this case is the tangent to the ellipse passing through the vertex.

3. Should the circle touch the three lines as in this figure, retaining the same symbols,



$\text{FPF}' = y(\text{A} - c) = y'c$ (1).

Also, since

$$\tan. \tfrac{1}{2}(\pi - \text{PFF}') = \frac{\sin. \text{PFF}'}{1 - \cos \text{PFF}'} = \frac{\text{A}y'}{(\text{A} - c)\ (\text{A} + x')}.$$

And, since FC bisects the angle below the base,

$$\tan. \tfrac{1}{2}(\pi - \text{PFF}') = \frac{y}{c - x};\ \text{hence,}$$

$$\frac{y}{c - x} = \frac{\text{A}y'}{(\text{A} - c)\ (\text{A} + x')}.$$

By this and the equation (1) we find

$$y' = \frac{y(\text{A} - c)}{c},$$

$$x' = -\frac{\text{A}x}{c}.$$

Substituting these in the equation of the ellipse, the result after reduction is

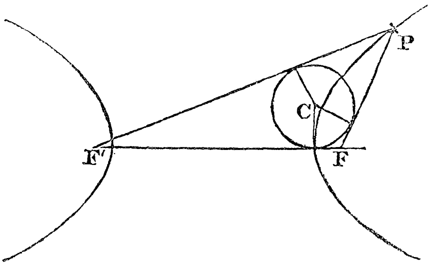
$$(\text{A} - c)^2 y^2 + \text{B}^2 x^2 = \text{B}^2 c^2,$$

which is the equation of an ellipse, whose semiaxes A', B', are

$$\text{A}' = c, \qquad \text{B}' = \frac{\text{B}c}{\text{A} - c}.$$

PROP. CXLI.

(291.) *Lines being drawn connecting any point in an hyperbola with the foci, to find the locus of the centre of a circle, which shall touch these lines and the axis.*



1. Let the circle touch, as in this figure. The same symbols being used as in the last Proposition,

$$\text{FPF}' = y(z + \text{A} + c) = y'c,$$

$$\text{tang. CFF}' = \frac{y}{c - x};$$

but $\text{CFF}' = \frac{1}{2}\,\text{PFF}'$, ∵

$$\text{tang. CFF}' = \frac{\sin.\ \text{PFF}'}{1 + \cos.\ \text{PFF}'} = \frac{y'}{z - x' + c}.$$

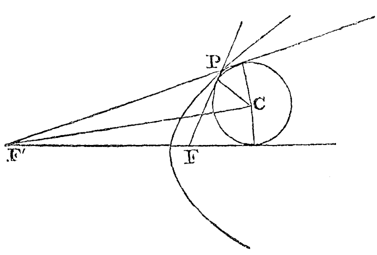
Hence, since $z = \dfrac{cx'}{\text{A}} - \text{A}$,

$$\frac{y}{c - x} = \frac{\text{A}y'}{(c - \text{A})(\text{A} + x')}.$$

The co-ordinates $y'x'$ being eliminated by this and the first equation, we find

$$x - \text{A} = 0,$$

which is the equation of the tangent through the vertex of the hyperbola, and which is therefore the locus of the centre in this case.



2. If the circle touch, as in this figure,

$$\text{FPF}' = y(c + \text{A}) = y'c.$$

The tang. CFF' having a similar value as above, a similar equation follows, which gives

$$y' = \frac{y(c+A)}{c},\ x' = \frac{Ax}{c}.$$

Making these substitutions in the equation

$$A^2y'^2 - B^2x'^2 = -A^2B^2,$$

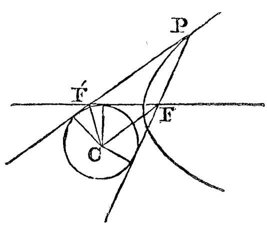
the result after reduction is

$$(c + A)^2y^2 - B^2x^2 = - B^2c^2,$$

which is the equation of an hyperbola, whose semiaxes are coincident with those of the given hyperbola, and whose values are

$$A' = c,\ B' = \frac{Bc}{A + c}\sqrt{-1}.$$

3. Let the circle touch, as in this figure,



$$FPF' = y(z + A - c) = y'c.$$

Also, $\frac{y}{c-x} = \text{tang.}\frac{1}{2}(\pi - PFF')$,

but

$$\text{tang.}\frac{1}{2}(\pi - PFF') = \frac{\sin. PFF'}{1 - \cos. PFF'} = \frac{y'}{z + x' - c}.$$

Eliminating z by $z = \frac{cx'}{A} - A$, and the values of y', x' being eliminated as before, we find

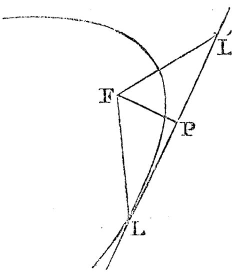
$$x + A = 0,$$

which shows that the locus is the tangent through the vertex of the opposite hyperbola.

PROP. CXLII.

(292.) *To find the locus of the vertex of a parabola, having a given point as focus, and touching a given right line.*

Let F be the given focus, and LL′ the given right line,



V the vertex, and FP a perpendicular from the focus on the right line $= a$.

By Prop. (CVI), $FL \times FV = a^2$. Let $FV = z$, and the angle $PFV = \omega$, ∵

$$z = a \cos. \omega.$$

If FP and PL' be taken as axes of co-ordinates,

$$z = \sqrt{y^2 + (a-x)^2},$$

$$\text{and } \cos. \omega = \frac{a-x}{\sqrt{y^2 + (a-x)^2}},$$

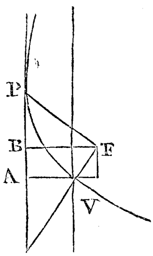
hence the equation of the locus sought is

$$y^2 + x^2 - ax = 0,$$

which is the equation of a circle passing through the points F and P, and whose diameter is FP.

PROP. CXLIII.

(293.) *To find the locus of the focus of a parabola, which has a given vertex, and which touches a given right line.*



Let V be the vertex, F the focus, AP the given line, and VA a perpendicular to it. This perpendicular being taken as axes of x, and a parallel to AP through V as axis of y, let the co-ordinates of F be yx. By (252), $FV . FP = FB^2$, but $FV^2 = y^2 + x^2$, $FB = a + x$, AV being expressed by a, and $FP = FB . \frac{FV}{x}$. Hence the equation of the locus of F is, after reduction,

$$y^2 - ax = 0.$$

The locus is therefore a parabola, whose vertex is the point V, whose axis coincides with AV, and whose parameter is AV.

PROP. CXLIV.

(294.) *Given a diameter of a parabola, the point where the curve intersects it, and its parameter, to find the locus of the focus.*

The distance of the vertex of any diameter of a parabola from the focus is a fourth part of the parameter of that diameter. This being given, the locus sought is a circle, of which the point, where the curve meets its diameter, is the centre, and a fourth part of the parameter the radius.

PROP. CXLV.

(295.) *Given the point where a parabola intersects a given diameter, and also the parameter of that diameter, to find the locus of the vertex of the curve.*

Let the given diameter and a perpendicular through its vertex be assumed as axes of co-ordinates. The equation of the parabola related to a diameter, and a tangent through its vertex as axes of co-ordinates being $y^2 - px = 0$, if the angle under the tangent and diameter be θ, and the axis of y changed to a perpendicular to the diameter, the equation becomes

$$y^2 + \tfrac{1}{2}p \sin. 2\theta . y - \sin.^2 \theta . px = 0.$$

The co-ordinates $y''x''$ of the vertex are

$$y'' = - \tfrac{1}{4}p \sin. 2\theta, \; x'' = - \tfrac{1}{4}p \cos.^2 \theta.$$

Eliminating from these equations the angle θ, we find

$$y''^2 + 4x''^2 + px'' = 0,$$

which is the equation of an ellipse, whose transverse axis coincides with the given diameter, and is equal to $\frac{p}{2}$, and whose conjugate axis equals $\frac{p}{4}$.

PROP. CXLVI.

(296.) *Given the diameter of a parabola, and a tangent through its vertex, to find the locus of the vertex.*

The axes of co-ordinates being placed as before, let p be eliminated by the values of the co-ordinates of the vertex. The result

$$y - 2 \text{ tang. } \theta \cdot x = 0,$$

shows that the locus is a right line.

PROP. CXLVII.

(297.) *On the same conditions to find the locus of the focus.*

The axes of co-ordinates remaining the same, the co-ordinates $y'x'$ of the focus are

$$y' = -\tfrac{1}{4}p \text{ sin. } 2\theta,\ x' = -\tfrac{1}{4}p \text{ cos. } 2\theta.$$

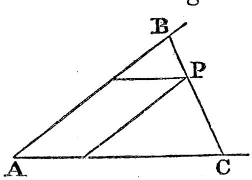
Eliminating p from these, the result is

$$y' - \text{tang. } 2\theta \cdot x' = 0,$$

which shows that the locus sought is a right line.

PROP. CXLVIII.

(298.) *A right line of a given length is terminated in the sides of a given angle, to find the locus of a point which divides it in a given ratio.*



Let the sides of the given angle BAC $= \theta$ be taken as axes of co-ordinates, and the co-ordinates of P being yx, and $\frac{\text{BP}}{\text{CP}} = \frac{m}{n}$,

$$\text{AB} = \frac{(m+n)y}{n},$$

$$\text{AC} = \frac{(m+n)x}{m}.$$

But $AB^2 + AC^2 - 2AB \cdot AC \cdot \cos.\theta = BC^2$. Hence, after reduction

$$m^2y^2 - 2mn \cos.\theta \cdot yx + n^2x^2 = m^2n^2,$$

which is the equation of an ellipse, since

$$B^2 - 4AC = 4m^2n^2(\cos.^2\theta - 1) = - 4m^2n^2 \sin.^2\theta < 0.$$

If $\theta = \frac{\pi}{4}$, the equation becomes

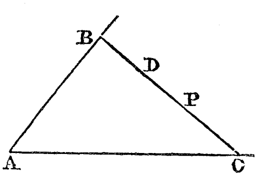
$$m^2y^2 + n^2x^2 = m^2n^2,$$

which is the equation of an ellipse, whose axes are equal to the segments of the given line, and coincide with the sides of the given angle.

If $m = n$ the locus is a circle in this case.

PROP. CXLIX.

(299.) *A right line passes through a given point, and is terminated in the sides of a given angle, to find the locus of the point which divides it in a given ratio.*



Let the sides of the given angle BAC be taken as axes of co-ordinates, and let the co-ordinates of the given point D be $y'x'$, those of the point P be yx, the equation of BC is

$$A(y - y') + B(x - x') = 0.$$

In this, if y and x be successively supposed $= 0$, we find

$$AC = \frac{Ay' + Bx'}{B},\ AB = \frac{Ay' + Bx'}{A}.$$

Let the ratio of the segments BP, PC be $m : n$,

$$\frac{y}{AB} = \frac{n}{m+n},\quad \frac{x}{AC} = \frac{m}{m+n}.$$

Dividing the first by the second, and substituting for $\frac{AC}{AB}$ its value $\frac{A}{B}$,

$$\frac{Ay}{Bx}=\frac{n}{m}, \therefore \frac{A}{B}=\frac{nx}{my};$$

hence the equation of the locus sought is

$$(m+n)xy - mx'y - ny'x = 0.$$

This is the equation of an hyperbola, the axes of co-ordinates being parallel to the asymptotes (123).

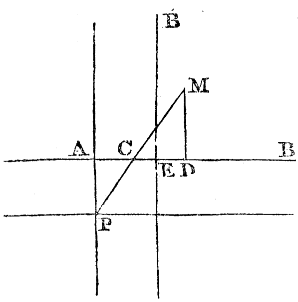
The co-ordinates of the centre being

$$x=\frac{mx'}{m+n}, \; y=\frac{ny'}{m+n},$$

show that if the co-ordinates of the given point be divided each in the given ratio, parallels to the sides of the given angle drawn through the points where they are divided thus, are the asymptotes.

PROP. CL.

(300.) *Given in position a right line* (AB), *and a point* (P) *outside it, a right line* (PM) *is drawn intersecting the given right line; from the extremity* M *of which, a perpendicular to the given right line intercepts* CD *of a given magnitude* (a), *to find the locus of the point* M.



By the conditions of the question, if PA be perpendicular to AB,

$$\frac{MD}{CD}=\frac{PA}{AC}.$$

Now, if P be the origin of rectangular co-ordinates, parallel and perpendicular to AB, this condition is expressed by

$$\frac{y-b}{a}=\frac{b}{x-a},$$

where b = PA. Hence the equation of the locus is

$$yx - ay - bx = 0.$$

The curve is therefore an equilateral hyperbola.

To find the centre, substitute in the formulæ (94) the values of the terms in this case, and we find

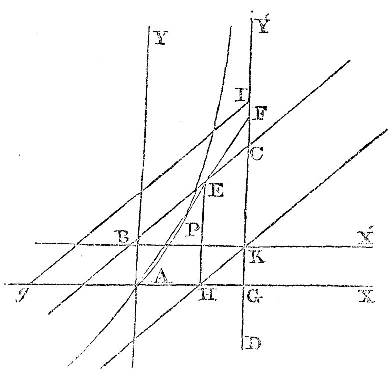
$$y'' = b, \ x'' = a.$$

Hence if AE $= a$, E is the centre, and EB and EB$'$ are the asymptotes.

PROP. CLI.

(301.) *From a given point* A *a right line* AF *is drawn, intersecting two right lines* BC *and* CD *given in position, and a part* AP *is assumed on this line from the given point* A, *always equal to the part* EF *intercepted between the given right lines* BC *and* CD, *it is required to find the locus of the point* P.

Let the origin of co-ordinates be assumed at A, and lines parallel and perpendicular to CD be assumed as axes of co-ordinates. Let AG $= x'$, and the equation of BC and AF be respectively



$$\text{A}y + \text{B}x + \text{C} = 0, \ (1),$$

$$\text{A}'y + \text{B}'x = 0, \ (2).$$

Eliminating y from these equations, we find the value of x for the point E,

$$\text{AH} = -\frac{\text{CA}'}{\text{BA}' - \text{B}'\text{A}},$$

and therefore

$$\text{HG} = x' + \frac{\text{CA}'}{\text{BA}' - \text{B}'\text{A}};$$

but by the conditions of the question, if yx be the co-ordinates of P, $x =$ HG; hence

$$x = x' + \frac{\text{C}\text{A}'}{\text{B}\text{A}' - \text{B}'\text{A}}.$$

By this and (2) $\frac{\text{B}'}{\text{A}'}$ being eliminated, the equation of the locus sought is

$$\text{A}yx + \text{B}x^2 - \text{A}x'y - (\text{B}x' + \text{C})x = 0,$$

or since $\text{A}y' = -(\text{B}x' + \text{C})$, where $y' = \text{GC}$,

$$\text{A}yx + \text{B}x^2 - \text{A}x'y + \text{A}y'x = 0;$$

and since $-\frac{\text{B}}{\text{A}} = \cot.\,\varphi$, where φ is the angle BCD,

$$yx - \cot.\,\varphi\,.\,x^2 - x'y + y'x = 0.$$

The locus sought is therefore an hyperbola.

The co-ordinates of the centre are

$$x = x',\ y = 2\cot.\,\varphi\,.\,x' - y'.$$

The origin of co-ordinates being removed to this point, the equation becomes

$$yx - \cot.\,\varphi\,.\,x^2 - x'(\cot.\,\varphi\,.\,x' - y') = 0.$$

Hence (121) the line CG is an asymptote, and the other asymptote is a right line, related (105) to the latter system of co-ordinates by the equation

$$y = \cot.\,\varphi\,.\,x.$$

Hence if Ag = AG, and gI be drawn parallel to BC, and GK = IC, the point K is the centre of the hyperbola, and a line through K parallel to BC is one asymptote, and CG the other.

PROP. CLII.

(302.) *If through the vertices of two similar lines of the second degree, whose axes coincide, two right lines be drawn intersecting them, they will be cut proportionally by those curves.*

Let the equations of the two curves be

$$y^2 = px + qx^2,$$
$$y^2 = p'x + qx^2.$$

Since $q = \frac{B^2}{A^2} = \frac{B'^2}{A'^2}$, these curves are similar. Let the equation of a right line intersecting them be $y = ax$, which, being substituted in each of the equations, gives

$$x = \frac{p}{a^2 - q},$$
$$x' = \frac{p'}{a^2 - q};$$

and dividing the one by the other,

$$\frac{x}{x'} = \frac{p}{p'}.$$

Hence the intercepts of the intersecting right line between the origin and the points where it meets the curves are proportional to the principal parameters, and therefore the ratio is independent of the inclination of the secant to the axis.

Cor. This question applied to the circle will furnish solutions for the following problems:

1°. *To describe a circle passing through two given points and touching a given circle.*

2°. *To describe a circle passing through a given point and touching two given circles.*

3°. *To describe a circle touching three given circles.* See *Puissant Propositions de Geometrie*, pp. 119, 120.—*Recreations Mathematiques* of Ozanam, tom. i. p. 377.—No. 6, *Correspondence sur l'Ecole Polytechnique.*

PROP. CLIII.

(303.) *Let two similar ellipses or hyperbolas have a common centre and coincident axes, and through the vertex of the smaller let a tangent be drawn intersecting the other; any two chords of the greater passing through the point where this tangent meets it, and equally inclined to this tangent, are together equal to two chords of the smaller ellipse parallel to them, and passing through the vertex.*

Let the equation of the smaller be

$$A^2y^2 + B^2x^2 - 2B^2Ax = 0,$$

the origin being at the vertex; this changed into a polar equation, gives

$$(A^2 \sin.^2 \omega + B^2 \cos.^2 \omega)r - 2B^2A \cos. \omega = 0;$$

or if e be the eccentricity,

$$(1 - e^2 \cos.^2 \omega)r - p \cos. \omega = 0;$$

and hence

$$r = \frac{p \cos. \omega}{1 - e^2 \cos.^2 \omega}.$$

Let the equation of the greater curve, the origin being at the centre, be

$$A'^2y^2 + B'^2x^2 = A'^2B'^2.$$

If the origin be removed to the point where the tangent intersects it, and whose co-ordinates are therefore

$$x = -A, \text{ and } y = \frac{B'}{A'}\sqrt{A'^2 - A^2},$$

the equation will be

$$A'^2y^2 + B'^2x^2 + 2A'B'\sqrt{A'^2 - A^2} \,.\, y - 2B'^2Ax = 0.$$

Since the ellipses are similar, their eccentricities are equal, and therefore this equation becomes, by dividing by A'^2, and observing that $\frac{B'^2}{A'^2} = 1 - e^2$, and $2(1 - e^2)A = p$,

$$y^2 + (1 - e^2)x^2 + 2\sqrt{1 - e^2}\sqrt{A'^2 - A^2} \,.\, y - px = 0.$$

This changed into a polar equation, and solved for r', gives

$$r' = \frac{p \cos.\, \omega - 2\sqrt{1-e^2}\ \sqrt{A'^2 - A^2} \sin.\, \omega}{1 - e^2 \cos.^2 \omega}.$$

The two values of r', which make equal angles with the axis of y, differ only in the sign of sin. ω, and therefore representing them by r' and r'',

$$r' + r'' = \frac{2p \cos.\, \omega}{1 - e^2 \cos.^2 \omega};$$

hence $r' + r'' = 2r$.

Cor. This proposition will apply also to two parabolas if they be equal.

This proposition is given by *Clairaut* in his *Theorie de la Terre*, and is the principle by which he proceeds in his investigation of the figure of the planets, when they are supposed to be homogeneous.

PROP. CLIV.

(304.) *Three unequal circles being given, if to every two of them common tangents be drawn, the three points of intersection of the tangents to each pair of circles will lie in the same straight line.*

Let C, C$'$, C$''$, be the centres of the circles, P, P$'$, P$''$ the three points of intersection of the tangents, r, r', r'', the three radii, and let the lines P$'$P$''$ and P$'$C be taken as axes of co-ordinates. Let P$'$C$''$ $= y''$, and let the co-ordinates of the centre C$'$ be $y'x'$. The ratio $y'' : y'$ may be considered as compounded of y'' : P$'$C, or $r'' : r$, and of P$'$C : y', or $r : r'$, therefore $y'' : y' :: r'' : r'$; but C$''$P : C$'$P :: $r'' : r'$. Hence P is on the axis of x.

PROP. CLV.

(305.) *Two circles being given in magnitude and position, let a tangent to one of them intersect the other, to find the locus of the intersection of tangents to the second passing through the points where the tangent to the first meets it.*

Let the centres be C, C′, the radii R, R′, p the point of contact of the tangent to the first, and P the point whose locus is sought. Let CC′ be the axis of x, and a perpendicular to it through C, the axis of y: let the co-ordinates of p be $y'x'$ and C′P $= r$, CC′$=x''$, and the angle PC′X $= \omega$.

Since the equation of the tangent through p is

$$yy' + xx' = R^2;$$

and C′P is perpendicular to the tangent, therefore the portion of C′P between C′ and the tangent, (b), is

$$-\frac{x'x'' - R^2}{R};$$

but $r = \frac{R'^2}{b}$, and $x' = R\cos.\omega$, therefore

$$r = \frac{R'^2}{R - x''\cos.\omega}.$$

This is the polar equation of a line of the second degree, the pole being the focus, and the values of ω measured from the axis. The parameter and eccentricity are given by the equations,

$$p = \frac{2R'^2}{R},$$

$$e = \frac{x''}{R}.$$

The locus is therefore a *parabola*, *ellipse*, or *hyperbola*, according as $x'' = R$, $x'' < R$, or $x'' > R$.

If the locus be an *ellipse* or *hyperbola*, the axes are determined by the equations,

$$\frac{B^2}{A} = \frac{R'^2}{R},$$

$$\frac{A^2 - B^2}{A^2} = \frac{x''^2}{R^2}.$$

Hence it follows that

$$B^2 = \frac{R'^4}{R^2 - x''^2},$$

$$A = \frac{RR'^2}{R^2 - x''^2};$$

the ratio of the axes are therefore $\sqrt{R^2 - x''^2} : R^2$.

The locus will be a *circle* if $x'' = 0$, scil. if the two circles are concentrical.

If the centre of the second circle be within the first, the locus is the *ellipse;* if it be on its circumference, it is the *parabola;* and if it be outside it, it is the *hyperbola.*

PROP. CLVI.

(306.) *To find the equation of a line of the second degree, touching the three sides of a given triangle.*

Let the sides of the given triangle be represented by the equations

$$\left.\begin{aligned} ay + bx + c &= 0 \\ a'y + b'x + c' &= 0 \\ a''y + b''x + c'' &= 0 \end{aligned}\right\} (A).$$

Let y be eliminated by each of these equations and the general equation of the second degree, and the results arranged by the dimensions of x, are

$$\left.\begin{aligned} &(Ab^2 - Bab + Ca^2)x^2 + (2Abc - Bac - Dab + Ea^2)x \\ &\qquad + Ac^2 - Dac + Fa^2 = 0 \\ &(Ab'^2 - Ba'b' + Ca'^2)x^2 + (2Ab'c' - Ba'c' - Da'b' + Ea'^2)x \\ &\qquad + Ac'^2 - Da'c' + Fa'^2 = 0 \\ &(Ab''^2 - Ba''b'' + Ca''^2)x^2 + (2Ab''c'' - Ba''c'' - Da''b'' + Ea''^2)x \\ &\qquad + Ac''^2 - Da''c'' + Fa''^2 = 0 \end{aligned}\right\} (B).$$

That the three sides of the triangle may be tangents, the roots of each of these equations must be real and equal, which furnishes the conditions:

$$\left.\begin{array}{l}(B^2-4AC)c^2+(D^2-4AF)b^2+(E^2-4CF)a^2-\\ \quad 2(BD-2AE)bc-2(BE-2CD)ac-2(DE-2BF)ab=0,\\ (B^2-4AC)c'^2+(D^2-4AF)b'^2+(E^2-4CF)a'^2-\\ \quad 2(BD-2AE)b'c'-2(BE-2CD)a'c'-2(DE-2BF)a'b'=0,\\ (B^2-4AC)c''^2+(D^2-4AF)b''^2+(E^2-4CF)a''^2-\\ \quad 2(BD-2AE)b''c''-2(BE-2CD)a''c''-2(DE-2BF)a''b''=0.\end{array}\right\}(C)$$

These three equations are sufficient to eliminate three of the coefficients of the general equation, and the remaining ones continue indeterminate.

If the two sides of the triangle represented by the second and third equations in (A) be taken as axes of co-ordinates, these equations must become respectively $y = 0$ and $x = 0$, and therefore $b' = 0$, $c' = 0$, $a'' = 0$, $c'' = 0$, and hence the conditions (C) become in this case

$$(B^2 - 4AC)c^2 - 2(BD - 2AE)bc - 2(BE - 2CD)ac \\ - 2(DE - 2BF)ab = 0,$$

$$E^2 - 4CF = 0,$$

$$D^2 - 4AF = 0.$$

The co-ordinates of the points where the curve touches the axes of co-ordinates, are in this case

$$y = -\frac{D}{2A} \text{ and } x = -\frac{E}{2C}.$$

PROP. CLVII.

(307.) *To find the equation of the locus of the centre of a line of the second degree, which touches the sides of a given angle in two given points.*

Let the sides of the given angle be assumed as axes of co-ordinates, and let the distances of the points of contact from the origin be respectively y' and x'. If the equation of the curve be

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

The conditions of the question give the equations,

$$D^2 - 4AF = 0,$$

$$E^2 - 4CF = 0,$$

$$y' = -\frac{D}{2A},$$

$$x' = -\frac{E}{2C}.$$

The co-ordinates of the centre are

$$x = -\frac{BD - 2AE}{B^2 - 4AC},$$

$$y = -\frac{BE - 2CD}{B^2 - 4AC}.$$

The quantities C, D, E, F, being eliminated from these by means of the former equations, the results are

$$y = \frac{2Ay'^2}{2Ay' + Bx'},$$

$$x = \frac{2Ay'x'}{2Ay' + Bx'}.$$

The equation therefore of the locus sought is found by eliminating B and A from these, which is done by dividing the one by the other, and gives

$$yx' - xy' = 0.$$

The locus is therefore a right line passing through the vertex of the given angle, and bisecting the line joining the points of contact. Since

$$B^2 - 4AC = \frac{4A^2y'^2}{x'^2} \cdot \frac{(y'-y)^2 - y^2}{x^2},$$

the curve is an ellipse or hyperbola, according as $y' < 2y$, or $> 2y'$, and it is a parabola if the centre be at an infinite distance. The species of the curve therefore depends on the side of the line joining the points of contact at which the centre is assumed; if it be at the same side with the vertex of the given angle it is an hyperbola, and if at a different side an ellipse.

If $y' = x'$, the locus is the bisector of the given angle, which is the common axis of all the curves.

PROP. CLVIII.

(308.) *To inscribe an ellipse or hyperbola in a triangle so as to touch its base at the point of bisection, and also to touch one of the sides in a given point.*

By the last Proposition, the centre must be upon the line through the point of bisection of the base, and the vertex of the opposite angle. And the line joining the points of contact of the other two sides must be parallel to the base; hence may be found the point of contact with the other side, and the solution of the problem is evident; if the given point of contact with the side be in the production of the side, the curve is an *hyperbola*, if otherwise, an *ellipse*.

PROP. CLIX.

(309.) *To find the locus of the centre of ellipses or hyperbolas which touch the three sides of a triangle, and touch one in a given point.*

Let two sides c, c', of the triangle be assumed as axes of co-ordinates, and the equation of the third side (c'') is

$$c'y + cx - cc' = 0.$$

The condition of contact with the axes of co-ordinates and this line are

$$\mathrm{E}^2 - 4\mathrm{CF} = 0,$$

$$\mathrm{D}^2 - 4\mathrm{CF} = 0;$$

$$cc' - 2\frac{\mathrm{BD} - 2\mathrm{AE}}{\mathrm{B}^2 - 4\mathrm{AC}}c - 2\frac{\mathrm{BE} - 2\mathrm{CD}}{\mathrm{B}^2 - 4\mathrm{AC}}c' - 2\frac{\mathrm{DE} - 2\mathrm{BF}}{\mathrm{B}^2 - 4\mathrm{AC}} = 0.$$

Let the distance of the point of contact with the axis of x from the origin be x', $\because x' = -\frac{\mathrm{E}}{2\mathrm{C}}$.

The co-ordinates of the centre being

$$y = -\frac{\text{BE} - 2\text{CD}}{\text{B}^2 - 4\text{AC}},$$

$$x = -\frac{\text{BD} - 2\text{AE}}{\text{B}^2 - 4\text{AC}}.$$

We find, after elimination, the equation of the locus sought,

$$2(c' - x')y + 2cx - cc' = 0,$$

which proves the locus sought to be a right line.

If $y = 0$, $x = \frac{c'}{2}$, and if $x = \frac{x'}{2}$, $y = \frac{c}{2}$. Hence it appears, that if a right line be drawn connecting the given point of contact with the vertex of the opposite angle, the right line which is the locus sought bisects this line, and the side of the triangle on which the given point of contact lies.

PROP. CLX.

(310.) *To find the locus of the vertex of a triangle constructed on a given base, one of whose base angles is double the other.*

The extremity of the base being taken as origin, and the base as axis of x, let one base angle be A, and the other 2A, and the co-ordinates of the vertex yx. By trigonometry,

$$\tan. 2\text{A} = \frac{2\tan. \text{A}}{1 - \tan.^2 \text{A}};$$

but $\tan. \text{A} = \frac{y}{x}$, and $\tan. 2\text{A} = \frac{y}{x' - x}$, where x' is the base.

Hence, after reduction, the equation of the curve sought is

$$y^2 - 3x^2 + 2x'x = 0,$$

which is the equation of an hyperbola, whose transverse axis is two-thirds of the base.

PROP. CLXI.

(311.) *Given in magnitude and position the vertical angle of a triangle, whose area is also given, to find the locus of a point which divides the opposite side in a given ratio.*

Let the sides of the given angle be assumed as axes of co-ordinates. The co-ordinates of the point, whose locus is sought, being yx, the equation furnished by the conditions of the question, after the requisite reduction, is

$$yx = \frac{2\text{A}}{\sin.\ \phi} \cdot \frac{m \cdot n}{(m+n)^2},$$

where ϕ = the given angle, A the given area, and $m : n$ the given ratio.

The locus is therefore an hyperbola, whose asymptotes are the sides of the given angle.

PROP. CLXII.

(312.) *To find the locus of the extremity of a portion, assumed upon the sine of an arc, equal to the sum or difference of its chord and versed sine.*

By the conditions expressed, the equation of the sought locus is

$$y = \sqrt{2rx} \pm x,$$

where r is the radius; which, when disengaged from the radical, becomes

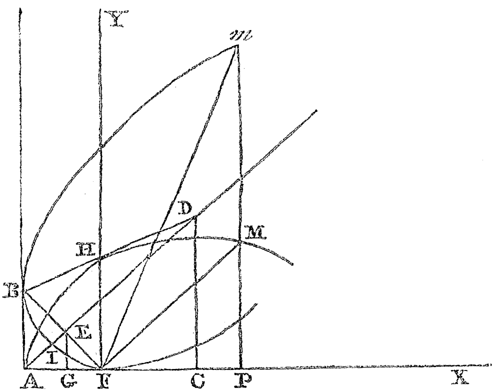
$$y^2 \mp 2yx + x^2 - 2rx = 0,$$

which is the equation of a parabola.

It is evident that the axis of the parabola is inclined at an angle of 45° to the diameter of the circle.

PROP. CLXIII.

(313.) *The ordinate to the axis of a line of the second degree being produced to until the part produced equals the distance of the point where it meets the curve from the focus, to find the locus of the extremity of the produced part.*



Let the ordinate PM be produced until Mm equals FM, F being the focus of the proposed curve: the object is to find the locus of the point m.

The polar equation of a line of the second degree is

$$r = \frac{p}{2(1 - e \cos. \omega)},$$

which represents an ellipse, hyperbola, or parabola, according as $e < 1$, > 1, or $= 1$.

Let Fm be drawn. By the conditions of the question

$$\text{F}m = 2r \cos. m\text{FM} = 2r \sin. m\text{FP}.$$

If yx be the rectangular co-ordinates of the point m, related to FY and FX, as axes of co-ordinates,

$$\text{F}m = \sqrt{y^2 + x^2},$$

$$\sin. m\text{FP} = \frac{y}{\sqrt{y^2 + x^2}},$$

and since MFP $= 90^\circ - 2m$FM,

$$\sqrt{y^2 + x^2} = \frac{p}{(1 - 2e \sin. m\text{FP} \cos. m\text{FP})} \cdot \frac{y}{\sqrt{y^2 + x^2}},$$

which reduced becomes

$$y^2 - 2eyx + x^2 - py = 0,$$

which is the equation of an ellipse, hyperbola, or parabola, according as $e < 1$, > 1, or $= 1$. The locus sought is therefore a line of the second degree of the same species as the proposed.

The solution of the equation for x shows that the curve touches the axis of x at F.

If the equation be solved for y, the roots are

$$y = ex + \tfrac{1}{2}p \pm \sqrt{(e^2 - 1)x^2 + pex + \tfrac{1}{4}p^2}.$$

To find the values of y, which touch the curve, let the values of x, which render the radical $= 0$, be found, and the corresponding values of y are those sought. These values of x are

$$x = -\frac{p}{2(1 + e)},$$

$$x = -\frac{p}{2(1 - e)};$$

and the corresponding values of y are equal to these respectively. These being the distances of the vertices of the proposed curve from the focus, indicate the following circumstances with respect to the position of the proposed locus.

If a perpendicular to AX, the transverse axis of the proposed curve, be drawn through its vertex A, and AB $=$ AF, the locus sought touches AY and AX at B and F.

If BF be drawn, and bisected at E, a right line passing through A and E is the axis of the locus. The line BH, the focal tangent of the proposed curve, is a diameter of the locus whose ordinates are parallel to AY.

The axes of the locus are inclined at an angle of 45° to those of the proposed curve.

If the proposed curve be a parabola, E will be the focus, and BF the parameter of the locus.

If CD be drawn through the centre of the proposed curve perpendicular to AX, and intersecting AE in D, D is the centre of the locus.

If the proposed curve be a parabola, whose parameter is p, the parameter of the locus $= \frac{p}{2^{\frac{3}{2}}}$.

If the proposed curve be an ellipse or hyperbola, let its semiaxes be a and b, $\therefore$ AD $= a\sqrt{2}$. And since the tangents AF and AB are at right angles, AD $= \sqrt{a'^2 + b'^2}$, a' and b' being the semiaxes of the locus; also AD . DE $= a'^2$, and DE $=$ CG . $\sqrt{2} = \frac{(a+c)}{\sqrt{2}}$. Hence it follows that

$$a\sqrt{2} - \frac{a+c}{\sqrt{2}} = a'^2,$$

$$\therefore a(a+c) = a'^2,$$

$$\therefore a(a-c) = b'^2.$$

It will appear by Sect. XVIII. that the areas of the two curves are equal.

PROP. CLXIV.

(314.) *To find the locus of the point of bisection of the normal to a line of the second degree.*

Let the equation of the line related to its axis and vertical tangent as axes of co-ordinates be

$$A^2y'^2 + B^2x'^2 - 2B^2Ax' = 0.$$

Let the co-ordinates of the point of bisection of the normal be yx. By the conditions of the question

$$y' = 2y$$

$$x - x' = \frac{B^2(A - x')}{2A^2},$$

since the subnormal is equal to $-\frac{B^2(A - x')}{A^2}$.

The co-ordinates $y'x'$ being eliminated by means of these equations, and the result arranged according to the dimensions of y and x, we find

$$(B^2 - 2A^2)^2 y^2 + B^2A^2x^2 - 2B^2A^3x - \tfrac{1}{4}B^4(B^2 - 4A^2) = 0,$$

the equation of the locus, which is therefore a line of the second degree, of the same kind as the given one.

If the given curve be a parabola, the equation of the locus (since A is infinite), becomes $16y^2 - 4px + p^2 = 0$, which is the equation of a parabola, whose vertex passes through the focus of the given one, and whose parameter is equal to a fourth of the parameter of the given parabola.

If the given curve be an ellipse or hyperbola, let the origin of co-ordinates be removed to the centre, and the equation of the locus becomes

$$(2A^2 - B^2)^2y^2 + A^2B^2x^2 = \tfrac{1}{4}B^2(2A^2 - B^2)^2.$$

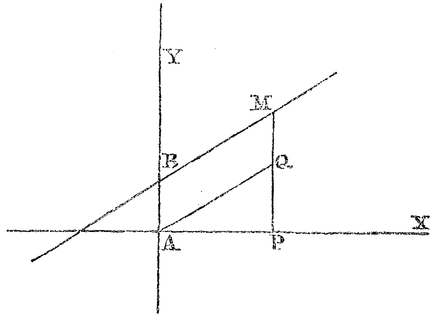
Hence the semiaxes A', B' of the locus are

$$A' = A - \frac{B^2}{2A},$$

$$B' = \tfrac{1}{2}B.$$

PROP. CLXV.

(315.) *A right line* (BM) *being related by its equation to rectangular co-ordinates, if a right line be drawn from the origin* (A), *meeting the ordinate of the proposed right line at* (Q), *so that* AQ = PM, *to find the locus of the point* (Q).



Let AQ $= r$, and QAP $= \omega$; let the equation of the right line be $y - ax - b = 0$. Since by hypothesis $r = y$, and $x = r \cos. \omega$, the equation of the

$$r = \frac{b}{(1 - a\cos.\omega)}.$$

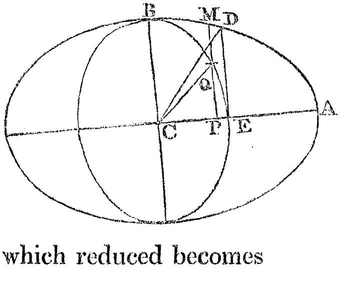
The locus is therefore a line of the second degree, whose parameter is $2b$, and eccentricity $= a$. It is obvious also that the right line BM is the focal tangent.

PROP. CLXVI.

(316.) *If from the centre* (C) *of an ellipse a line* (CQ) *be inflected on the ordinate* (PM) *to the axis, so that* CQ = PM, *to find the locus of the point* Q.

Let the equation of the given ellipse be

$$A'^2y'^2 + B'^2x'^2 = A'^2B'^2,$$



and let the co-ordinates of the point Q be yx. By the conditions of the question, the equation of the locus is

$$y^2 + x^2 = \frac{B'^2}{A'^2}(A'^2 - x^2),$$

which reduced becomes

$$A'^2y^2 + (A'^2 + B'^2)x^2 = A'^2B'^2.$$

Hence the locus is an ellipse, whose axes coincide with those of the given one. Let the semiaxes of the locus be A, B,

$$A = \frac{A'B'}{\sqrt{A'^2 + B'^2}},$$

$$B = B'.$$

Hence, if the angle BCA be bisected by CD, and DE be drawn perpendicular to CA, CE = A.

SECTION XV.

Of the application of the differential and integral calculus to curves.

Of tangents, normals, &c.

(317.) The *differential* and *integral calculus* is peculiarly adapted to the analytical investigation of the properties of curves; and the application of that science to this purpose cannot but be considered as one of the most interesting and useful parts of *Algebraic Geometry*. We shall therefore in the present section proceed to apply the *calculus* to the discovery of those properties to which it is particularly adapted, and in which the principles of common algebra, used in the preceding sections, are either inadequate or incommodious.

PROP. CLXVII.

(318.) *To determine the position of a tangent passing through a given point* $(y'x')$ *on a curve, whose equation is* $\mathrm{F}(yx) = 0$.

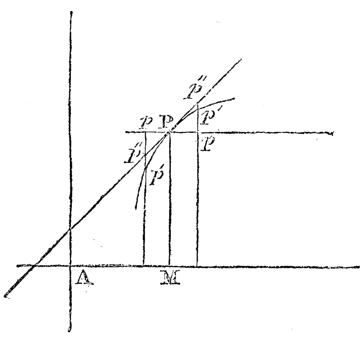
Let the equation of the tangent sought be

$(y-y')-a(x-x')=0,$

where $a = \frac{\sin.\ lx}{\sin.\ ly}$. Let P be the given point, and

$\mathrm{P}p = \pm \Delta x,$

then by Taylor's theorem,



$$pp' = \pm \frac{dy}{dx} \cdot \frac{\Delta x}{1} + \frac{d^2y}{dx^2} \cdot \frac{\Delta x^2}{1 \cdot 2} \pm \frac{d^3y}{dx^3} \cdot \frac{\Delta x^3}{1 \cdot 2 \cdot 3}, \text{ \&c.}$$

If $a = \frac{dy}{dx}$, $\because$ $pp'' = \pm \frac{dy}{dx} \cdot \frac{\Delta x}{1}$, $\because$

$$p'p'' = +\frac{d^2y}{dx^2}\cdot\frac{\Delta x^2}{1\cdot 2}\pm\frac{d^3y}{dx^3}\cdot\frac{\Delta x^3}{1\cdot 2\cdot 3}, \text{ \&c.}$$

In this series such a value may be assigned to Δx as will render the first term greater than the remainder of the series, and the same term will be greater than the remainder of the series for all values of Δx between that and zero. Hence if $\Delta x = \text{P}p$ render the first term greater than the remainder, $p'p'' = pp' - pp''$ will have the same sign with $\frac{d^2y}{dx^2}$, since Δx^2 is positive whatever be the sign of Δx, and the same will be true for all points between P and p. Hence it follows, that at each side of the point P the curve lies at the same side of the right line, and that it lies above or below it according as $\frac{d^2y}{dx^2}$ and y have the same or different signs. The case in which $\frac{d^2y}{dx^2} = 0$ shall be considered hereafter. The curve is ∴ *convex* towards the axis of x, if $\frac{d^2y}{dx^2}$ has the same sign with y, and *concave* if they have different signs. Any other right line passing through the point P must intersect the curve; for let its equation be

$$(y - y') - a'(x - x') = 0,$$

∵ $pp'' = a' \cdot \Delta x$, ∴

$$p'p'' = \pm(\frac{dy}{dx} - a')\frac{\Delta x}{1} + \frac{d^2y}{dx^2}\cdot\frac{\Delta x^2}{1\,.\,2}\pm\frac{d^3y}{dx^3}\cdot\frac{\Delta x^3}{1\,.\,2\,.\,3}.$$

In this series such a value may be assigned to Δx as will render the first term equal to the remainder of the series, and therefore if the sign of Δx be in that case different from that of the remainder of the series, the value of $p'p''$ will vanish, and the right line will meet the curve at that point, and for every point between that and P the right line will lie within the curve.

Hence the equation of the tangent through the point $y'x'$ is

$$(y-y') - \frac{dy'}{dx'}(x - x') = 0.$$

(319.) *Cor.* 1. A point can be found on a curve, through which a tangent shall be parallel to a right line given in position. Let the equation of the right line be

$$\text{A}y + \text{B}x + \text{C} = 0.$$

The co-ordinates of the point of contact may be found by the equations

$$\frac{dy'}{dx'} = -\frac{\text{B}}{\text{A}},$$
$$\text{F}(y'x') = 0,$$

the latter being the equation of the curve.

(320.) *Cor.* 2. If $\frac{dy'}{dx'} = 0$, the tangent is parallel to the axis of x, and *vicè versâ.*

(321.) *Cor.* 3. If $\frac{dy}{dx} = \infty$, the tangent is parallel to the axis of y, and *vice versâ.*

(322.) *Cor.* 4. The equation of a tangent to a given point on a line of the second degree, may be found by differentiating the equation

$$\text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0,$$

which gives

$$\frac{dy}{dx} = -\frac{2\text{C}x + \text{B}y + \text{E}}{2\text{A}y + \text{B}x + \text{D}},$$

and therefore the equation of the tangent is

$$(2\text{A}y' + \text{B}x' + \text{D})\,(y-y') + (2\text{C}x' + \text{B}y' + \text{E})\,(x - x') = 0,$$

which is the same with the result of (133).

PROP. CLXVIII.

(323.) *To find the subtangent to a given point on a curve.*

In the equation of the tangent let $y = 0$, and the value of the subtangent s is $x' - x$, ∴

$$\text{s} = \frac{y' \,.\, dx'}{dy'},$$

and, in like manner, the value of the subtangent measured on the axis of y, is

$$\text{s}' = \frac{x'dy'}{dx'}.$$

(324.) *Cor.* If the length of the tangent be T,

$$\text{T}^2 = \frac{y'^2(dy'^2 + dx'^2 + 2dy'dx' \cos. yx)}{dy'^2},$$

which, when $yx = \frac{\pi}{2}$, becomes

$$\text{T}^2 = y'^2 \,.\, (1 + \frac{dx'^2}{dy'^2}).$$

PROP. CLXIX.

(325.) *To find the equation of the normal and the subnormal.*

The equation of a line perpendicular to the tangent is, by (39),

$$(\cos. yx + \frac{dy'}{dx'})\ (y - y') + (\frac{dy'}{dx'} \cos. yx + 1)\ (x - x') = 0,$$

which, when $yx = \frac{\pi}{2}$, becomes

$$(y - y') + \frac{dx'}{dy'}(x - x') = 0.$$

The subnormal, taken relatively to each axis of co-ordinates, may be found by supposing y and x successively $= 0$ in these equations, which gives

$$s = -y' \cdot \frac{dy' + \cos. yx \, . \, dx'}{dx' + \cos. yx \, . \, dy'},$$

$$s' = -x' \cdot \frac{dx' + \cos. yx \, . \, dy'}{dy' + \cos. yx \, . \, dx'},$$

which, when $yx = \frac{\pi}{2}$, become

$$s = -\frac{y'dy'}{dx'},$$

$$s' = -\frac{x'dx'}{dy'}.$$

PROP. CLXX.

(326.) *To transform any expression involving the co-ordinates* YX *of any point, and their differentials, into one involving the polar co-ordinates* Z, ω, *and their differentials.*

The angle yx may in this case be taken as a right angle, to avoid the complexity of the expressions which would result from any other supposition. Any formula related to oblique angled co-ordinates may be transformed first to rectangular, and then to polar co-ordinates.

The angle yx being a right angle, the point $y'x'$ the pole, and the angle ω being measured from a line which makes with the axis of x an angle ω',

$$y = z \sin. (\omega + \omega'),$$

$$x = z \cos. (\omega + \omega'),$$

$$dy = z \cos. (\omega + \omega')d\omega + \sin. (\omega + \omega')dz,$$

$$dx = \cos. (\omega + \omega')dz - z\sin. (\omega + \omega')d\omega,$$

$$\therefore \frac{dy}{dx} = \frac{zd\omega + \tan. (\omega + \omega')dz}{dz - \tan. (\omega + \omega')zd\omega},$$

$$d^2y = \sin. (\omega + \omega')d^2z + 2 \cos. (\omega + \omega')dzd\omega - z \sin. (\omega + \omega')d\omega^2,$$

$$d^2x = \cos. (\omega + \omega')d^2z - 2 \sin. (\omega + \omega')dzd\omega - z \cos. (\omega + \omega')d\omega^2.$$

By these formulæ any expression involving yx and their first and second differentials, can be converted into an expression involving $z\omega$, and their first and second differentials; and in like manner, by continuing the process, the substitution necessary for the differentials of higher orders may be found.

PROP. CLXXI.

(327.) *To express the angle under the radius vector of a curve, whose equation is* $\mathrm{z} = \mathrm{F}(\omega)$, *and a tangent through any point* $\mathrm{z}\omega$.

Let the angle under the radius vector and fixed axis be zx, and that under the tangent and the same tx, and the angle under the tangent and radius vector tz. Now,

$$\tan.\,tz = \frac{\tan.\,tx - \tan.\,rx}{1 + \tan.\,tx\,.\,\tan.\,rx},$$

but

$$\tan.\,zx = \frac{y}{x}, \qquad \tan.\,tx = \frac{dy}{dx};$$

hence

$$\tan.\,tz = -\frac{ydx - xdy}{xdx + ydy}.$$

Substituting in this expression for y, x, dy, dx, the values found in (326), the result is

$$\tan.\,tz = \frac{z^2 d\omega \left\{ \sin.^2(\omega + \omega') + \cos.^2(\omega + \omega') \right\}}{zdz \left\{ \sin.^2(\omega + \omega') + \cos.^2(\omega + \omega') \right\}},$$

$$\therefore \tan.\,tz = \frac{zd\omega}{dz};$$

hence also

$$\sin.\,tz = \frac{zd\omega}{(z^2 d\omega^2 + dz^2)^{\frac{1}{2}}}.$$

PROP. CLXXI.

(328.) *Given the polar equation* $z = F(\omega)$ *of a curve, to express the polar subtangent.*

Let the polar subtangent be P, ∵ $P = z \tan. tz$, ∵

$$P = \frac{z^2 d\omega}{dz}.$$

Of rectification and quadrature.

PROP. CLXXII.

(329.) *The equation of a curve being given, to find the length of any arc of it.*

1. If the equation be related to fixed axes of co-ordinates yx, let A be the arc, and it is plain that

$$dA = (dy^2 + dx^2 + 2dy\,dx \cos. yx)^{\frac{1}{2}},$$

$$\therefore A = \int (dy^2 + dx^2 + 2dy\,dx \cos. yx)^{\frac{1}{2}} + C,$$

in which the value of the constant C is determined by the co-ordinates of the extremities of the arc sought.

If $yx = \frac{\pi}{2}$,

$$A = \int (dy^2 + dx^2)^{\frac{1}{2}} + C.$$

2. If the curve be expressed by a polar equation $r = F(\omega)$, let the values of dy and dx (326) be substituted in the preceding equation, and the result is

$$A = \int (z^2 d\omega^2 + dz^2)^{\frac{1}{2}} + C,$$

where C is determined by the values of z and ω for the extremities of the proposed arc.

The determination of the length of an arc is usually called the rectification of the curve.

PROP. CLXXIII.

(330.) *To find the area included by two values of* Y, *the curve and the axis of* X, *or by two radii vectores, if the curve be expressed by a polar equation.*

1. Let the equation be $F(yx) = 0$, and A' the sought area,

$$dA' = ydx \ . \ \sin. yx,$$

$$\therefore A' = \int ydx \ . \ \sin. yx + C;$$

and if the co-ordinates be rectangular,

$$A' = \int ydx + C;$$

where C is determined by the values of y, which include the area.

2. If the curve be expressed by a polar equation,

$$dA' = \tfrac{1}{2}\sin. tz \ . \ zdA,$$

where $dA = (z^2d\omega^2 + dz^2)^{\frac{1}{2}}$ (329), and

$$\sin. tz = \frac{zd\omega}{(z^2d\omega^2+dz^2)^{\frac{1}{2}}}; \ \therefore$$

$$dA' = \tfrac{1}{2}z^2d\omega,$$

$$\therefore A' = \tfrac{1}{2}\int z^2d\omega + C,$$

where C, as before, is determined by the values of z, which include the proposed area.

The determination of the area is usually called the quadrature of the curve.

Of osculating circles and evolutes.

(331.) The principles on which the investigation of a line touching a curve is founded being generalised, produce some results of considerable importance in the analysis of curves. The object sought in that case, was a right line meeting the curve in such manner, that no other right line passing through the same point could pass between it and the curve, but must pass at the same side of both. Now a circle may be sought fulfilling similar conditions, scil. so meeting the curve, that no other circle through the same point can pass between it and the curve. Let the equation of the curve and that of the sought circle be

$$F(yx) = 0 \quad (1),$$

$$(y - y')^2 + (x - x')^2 - R^2 = 0 \quad (2),$$

where $y'x'$ are the co-ordinates of the centre of the circle, and R is the radius. In order to limit the circle to touch the curve at the point yx, it is necessary that the first differential coefficient in the two equations be equal to each other, for, in that case, the same right line shall touch them both at the point P. By differentiating the equation of the circle, the result is

$$(y - y')dy + (x - x')dx = 0 \quad (3).$$

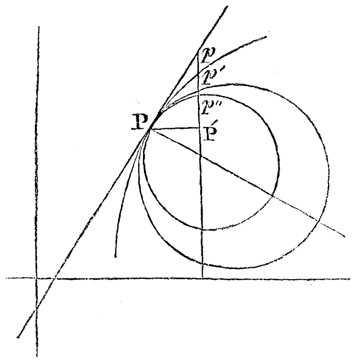
The value of $\frac{dy}{dx}$ resulting from equation (1) being substituted in this, and $y'x'$ being supposed variable, and yx constant, it is the equation of the locus of the centre of a circle touching the curve at the point, and shows that the centres of all such circles are on the normal (39). The question then is, among those circles to determine that between which and the curve none of the others pass. For this purpose, if the equation (3) be differentiated,

$$(y - y')d^2y + (x - x')d^2x + dy^2 + dx^2 = 0 \quad (4);$$

this and (3) will determine the centre of the sought circle.

Let the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. for the equation $F(yx) = 0$ of the curve Pp be A', A'', A''', &c., and their values for the circle (Pp') determined by (2), (3), (4), be B', B'', B''', &c.; and the values for any other circle Pp'' be c', c'', c''', &c.

Let $PP' = \Delta x$, $P'p = \Delta y$, $P'p' = \Delta y'$, $P'p'' = \Delta y''$; then by Taylor's theorem,



$$\Delta y = A' \,.\, \frac{\Delta x}{1} + A'' \frac{\Delta x^2}{1.2} + A''' \frac{\Delta x^3}{1.2.3}, \text{ \&c.}$$

$$\Delta y' = B' \,.\, \frac{\Delta x}{1} + B'' \frac{\Delta x^2}{1.2} + B''' \frac{\Delta x^3}{1.2.3}, \text{ \&c.}$$

$$\Delta y'' = C' \,.\, \frac{\Delta x}{1} + C'' \frac{\Delta x^2}{1.2} + C''' \frac{\Delta x^3}{1.2.3}, \text{ \&c.}$$

Now, since $A' = B' = C'$, and $A'' = B''$, by the conditions already laid down, $\because$

$$\Delta y - \Delta y'' = (A'' - C'') \frac{\Delta x^2}{1.2} + (A''' - C''') \frac{\Delta x^3}{1.2.3}, \text{ \&c.}$$

$$\Delta y' - \Delta y'' = (B'' - C'') \frac{\Delta x^2}{1.2} + (B''' - C''') \frac{\Delta x^3}{1.2.3}, \text{ \&c.}$$

The value of Δx may be taken so small, that the first term of each of these series shall surpass the value of the sum of the remaining terms, and therefore the sign of the whole series will be that of the first term in each; but since $A'' = B''$, $\because A'' - C'' = B'' - C''$, hence the signs of $\Delta y - \Delta y''$, and $\Delta y' - \Delta y''$ are the same, and therefore the point p'' cannot lie between the points p and p', that is to say, the part of the circle Pp'' flowing immediately from the point P, must lie at the same side of the curve Pp and the circle Pp'.

(332.) *Def.* The circle thus determined, is called *the osculating circle* to the point P.

PROP. CLXXIV.

(333.) *To express the co-ordinates of the centre and the radius of the osculating circle.*

Let the values of $y'x'$ and R be determined by the equations (2), (3), (4); whence,

$$y' = y + \frac{(dy^2 + dx^2)dx}{d^2y\, dx - d^2x\, dy},$$

$$x' = x + \frac{(dy^2 + dx^2)dy}{d^2x\, dy - d^2y\, dx},$$

$$R = \pm \frac{(dy^2 + dx^2)^{\frac{3}{2}}}{d^2y\, dx - d^2x\, dy}.$$

The value of R being a square root, is susceptible of two signs: which we should employ is conventional. If the concavity of the curve be turned towards the axis of x, the radius of the circle which passes through the point of contact will also be in that direction. If the radius thus situate be considered positive, the value of R given above must in that case be taken with a negative sign, because d^2y will in that case be negative (318), y being supposed positive. We shall therefore consider the value of R to have the negative sign prefixed.

(334.) In the preceding investigation we have considered both dy and dx as variable, for the sake of generality, and also because it preserves more symmetry in the expressions. If dx, however, be considered constant, $d^2x = 0$, and the expressions therefore become

$$y' = y + \frac{dy^2 + dx^2}{d^2y},$$

$$x' = x - \frac{(dy^2 + dx^2)dy}{d^2y\, dx},$$

$$R = - \frac{(dy^2 + dx^2)^{\frac{3}{2}}}{d^2y\, dx}.$$

(335.) The *osculating circle* is known by the name of the *circle of curvature*, and its radius is called the *radius of curvature*. It received this name probably from the supposition that it has the same curvature with the curve at the point of contact; but this is not strictly the case, as there are an infinite number of other curves which may pass between it and the given curve, and whose curvatures therefore approach nearer to that of the curve than the curvature of the *osculating circle*, as will be shown hereafter. The curvature of this circle, however, approaches nearer to that of the curve than the curvature of *any other circle*, and in this sense the name of the circle of curvature may not be inapplicable.

PROP. CLXXV.

(336.) *A curve being expressed by a polar equation*, $z = F(\omega)$, *to find the radius of the osculating circle.*

In the value of R in the equation

$$R = -\frac{(dy^2 + dx^2)^{\frac{3}{2}}}{d^2y\,dx - d^2x\,dy},$$

let the values of dy, dx, d^2y, d^2x, be substituted, and the result is

$$R = -\frac{(z^2d\omega^2 + dz^2)^{\frac{3}{2}}}{(z^2d\omega^2 + 2dz^2 - zd^2z)d\omega}.$$

(337.) *Def.* The *osculating circle* varying its *position* and *magnitude* for the different points of the curve, the *locus* of its centre is a line whose nature and properties depend on, and are derivable from, those of the given curve. This locus is called the *evolute* of the *curve*, and the curve is its *involute*.

PROP. CLXXVI.

(338.) *The equation of a curve* F$(yx) = 0$ *being given, to find that of its evolute.*

By the equations

$$y' - y = \frac{(dy^2 + dx^2)dx}{d^2y\,dx - d^2x\,dy},$$

$$x' - x = \frac{(dy^2 + dx^2)dy}{d^2x\,dy - d^2y\,dx},$$

united with that of the curve and its first and second differentials, the quantities y, x, dy, dx, d^2y, and d^2x, may be eliminated, and an equation will be thence found expressing the relation between $y'x'$, the co-ordinates of the centre of the osculating circle, and the constants of the equation F$(yx) = 0$ of the curve. This relation is independent of the values of y and x since they were eliminated, and therefore expresses a relation between y' and x' common to all the points of the curve, and is therefore the equation of the locus of the centre of curvature.

(339.) The principle here used is one of the most extensive power and utility in analytical and geometrical investigations. The elimination of several variables by several equations always gives an equation or equations which express the relation between those which remain, and which, being independent of any particular values of those which have been eliminated, is common to all values of them. We cannot advance a step in analytical investigations without being sensible of the power with which this invests us.

PROP. CLXXVII.

(340.) *To find the equation of a tangent to the evolute drawn from a point* (yx) *on the curve.*

By (318) the equation is

$$y' - y = \frac{dy'}{dx'}(x' - x);$$

the object is therefore to express $\frac{dy'}{dx'}$ as a function of yx.

Let the equation

$$(y - y')dy + (x - x')dx = 0$$

be differentiated, $y'x'$ being considered variable; the result is

$$(y-y')d^2y + (x-x')d^2x + dy^2 + dx^2 - dy\,dy' - dx\,dx' = 0,$$

which being subtracted from

$$(y - y')d^2y + (x - x')d^2x + dy^2 + dx^2 = 0,$$

gives

$$dy\,dy' + dx\,dx' = 0,$$

$$\text{or } \frac{dy'}{dx'} = -\frac{dx}{dy}.$$

Hence the equation of the tangent sought is

$$(y' - y)dx + (x' - x)dy = 0.$$

(341.) *Cor.* Hence (325) the tangent to the evolute drawn from any point in the curve coincides with the normal of the curve through the same point, and therefore (337), the centre of the *osculating circle* is the point of contact; and the length of the tangent, from the point on the curve to the point of contact, is the *radius* of the *osculating circle.*

PROP. CLXXVIII.

(342.) *To find the length of an arc of the evolute to a given curve.*

If the equation

$$(y - y')^2 + (x - x')^2 = R^2$$

be differentiated, considering $y'x'$ and R as variables, the result is

$$(y - y')(dy - dy') + (x - x')(dx - dx') = \text{R}d\text{R};$$

but since $(y - y')dy + (x - x')dx = 0$, ∵

$$-(y - y')dy' - (x - x')dx' = \text{R}d\text{R},$$

by this and the equations

$$(y - y')^2 + (x - x')^2 = \text{R}^2,$$
$$(y - y')dx' - (x - x')dy' = 0,$$

the quantities $(y - y')$ and $(x - x')$ being eliminated, we find

$$(d\text{R})^2 = dy'^2 + dx'^2,$$
$$d\text{R} = (dy'^2 + dx'^2)^{\frac{1}{2}};$$

the latter member of this equation being the differential of the arc of the evolute, it follows that this arc and the radius of curvature increase by equal differences. Let VV′ be the evolute of the curve MM′, and V the centre of the osculating circle corresponding to the point M; the line MV therefore touches the evolute at the point V. In like manner, let M′V′ be the radius of the osculating circle at the point M′ touching the evolute at V′. By what has been proved, the arc VV′ of the evolute is equal to the difference between the lines MV and M′V′. Hence it follows, that if MV be supposed a flexible string wrapped upon the curve VV′ as it unwinds itself from off VV′ its extremity M will trace out the curve MM′.

(343.) The analogy between this manner of conceiving the involute to be described, and the description of a circle is manifest. The evolute may be conceived to act as centre, and the radius, instead of being a constant length, to be variable.

(344.) *Cor.* It follows also, that if the involute be an algebraic curve, the evolute is *rectifiable.* For any arc of it is equal to the difference between the radii of the osculating circles at the points of the involute corresponding to the extremities of the arc of the evolute.

Of asymptotes.

(345.) Two lines are said to be asymptotes to each other when extending indefinitely they continually approach each other, and approximate closer than any assignable distance, and yet never intersect or touch. Thus, if two curves be represented by the equations $\text{F}(yx) = 0$ and $\text{F}'(y'x') = 0$, and for the same value of x the value of $(y - y')$ diminishes without limit as x increases, but that condition $y - y' = 0$ can only be fulfilled by supposing x infinite, the curves are said to be *asymptotes* to each other.

PROP. CLXXIX.

(346.) *To find a right line which is an asymptote to a curve, whose equation is* $\text{F}(yx) = 0$.

This problem may be solved by considering the limit of the position of a tangent when the point of contact is removed to an infinite distance. The equation of a tangent through a point $y'x'$ is

$$(y - y') = \frac{dy}{dx}(x - x').$$

If in this equation $y = 0$, the corresponding value of x will be

$$\text{AB} = \frac{x'dy - y'dx}{dy};$$

and if $x = 0$, the corresponding value of y will be

$$\text{AC} = \frac{y'dx - x'dy}{dx}.$$

If when x' is increased without limit, these quantities have limits, the curve has asymptotes, and they will be determined by these limiting values of AB and AC.

If AB have a limit, but AC none, the asymptote is parallel to AX; and if AB have a limit, but AC none, the asymptote is parallel to AY.

If neither have a limit, the curve has no asymptote; or it may be conceived to have asymptotes infinitely removed.

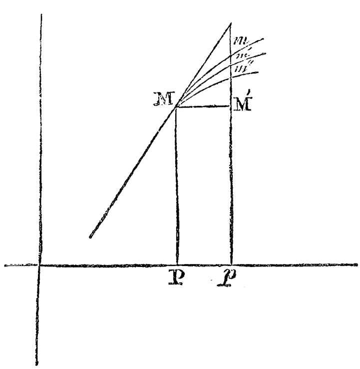
If the limits be impossible, the curve has no asymptotes.

If the limit of AB$=0$, the axis of y is an asymptote; and if the limit of AC $=0$, the axis of x is an asymptote. If both limits $=0$, the asymptotes pass through the origin, and their direction may be found by the limiting value of $\frac{dy}{dx}$, as x is indefinitely increased.

SECTION XVI.

Of the general principles of contact and osculation.

(347.) The principles which have been already explained relative to the contact of right lines and circles with curves, and also those on which the osculation of the circle with a curve has been founded, may be considerably generalised by the powers which the differential and integral calculus gives us.



Let three curves (Mm, Mm', Mm'') having a common point M, be represented by the equations,

$$F(yx) = 0,$$
$$F'(y'x') = 0,$$
$$F''(y''x'') = 0.$$

Let Pp = Mm = Δx, and M$'m$ = Δy, M$'m'$ = $\Delta y'$, and M$'m''$ = $\Delta y''$. Hence by Taylor's theorem,

$$\Delta y = \frac{dy}{dx} \cdot \frac{\Delta x}{1} + \frac{d^2y}{dx^2} \cdot \frac{\Delta x^2}{1 \,.\, 2} + \frac{d^3y}{dx^3} \cdot \frac{\Delta x^3}{1 \,.\, 2 \,.\, 3} + \frac{d^4y}{dx^4} \cdot \frac{\Delta x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4}, \text{ \&c. } (1),$$

$$\Delta y' = \frac{dy'}{dx'} \cdot \frac{\Delta x}{1} + \frac{d^2y'}{dx'^2} \cdot \frac{\Delta x^2}{1 \,.\, 2} + \frac{d^3y'}{dx'^3} \cdot \frac{\Delta x^3}{1 \,.\, 2 \,.\, 3} + \frac{d^4y'}{dx'^4} \cdot \frac{\Delta x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4}, \text{ \&c. } (2),$$

$$\Delta y'' = \frac{dy''}{dx''} \cdot \frac{\Delta x}{1} + \frac{d^2y''}{dx''^2} \cdot \frac{\Delta x^2}{1 \,.\, 2} + \frac{d^3y''}{dx''^3} \cdot \frac{\Delta x^3}{1 \,.\, 2 \,.\, 3} + \frac{d^4y''}{dx''^4} \cdot \frac{\Delta x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4}, \text{ \&c. } (3).$$

If in (1) and (2) $\frac{dy}{dx} = \frac{dy'}{dx'}$, these two curves M$m$ and Mm' will have a common rectilinear tangent at M and any other curve Mm'' not fulfilling the same condition, must lie at the same side of the two curves Mm, Mm', so touching at M, and cannot pass between them. This has been already established (318).

If in (1), (2), and (3),

$$\frac{dy}{dx} = \frac{dy'}{dx'} = \frac{dy''}{dx''},$$

the three curves tôuch at M; but if also the condition

$$\frac{d^2y}{dx^2} = \frac{d^2y'}{dx'^2},$$

the curve Mm' must pass between Mm and Mm''.

For by subtracting (3) from (1) and (2),

$$\Delta y - \Delta y'' = \left\{ \frac{d^2y}{d^2x} - \frac{d^2y''}{dx''^2} \right\} \frac{\Delta x^2}{1 \,.\, 2} + \left\{ \frac{d^3y}{dx^3} - \frac{d^3y''}{dx''^3} \right\} \frac{\Delta x^3}{1 \,.\, 2 \,.\, 3}, \text{ \&c.}$$

$$\Delta y' - \Delta y'' = \left\{ \frac{d^2y'}{dx'^2} - \frac{d^2y''}{dx''^2} \right\} \frac{\Delta x^2}{1 \,.\, 2} + \left\{ \frac{d^3y'}{dx'^3} - \frac{d^3y''}{dx''^3} \right\} \frac{\Delta x^3}{1 \,.\, 2 \,.\, 3}, \text{ \&c.}$$

Such a value Pp may be assigned to Δx as will render the first terms of these series greater than the sum of the remaining terms, and the same condition will hold good for all values of Δx between Pp and zero; therefore the sign of the entire series will be in each case that of the coefficient of $\frac{\Delta x^2}{1 \,.\, 2}$ in the first term, which coefficients being equal by the condition $\frac{d^2y}{dx^2}=\frac{d^2y'}{dx'^2}$, the terms $\Delta y - \Delta y''$ and $\Delta y' - \Delta y''$ will have the same sign. Hence the arc Mm'', intercepted between PM and pm'', must lie at the same side of the curves Mm and Mm', and therefore the contact of these two must be more intimate than that of Mm'' with either of them.

(348.) From what has been said, it appears that curves may have with each other different degrees of contact, and the principles on which the theory of contacts, in its most general form is founded, are embraced in the following theorem.

PROP. CLXXX.

(349.) *Let three curves* (MM, MM$'$, MM$''$,) *having a common point* (M), *be represented by the equations* F(yx) = 0, F$'$(y$'$x$'$) = 0, F$''$(y$''$x$''$) = 0, *and let the successive differential coefficients of these equations, from the* 1*st to the* p*th, be equal each to each; and also let the successive differential coefficients of the first two equations, from the* p*th to the* n*th, be equal each to each. Under these conditions the part of the curve* MM$''$ *next the point* M, *must lie at the same side of the two curves* Mm *and* Mm'.

For, by hypothesis, the terms of the three series (1), (2), (3), as far as the pth term, are equal each to each; therefore, if (3) be subtracted from (1) and (2), the result is

$$\Delta y - \Delta y'' = \left\{\frac{d^{p+1}y}{dx^{p+1}} - \frac{d^{p+1}y''}{dx''^{p+1}}\right\}\frac{\Delta x^{p+1}}{1\,.\,2\ldots(p+1)}$$

$$+\left\{\frac{d^{p+2}y}{dx^{p+2}} - \frac{d^{p+2}y''}{dx''^{p+2}}\right\}\frac{\Delta x^{p+2}}{1\,.\,2\ldots(p+2)}$$

$$+\left\{\frac{d^{p+3}y}{dx^{p+3}} - \frac{d^{p+3}y''}{dx''^{p+3}}\right\}\frac{\Delta x^{p+3}}{1\,.\,2\ldots(p+3)}, \text{ \&c.}$$

$$\Delta y' - \Delta y'' = \left\{ \frac{d^{p+1}y'}{dx'^{p+1}} - \frac{d^{p+1}y''}{dx''^{p+1}} \right\} \frac{\Delta x^{p+1}}{1\,.\,2 \ldots (p+1)}$$
$$+ \left\{ \frac{d^{p+2}y'}{dx'^{p+2}} - \frac{d^{p+2}y''}{dx''^{p+2}} \right\} \frac{\Delta x^{p+2}}{1\,.\,2 \ldots (p+2)}$$
$$+ \left\{ \frac{d^{p+3}y'}{dx'^{p+3}} - \frac{d^{p+3}y''}{dx''^{p+3}} \right\} \frac{\Delta x^{p+3}}{1\,.\,2 \ldots (p+3)}, \text{ \&c.}$$

By hypothesis, the sum of the first $(n - p)$ terms of these series are equal: let this sum be (s), therefore

$$\Delta y - \Delta y'' = \text{s} + \left\{ \frac{d^{n+1}y}{dx^{n+1}} - \frac{d^{n+1}y''}{dx''^{n+1}} \right\} \frac{\Delta x^{n+1}}{1\,.\,2 \ldots (n+1)}$$
$$+ \left\{ \frac{d^{n+2}y}{dx^{n+2}} - \frac{d^{n+2}y''}{dx''^{n+2}} \right\} \frac{\Delta x^{n+2}}{1\,.\,2 \ldots (n+2)}$$
$$+ \left\{ \frac{d^{n+3}y}{dx^{n+3}} - \frac{d^{n+3}y''}{dx''^{n+3}} \right\} \frac{\Delta x^{n+3}}{1\,.\,2 \ldots (n+3)}, \text{ \&c.}$$

$$\Delta y' - \Delta y'' = \text{s} + \left\{ \frac{d^{n+1}y'}{dx'^{n+1}} - \frac{d^{n+1}y''}{dx''^{n+1}} \right\} \frac{\Delta x^{n+1}}{1\,.\,2 \ldots (n+1)}$$
$$+ \left\{ \frac{d^{n+2}y'}{dx'^{n+2}} - \frac{d^{n+2}y''}{dx''^{n+2}} \right\} \frac{\Delta x^{n+2}}{1\,.\,2 \ldots (n+2)}$$
$$+ \left\{ \frac{d^{n+3}y'}{dx'^{n+3}} - \frac{d^{n+3}y''}{dx''^{n+3}} \right\} \frac{\Delta x^{n+3}}{1\,.\,2 \ldots (n+3)}, \text{ \&c.}$$

The succeeding terms of the series being supposed to be finite, such a value (MM′) can be assigned to Δx as will render (s) greater than the sum of the remaining terms of either of these series, and therefore $\Delta y - \Delta y''$ and $\Delta y' - \Delta y''$ will both have the sign of (s), and Δy and $\Delta y'$ will be both greater or both less than $\Delta y''$, for this and every value of Δx between MM′ and zero. Hence all the corresponding points of the curve Mm'' lie above both Mm and Mm', or below both, according as (s) is negative or positive, and therefore the curve Mm'' can in no case lie between Mm and Mm'.

(350.) *Cor.* 1. Hence, in general, if any two curves have a common point (M), and the co-ordinates of that point being substituted for yx in the successive differential coefficients, beginning from the first, render them respectively equal each to each, no curve in which the same equality takes place for a less number of differential coefficients can pass between them at the point (M), and every curve in which the same equality takes place for a greater number of differential coefficients, must pass between them at that point.

(351.) *Cor.* 2. The greater the number of differential coefficients of the equations of two curves are equal the more intimate the contact.

(352.) *Def.* The contact involved in the conditions

$$y = y', \quad \frac{dy}{dx} = \frac{dy'}{dx'},$$

is called *contact of the first order.* That involved in the conditions

$$y = y', \frac{dy}{dx} = \frac{dy'}{dx'}, \quad \frac{d^2y}{dx^2} = \frac{d^2y'}{dx'^2},$$

is called *contact of the second order.* And in general the contact involved in the conditions

$$y = y', \frac{dy}{dx} = \frac{dy'}{dx'}, \quad \frac{d^2y}{dx^2} = \frac{d^2y'}{dx'^2}, \quad \ldots \quad \frac{d^ny}{dx^n} = \frac{d^ny'}{dx'^n},$$

is called *contact of the* n*th order.*

PROP. CLXXXI.

(353.) *To find that curve of a given species* F′(y′x′) = 0, *which has the highest order of contact with a given curve* F(xy) = 0.

Let the number of constants in the equation $F'(y'x') = 0$ be n. The equations being differentiated $n-1$ times, and the values of the constants of the equation $F'(y'x') = 0$ found from the equations

$$y = y', \frac{dy}{dx} = \frac{dy'}{dx'}, \quad \frac{d^2y}{dx^2} = \frac{d^2y'}{dx'^2} \ldots \frac{d^{n+1}y}{dx^{n+1}} = \frac{d^{n-1}y'}{dx'^{n-1}},$$

being substituted in that equation, will give the equation of the curve sought. For the number of constants being by hypothesis n, will be sufficient to fulfil these conditions, and therefore the contact may be of the $(n-1)$th order; but it cannot be of a higher order, as n constants could not fulfil more than n equations.

(354.) *Def.* Of all curves of a given species, touching a given curve at a given point, that whose contact is of the highest order, is called the *osculating curve* of that species, and the contact is distinguished from the contact of other curves of the same kind by the name *osculation*. If the number of constants in the equation of the *osculating curve* be n, the osculation is said to be of the $(n-1)$th order.

(355.) When we speak of different degrees of contact and osculation, it should not be understood that the curve, which is said to touch another in a greater or less degree, is more or less coincident with the curve it is said to touch. The fact is, there is only one point of actual coincidence, namely, the point fulfilling the conditions $x = x'$, $y = y'$. But the portions of the curve flowing from this common point may be more or less distant from each other. Thus, as has been proved, a curve of a given species, meeting another in a given point, may be so situate that no curve of the same species can pass between them; but by this it is not at all to be imagined that any coincidence takes place between any arc of the one curve, and any arc of the other, how small soever these arcs may be supposed. Nay, so far from any such coincidence taking place, it follows from what has been already proved, that how high soever the order of contact of two curves may be, another curve can be found, whose contact, being of a higher order, will pass between them.

(356.) It appears also that the higher the degree of the equation of a curve is, the higher the order of its osculation, since it contains a greater number of constants; and that since the number of points necessary to determine a curve is always equal to the number of constants in its equation, as will appear by Sect. XXI., the order of its osculation is always one less than the number of points necessary to determine it.

(357.) The osculation of curves is sometimes explained by supposing the osculating curve first to intersect the given curve in n points, and then supposing these points to be united in one. But as the principles can be more clearly explained without this supposition, and as it is only calculated to mislead the student, and produce wrong ideas of what are called contact and osculation, we have rejected it.

(358.) From what has been said, it appears that the contact of a right line with a curve is both *contact* and *osculation* of the first order. For the equation of a right line

$$y - ax - b = 0,$$

involves but two constants, and therefore the highest order of contact of which it is susceptible is the first, and the equation of the osculating right line is, as has been already found,

$$(y - y')dx - (x - x')dy = 0,$$

$y'x'$ being the point common to it and the curve.

(359.) The equation of the circle

$$(y - y')^2 + (x - x')^2 = R^2,$$

involves three constants, the co-ordinates of the centre, and the radius. The highest order of contact of which this is susceptible is the second, and therefore the *osculation* of a circle is of the *second order*.

SECTION XVII.

Of the singular points of curves.

(360.) *Def.* Those points of a curve which possess any remarkable properties, which the adjacent points do not possess, are called *singular points*. The differential calculus enables us to discover these points, and in general to discover the figure of any curve whose equation is given.

(361.) The position of the tangent being determined by the equation

$$(y - y') - \frac{dy}{dx}(x - x') = 0,$$

if the co-ordinates of P satisfy the equation $\frac{dy}{dx} = 0$, the tangent at the point P must be parallel to the axis of x, for the equation of the tangent becomes in that case

$$y - y' = 0.$$

(362.) In like manner, if $\frac{dy}{dx} = \frac{1}{0}$, the equation of the tangent becomes

$$x - x' = 0,$$

and is therefore parallel to the axis of y.

(363.) If $\frac{d^2y}{dx^2} = 0$, the series in (318) gives

$$pp'' - pp' = \mp \frac{d^3y}{dx^3} \cdot \frac{\Delta x^3}{1.2.3} - \frac{d^4y}{dx^4} \cdot \frac{\Delta x^4}{1.2.3.4}$$
$$\mp \frac{d^5y}{dx^5} \cdot \frac{\Delta x^5}{1.2.3.4.5}, \text{ \&c.}$$

A value Pp of Δx being taken so small that the first term shall surpass the remainder of the series, the sign of

$pp'' - pp'$, resulting from $+\Delta x$, will be different from that which results from $-\Delta x$; and the same being true for all values of Δx between Pp and zero, it follows that the parts of the curve on either side of P lie at different sides of the tangent, and consequently that as the curve passes the point P, it changes the direction of its curvature. Such a point is called a *point of contrary flexure*, or *a point of inflexion.*

(364.) The principle is however more general. If several successive differential coefficients after the first vanish, when the co-ordinates of the point P are substituted for the variables in their expressions, let the first differential coefficient, which does not vanish, be $\frac{d^n y}{dx^n}$.

1. If n be an even number,

$$pp'' - pp' = -\frac{d^n y}{dx^n} \cdot \frac{\Delta x^n}{1 \cdot 2 \cdot 3 \ldots n}$$

$$\mp \frac{d^{n+1}y}{dx^{n+1}} \cdot \frac{\Delta x^{n+1}}{1 \cdot 2 \cdot 3 \ldots (n+1)} - \frac{d^{n+2}y}{dx^{n+2}} \cdot \frac{\Delta x^{n+2}}{1 \cdot 2 \ldots (n+2)}, \text{ \&c.}$$

As the sign of Δx does not affect that of Δx^n, such a value Pp may be assigned to Δx as will give $pp' - pp''$ the sign of $\frac{d^n y}{dx^n}$, both for $+\Delta x$ and $-\Delta x$, and the same is true for every value between Pp and zero. Hence the concavity is turned towards or from the axis of x, according as $\frac{d^n y}{dx^n}$ is < 0 or > 0.

2. If n be an odd number,

$$pp'' - pp' = \mp \frac{d^n y}{dx^n} \cdot \frac{\Delta x^n}{1 \cdot 2 \ldots n}$$

$$-\frac{d^{n+1}y}{dx^{n+1}} \cdot \frac{\Delta x^{n+1}}{1 \cdot 2 \ldots (n+1)} \mp \frac{d^{n+2}y}{dx^{n+2}} \cdot \frac{\Delta x^{n+2}}{1 \cdot 2 \ldots (n+2)}, \text{ \&c.}$$

By reasoning similar to that used before, it may be shown that the parts of the curve at either side of the point P lie at different sides of the tangent, and that therefore the point P is *a point of inflexion.*

(365.) In these cases the curve touches the tangent with a contact of the $(n-1)$th order; for the first differential coefficients of the equations of the curve and tangent are equal; and the succeeding differential coefficients of the equation of the tangent being respectively equal to zero, must be equal to the corresponding differential coefficients of the equation of the curve for the point P, as far as the $(n-1)$th differential coefficient, therefore the contact must be of the $(n-1)$th order.

(366.) It should be observed, that when $\frac{d^2y}{dx^2}=0$, the radius of the osculating circle becomes infinite (333), which shows that at such a point no circle can be described between which and the curve another may not pass.

(367.) If, at the same time that the conditions

$$\frac{d^2y}{dx^2}=0,\ \frac{d^3y}{dx^3}=0,\ \frac{d^4y}{dx^4}=0,\ \ldots\ \frac{d^ny}{dx^n}=0,$$

are fulfilled, the condition $\frac{dy}{dx}$ is also fulfilled, in addition to the circumstances already proved, the tangent through the point P will be parallel to the axis of x, and if $\frac{1}{\frac{dy}{dx}}=0$, it will be parallel to the axis of y.

(368.) It may happen that the co-ordinates of the point P may be such that $\frac{dy}{dx}$ may have two or more unequal values. This happens whenever the value of x, for the point P, causes a radical to vanish in the value of y, and yet does *not* cause the *same radical* to vanish in $\frac{dy}{dx}$. Whenever this takes place, there are always as many tangents to the same point of the curve as there are different values of $\frac{dy}{dx}$, and therefore as many branches of the curve must intersect at that point.

(369.) If the values of $\frac{dy}{dx}$ be equal, and $\frac{d^2y}{dx^2}$ have two or more unequal values, the curve will consist of as many different branches, which have a common tangent at that point.

(370.) Points where several branches of a curve meet are called *multiple points*. If two branches meet, they are called *double* points; if three, *triple*, &c.

(371.) The direction of the curvature of the different branches may be found, as was shown before, from the sign of the second differential coefficient.

(372.) If two branches have, at the same point, a common tangent, that point is called a *cusp*. It is said to be a *cusp of the first kind* if they lie at different sides of the tangent, and a *cusp of the second kind* if they lie at the same side.

(373.) The principle just laid down may be expressed more generally. If, for the values of yx corresponding to the point P, the nth differential coefficient have two or more values, the preceding coefficients having each but one, then two branches of the curve touch at the point P with the $(n-1)$th order of contact, and the species of cusp is the first, since $\frac{d^2y}{dx^2}$ is the same for both branches.

(374) If the value of any differential coefficient be impossible for the co-ordinates of the point P, that point can neither be preceded nor followed immediately by another,

and is an insulated point not continuously connected with the curve itself. Such are called *conjugate points*. They being thus detached from the curve, can only be considered algebraically to belong to it, because their co-ordinates fulfil its equation. But considered geometrically, they do not belong to the curve.

SECTION XVIII.

Of the rectification, quadrature, and curvature of lines of the second degree.

PROP. CLXXXII.

(375.) *Of the rectification of the circle.*

First method.

If x be any arc of a circle whose radius is unity, by expressing x in a series of powers of sin. x by M'Claurin's theorem,

$$x = \frac{\sin. x}{1} + \frac{\sin.^3 x}{1.2.3} + \frac{3^2 \sin.^5 x}{1.2.3.4.5} + \frac{3^2 . 5^2 \sin.^7 x}{1.2.3.4.5.6.7}$$
$$+ \frac{3^2.5^2.7^2 \sin.^9 x}{1.2.3 \ldots 9} +, \&c.$$

If $x = 30^\circ = \frac{\pi}{6}$, $\because$ sin. $x = \frac{1}{2}$,

$$\pi = 6 . \left\{ \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{1.2.3} + \frac{1}{32} \cdot \frac{3^2}{1.2.3.4.5} \right.$$
$$\left. + \frac{1}{128} \frac{3^2.5^2}{1.2.3.4 \ldots 7}, \&c. \right\}$$

This series was used by Newton for the calculation of the circumference of the circle, but does not converge with sufficient rapidity.

Second method.

By expressing x in a series of the powers of tan. x by the same theorem, we find

$$x = \frac{\tan. x}{1} - \frac{\tan.^3 x}{3} + \frac{\tan.^5 x}{5} - \frac{\tan.^7 x}{7}, \text{ \&c.}$$

If $x = \frac{\pi}{4}$, tan. $x = 1$, $\therefore$

$$\pi = 4\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9}, \text{ \&c.}\}$$

This series will also give the value of π, but is inconvenient for calculation, owing to its want of sufficiently rapid convergence.

This may be remedied thus:

let tan. $a = \frac{1}{5}$, $\therefore$ tan. $2a = \dfrac{2 \tan. a}{1 - \tan.^2 a} = \frac{5}{12}$, and therefore

tan. $4a = \dfrac{2 \tan. 2a}{1 - \tan.^2 2a} = \frac{120}{119}$. Hence,

$$\tan. (4a - \frac{\pi}{4}) = \tfrac{1}{239}.$$

Hence we find

$$4a - \frac{\pi}{4} = \frac{1}{239} - \frac{1}{3.(239)^3} + \frac{1}{5.(239)^5} - \frac{1}{7(239)^7}, \text{ \&c.};$$

but since tan. $a = \frac{1}{5}$,

$$a = \frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} +, \text{ \&c.}$$

therefore,

$$\pi = 4\left\{\begin{array}{l} 4\left(\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} +, \&c.\right) \\ -\left(\frac{1}{1.239} - \frac{1}{3.(239)^3} + \frac{1}{5.(239)^5} -, \&c.\right) \end{array}\right\}$$

This series converges with sufficient rapidity to afford great facility in calculating the value of π.

Let r be the radius of a circle whose circumference is c. since the circumferences of circles are as their radii,

$$r : 1 :: c : 2\pi;$$

hence $c = 2r\pi$; ∵ the circumference of a circle is equal to the diameter multiplied by the value of π found by the means above stated.

PROP. CLXXXIII.

(376.) *Of the quadrature of the circle.*

By the general formula for the quadrature of curves in (330), the area is

$$\int \frac{r^2 d\omega}{2};$$

but r^2 is in this case constant and integrating between the limits $\omega = 0$ and $\omega = 2\pi$, the whole area of the circle is $r^2\pi$.

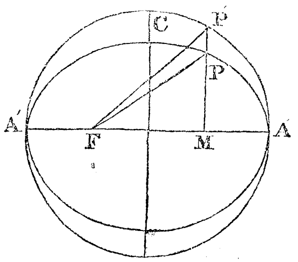
(377.) *Cor.* Since the semicircumference of the circle is $r\pi$, the area of the circle is equal to the rectangle under the radius and semicircumference.

PROP. CLXXXIV.

(378.) *To find the area of an ellipse.*

The equation of the ellipse related to its axes being solved for y, gives

$$y = \frac{B}{A}\sqrt{A^2 - x^2}.$$



If a circle ACA' be described on the axis AA' as diameter, any ordinate y' to the diameter of this circle is expressed by $\sqrt{\text{A}^2 - x^2}$, $\because$

$$y = \frac{\text{B}}{\text{A}}\, y'; \text{ hence}$$

$$ydx = \frac{\text{B}}{\text{A}} \cdot y'dx,$$

$$\int ydx = \frac{\text{B}}{\text{A}} \cdot \int y'dx;$$

but the value of $\int y'dx$ is the area of the circle; no constant is necessary, as ydx and $y'dx$ begin together. Hence, if A' be the area of the ellipse,

$$\text{A}' = \frac{\text{B}}{\text{A}} \cdot \text{A}^2\pi = \text{BA}\pi.$$

Hence the area of an ellipse is equal to that of a circle described with a radius, which is a mean proportional between its semiaxes.

(379.) *Cor.* 1. The circle described on the transverse axis as diameter, the ellipse and the circle described on the conjugate diameter, are in geometrical progression.

(380.) *Cor.* 2. The areas of ellipses are as the rectangles under their axes.

(381.) *Cor.* 3. If two ellipses have one axis common, the areas cut off by a common ordinate MPP' are as the other axes; for $d\text{A}' = \frac{\text{B}}{\text{A}}\int y'dx$ and $\frac{y'dx}{\text{A}}$ being the same for both $d\text{A}' \propto \text{B}$, $\because$ since the corresponding increments of the areas are in the ratio of the axes, the sum of any number of these will be in the same ratio.

(382.) *Cor.* 4. If any point F be taken on the transverse axis, the area FPA is to the area $\text{FP}'\text{A}$, (P' being on the circumscribed circle), as the conjugate to the transverse axis.

PROP. CLXXXV.

(383.) *To find the area intercepted between two ordinates to the asymptote of an hyperbola.*

The equation of an hyperbola related to its asymptotes being

$$yx = \frac{A^2 + B^2}{4},$$

$$ydx \sin. yx = \frac{A^2 + B^2}{4} \cdot \frac{dx}{x} \cdot \sin. yx;$$

$$\therefore A' = \frac{A^2 + B^2}{4} \sin. yx \cdot \int \frac{dx}{x} = \frac{A^2 + B^2}{4} \cdot \sin. yx \cdot (\log. x - \log. x')$$

If the area be supposed to begin when $x = 1$, the expression is simplified, and becomes

$$A' = \frac{A^2 + B^2}{4} \sin. yx \cdot \log. x.$$

The coefficient $(A^2 + B^2)$ is the square of the line joining the extremities of the axis. If half this line be taken as the linear unit, the expression is still farther simplified, and becomes

$$A' = \sin. yx \cdot \log. x;$$

and if instead of the neperian logarithm, a logarithm whose modulus is the cosec. yx be used, the expression is

$$A' = \log. x.$$

Hence, if a series of values of x be measured from the centre in geometrical progression, the areas intercepted by ordinates through their extremities will be equal, since the areas measured from $x = 1$ must be in arithmetical progression.

(384.) *Cor.* 1. If the values of x be taken to represent a series of numbers related to $\frac{1}{2}\sqrt{A^2 + B^2}$ as unity, the corresponding areas measured from the ordinate of the vertex of the curve will represent a system of logarithms of these numbers whose modulus is cosec. yx.

(385.) *Cor.* 2. If the hyperbola be equilateral,

$$yx = \frac{\pi}{2}, \because \text{cosec.}\, yx = 1;$$

therefore, the logarithms will be in this case neperian logarithms. It is for this reason that the neperian are sometimes called hyperbolic logarithms.

PROP. CLXXXVI.

(386.) *To find the area included by an arc of a parabola, a diameter through one extremity, and an ordinate to that diameter through the other.*

The diameter being axis of x, and a tangent through its vertex axis of y, the equation is

$$y^2 = px,$$

$$\therefore y = \sqrt{px};$$

$$\text{sin.}\, yx \,.\, ydx = \sqrt{px} \,.\, dx \text{ sin.}\, yx,$$

$$A' = \sqrt{p} \int \sqrt{x} \,.\, dx \text{ sin.}\, yx = \tfrac{2}{3} \sqrt{p} \,.\, x^{\frac{3}{2}} \text{ sin.}\, yx = \tfrac{2}{3} yx \text{ sin.}\, yx.$$

No constant is added, because the area and y are at the same time equal to zero.

Hence, the area sought is two-thirds of the parallelogram formed by y and x.

PROP. CLXXXVII.

(387.) *To find the radius of curvature to any given point $y'x'$ in an ellipse or hyperbola.*

The equation related to the axes being twice differentiated, gives

$$\frac{dy}{dx} = -\frac{B^2 x'}{A^2 y'},$$

$$\frac{d^2 y}{dx^2} = -\frac{B^4}{A^2 y'^3}.$$

Making these substitutions in the formula for the radius of curvature found in (333), we find, after reduction,

$$R = \frac{(A^4y'^2 + B^4x'^2)^{\frac{3}{2}}}{A^4B^4};$$

but $A^4y'^2 + B^4x'^2 = A^2B^2(A^2 - e^2x'^2)$, and $A^2 - e^2x'^2 = B'^2$ (167); hence

$$R = \frac{B'^3}{AB}.$$

(388.) *Cor.* 1. Since the curvature is a maximum when the radius of curvature is a minimum, and *vice versa*, the curvature of an ellipse is least at the extremities of the conjugate axis, and greatest at the extremities of the transverse axis. That of an hyperbola is greatest at the extremity of the transverse axis, and diminishes without limit. These follow obviously from the above expression for the radius of curvature.

(389.) *Cor.* 2. The maximum and minimum values of the radius of curvature are $\frac{A^2}{B}$ and $\frac{B^2}{A}$.

PROP. CLXXXVIII.

(390.) *To find the radius of curvature to a given point in a parabola.*

The equation of the parabola being twice differentiated, gives

$$\frac{dy}{dx} = \frac{p}{2y},$$
$$\frac{d^2y}{dx^2} = -\frac{p^2}{4y^3}.$$

By substituting these values in (333), we find

$$R = \frac{p'^{\frac{3}{2}}}{2p^{\frac{1}{2}}},$$

where p' = the parameter of the point, and p = the principal parameter.

(391.) *Cor.* Hence the point of greatest curvature in a parabola is the vertex.

PROP. CLXXXIX.

(392.) *To find the chord* (c) *of the osculating circle which coincides with the diameter through the point of contact in any line of the second degree.*

Let the angle under the diameter and tangent be θ,

$$c = 2R \sin. \theta;$$

but in the ellipse and hyperbola $\sin. \theta = \frac{AB}{A'B'}$, ∴

$$c = \frac{2B'^2}{A'},$$

and in the parabola $\sin. \theta = \frac{p^{\frac{1}{2}}}{p'^{\frac{1}{2}}}$, ∴

$$c = p'.$$

Hence the chord of the osculating circle which coincides with the diameter of a line of the second degree passing through the point of contact, is equal to the *parameter* of that diameter.

PROP. CXC.

(393.) *To find the equation of the evolute of an ellipse or hyperbola.*

The values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ derived from the equation of the curve being substituted in the general formulas found in (333), give

$$y - y' = \frac{y(A^4y^2 + B^4x^2)}{A^2B^4},$$

$$x - x' = \frac{x(A^4y^2 + B^4x^2)}{A^4B^2};$$

and, since by the equation of the curve,

$$B^2x^2 = A^2B^2 - A^2y^2, ∴$$

$$A^4y^2 + B^4x^2 = A^2(B^4 + C^2y^2);$$

also, since $A^2y^2 = A^2B^2 - B^2x^2$, $\therefore$

$$A^4y^2 + B^4x^2 = B^2(A^4 - C^2x^2).$$

These substitutions being made in the above equations, the results, after reduction solved for y and x, give

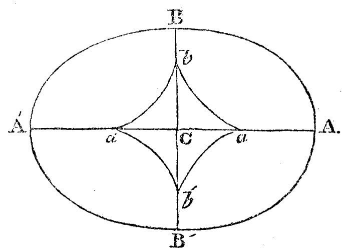
$$y = -\frac{B^{\frac{4}{3}}y'^{\frac{1}{3}}}{C^{\frac{2}{3}}},$$

$$x = \frac{A^{\frac{4}{3}}x'^{\frac{1}{3}}}{C^{\frac{2}{3}}}.$$

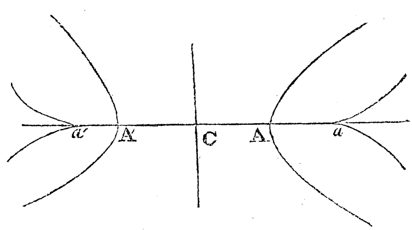
Substituting these values in the equation of the curve, and dividing the result by $\frac{A^2B^2}{C^{\frac{4}{3}}}$, we find

$$B^{\frac{2}{3}}y'^{\frac{2}{3}} \pm A^{\frac{2}{3}}x'^{\frac{2}{3}} = \pm C^{\frac{4}{3}},$$

where $+$ is taken for the ellipse, and $-$ for the hyperbola.



In this equation for the ellipse, all values of x between $x = +\frac{C^2}{A}$ and $x = -\frac{C^2}{A}$, give real values of y, and all values beyond these give impossible values of y. In like manner, all values of y between $y = +\frac{C^2}{B}$ and $y = -\frac{C^2}{B}$ give real values of x, and all beyond these impossible values of x; hence the evolute is confined within these limits. Also, it appears from the form of the equation, that the parts of the evolute included between the



four angles formed by the co-ordinates are similar and equal. The figure of the evolute is represented above. It is obvious, since the axes of an ellipse must be both tangents to the evolute at the points where it meets them, that the points aa', bb', are cusps of the first kind. The transverse axis of the hyperbola must be a tangent at the points aa', which are cusps of the first kind.

(394.) *Cor.* 1. The arc ab of the evolute of the ellipse is equal to $\text{B}'b-\text{A}a$ (342); but $\text{B}'b=\frac{\text{A}^2}{\text{B}}$, $\text{A}a=\frac{\text{B}^2}{\text{A}}$, therefore,

$$ab=\frac{\text{A}^3-\text{B}^3}{\text{AB}}.$$

(395.) *Cor.* 2. If $\text{A}'=ca$, $\text{B}'=cb$, ∵

$$\text{A}'=\frac{\text{C}^2}{\text{A}},\ \text{B}'=\frac{\text{C}^2}{\text{B}},\ \because\ \text{A}=\frac{\text{C}^2}{\text{A}'},\ \text{B}=\frac{\text{C}^2}{\text{B}'},$$

if the substitutions be made in the equation of the evolute, and the result multiplied by $\frac{\text{A}'^{\frac{2}{3}}\text{B}'^{\frac{2}{3}}}{\text{C}^{\frac{4}{3}}}$, the result is

$$\text{A}'^{\frac{2}{3}}y^{\frac{2}{3}}\pm\text{B}'^{\frac{2}{3}}x^{\frac{2}{3}}=\pm\text{A}'^{\frac{2}{3}}\text{B}'^{\frac{2}{3}},$$

which bears an obvious analogy to the equation of the curve itself.

PROP. CXCI.

(396.) *To find the equation of the evolute of a parabola.*

The values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, derived from the equation of the curve being substituted, as before, in the general formulas (333), give

$$y-y'=\frac{y(4y^2+p^2)}{p^2}=\frac{y(4x+p)}{p},$$

$$x - x' = -\frac{4x + p}{2}.$$

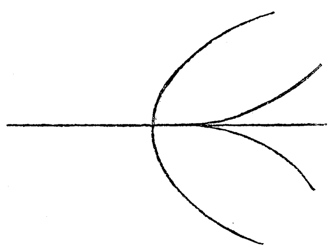
Hence we find

$$y = -\frac{p^{\frac{2}{3}} y'^{\frac{1}{3}}}{4^{\frac{1}{3}}}, \quad x = \tfrac{1}{3}(x' - \tfrac{1}{2}p).$$

Making these substitutions in the equation $y^2 = px$, and transforming the origin of co-ordinates to the point $y = 0$, $x = \frac{1}{2}p$, the equation becomes, after reduction,

$$py'^2 = \tfrac{16}{27} x'^3.$$

Hence y' is only real for the values of x' which have the same sign as p, and therefore the curve is extended indefinitely in the same direction as the parabola itself, touching



the axis of the parabola at a point whose distance from the vertex is half the principal parameter. This point of the evolute is a cusp of the first kind. The form of the evolute is represented in the figure.

This curve is called the *semicubical parabola*.

SECTION XIX.

Of the properties of the Logarithmic, Choncoid, Cissoid, and other curves, both algebraic and transcendental.

Of the logarithmic.

(397.) *Def.* The *logarithmic* is a curve expressed by the equation $y = a^x$ related to rectangular co-ordinates.

PROP. CXCII.

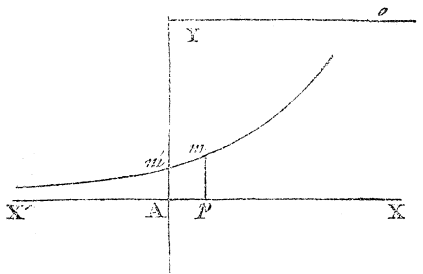
(398.) *Perpendiculars intercepting equal parts on the axis of x are in geometrical progression.*

For in this case x varies in arithmetical progression, and therefore a^x or y must vary in geometrical progression.

(399.) *Cor.* Hence, if any series of numbers be represented by the values of x, the values of y will represent their logarithms related to the base a. The curve has received its name from this property.

PROP. CXCIII.

(400.) The axis of x is an asymptote.



When $x=0, y=1$. Therefore if Am' be assumed to represent the linear unit, the curve intersects AY at m'. Let Ap=Am', $\therefore pm = a$.

1°. If $a > 1$, the values of y increase without limit for the increasing positive values of x, and decrease without limit for the increasing negative values of x. Hence on the negative side of A the curve is continually approaching AX$'$, and approaches it without limit, and on the positive side of A it is continually receding from AX, and recedes from it without limit.

2°. If $a < 1$, the value of y decreases without limit for the increasing positive values of x, and increases without limit for the increasing negative values. Hence it continually recedes from the line XX$'$ on the negative side of A, and continually approaches it, and approaches it without limit on the positive side of A.

Hence in both cases the line XX$'$ is an *asymptote.*

PROP. CXCIV.

(401.) *To find the equation of the tangent to a given point in the logarithmic.*

By taking the logarithms of the equation $y = a^x$, we have $ly = x \,.\, la$, which being differentiated is

$$dy \,.= \frac{la}{m} \,.\, y \,.\, dx,$$

m being the *modulus*. If a be the base $la = 1$, and the equation is

$$mdy - ydx = 0.$$

Hence the equation of a tangent through a point $y'x'$ is

$$m(y - y') - y'(x - x') = 0.$$

PROP. CXCV.

(402.) *To find the subtangent.*

By (323) $s = \frac{ydx}{dy} = m$. Hence the subtangent for all points on the same curve is the same, being the modulus of the logarithms, whose base is a.

PROP. CXCVI.

(403.) *To find the centre and radius of the osculating circle.*

The equation $y = a^x$ being differentiated twice, gives

$$\frac{dy}{dx} = \frac{y}{m}, \quad \frac{d^2y}{dx^2} = \frac{y}{m^2}.$$

These values being substituted in (333) give

$$R^2 = \frac{(m^2 + y^2)^3}{m^2y^2},$$

$$y' = \frac{2y^2 + m^2}{y}, \; x' = x - m - \frac{y^2}{m}.$$

PROP. CXCVII.

(404.) *To find the point of greatest curvature.*

The point of greatest curvature is that at which the radius of curvature is a minimum. To find this, let the value of R, found in the last proposition, be differentiated, and equated with zero. The result, divided by $\frac{1}{2}m(m^2+y^2)^{\frac{1}{2}}$, is

$$3yd(y^2+m^2)-2(y^2+m^2)dy=0,$$

which gives

$$2y^2=m^2, \therefore y=\frac{m}{\sqrt{2}}.$$

Hence the point sought is that whose ordinate is equal to the side of a square, whose diagonal is the subtangent.

PROP. CXCVIII.

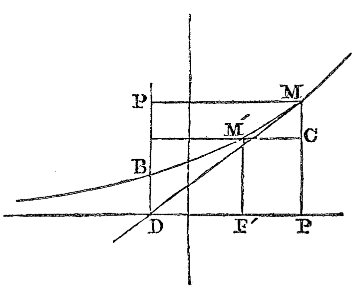
(405.) *Of the quadrature of the logarithmic.*

By (323), $A=\int ydx$, but $ydx=mdy$, $\therefore$

$$A=my+C.$$

To find C, suppose the area to commence from $y'=P'M'$, $\therefore$ when $y=y'$, $A=0$, $\therefore C=-my'$. Hence

$$A=m(y-y'),$$



that is, the area included between any two ordinates, PM and P'M', is equal to the rectangle under the subtangent, and the difference between the ordinates. The area PMM'P' = the rectangle CF.

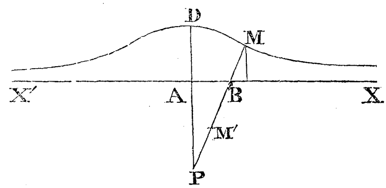
(406.) *Cor.* 1. The area included by the curve MM', extending indefinitely, and approaching the asymptote, the asymptot and the ordinate PM is equal to the rectangle DM under the subtangent and the ordinate: for in this case

$$y' = 0, \because A = my.$$

(407.) *Cor. 2.* The area extending from DB indefinitely, is equal to the space BMF.

On the conchoid of Nicomedes.

(408.) *Def.* A right line XX′ being given in position, another right line passing through a given point P revolves in the plane passing through the given right line and the given point. Let BM and BM′ be assumed of a constant magnitude, and the loci of the points M, M′ is called a *conchoid.* The locus of M is called the *superior*, and that of M′ the *inferior conchoid.*



The line XX′ is called the *rule of the conchoid.*

The line BM is called the *modulus of the conchoid.*

The point P is called the *pole of the conchoid.*

PROP. CXCIX.

(409.) *To find the equation of the conchoid.*

Let PM $= z$, BM $= m$, PA $= b$, APM $= \omega$. Hence

$$\text{PB} = (z \mp m), \because (z \mp m)\cos.\omega = b, \ (1),$$

which is the polar equation of the curve. The upper sign applies to the superior, and the lower to the inferior conchoid.

The equation related to rectangular co-ordinates, of which XX′ is axis of x and A the origin, may be found; for

$z^2 = (y+b)^2 + x^2$, and $\cos.\omega = \dfrac{y+b}{z}$, and by these substitutions we find

$$y^2x^2 + (y^2 - m^2)(y + b)^2 = 0.$$

This equation includes both superior and inferior conchoids, since both $+m$ and $-m$ are involved in m^2.

The conchoid is therefore a curve of the fourth degree.

PROP. CC.

(410.) *To find the equation of a tangent to the conchoid.*

Let the point on the curve through which the tangent passes be $y'x'$, and the equation being differentiated gives

$$\frac{dy'}{dx'} = -\frac{y'^2(m^2 - y'^2)^{\frac{1}{2}}}{y'^3 + m^2b}.$$

Hence the equation of the tangent is

$$(y - y')(y'^3 + m^2b) + (x - x')(m^2 - y'^2)^{\frac{1}{2}}y'^2 = 0.$$

PROP. CCI.

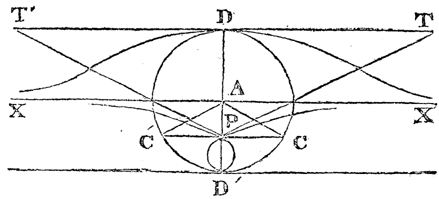
(411.) *To investigate the figure of the conchoid.*

1°. Let $m > b$. If $y = \pm m$, $x = 0$, and for all values of y beyond these x is impossible. Therefore, if AD $= +m$, AD$' = -m$, and through the points D, D$'$ parallels to XX$'$ be drawn, the entire curve will be included between these parallels. Also, if $y = -b$, $x = 0$, $\therefore$ the curve meets the axis of y at P the pole.

Since, for $y = \pm m$, $\frac{dy}{dx} = 0$, the parallels through D, D$'$ to the axis of x are tangents to the curve at the points D, D$'$. And since $y = 0$ renders x infinite, the axis of x is an asymptote to both inferior and superior conchoids.

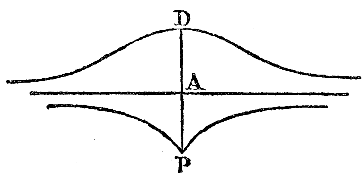
If $y = -b$, $\frac{dy}{dx} = \pm \frac{b}{(m^2 - b^2)^{\frac{1}{2}}}$; therefore the pole is a double point, and the values of $\frac{dy}{dx}$ for that point evidently show the geometrical method of determining them.

On DD′ as diameter, let a circle be described, and through the pole P let CC′ be drawn perpendicular to DD′, and let the



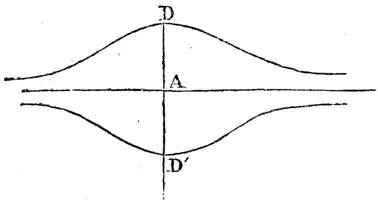
lines AC and AC′ be drawn. Lines drawn from the point P to the points of bisection of the lines AC and AC′ are tangents at the point P. For $\frac{b}{\sqrt{m^2 - b^2}} = \frac{\text{AP}}{\text{PC}} = \tan. \text{ACP} = \tan. \text{TPC}$: therefore PT is a tangent, and for the same reason PT′ is also a tangent. The figure of the conchoids is therefore in this case represented as in the preceding figure.

2. If $m = b$, as before, the curve is included between the parallels to the asymptote through D and P. If



$y = + m$, $\frac{dy}{dx} = 0$, ∵ the parallel through D is a tangent to the superior conchoid. If $y = - m$, $\frac{dy}{dx}$ is infinite, therefore the tangent through the point P is the line PD. This forms as it were the union of the two tangents, in the last case the oval PD′ being supposed to vanish, by its diameter $m - b$ becoming equal to zero. The point P is in this case a cusp of the first kind. The figure of the conchoids in this case is represented in the preceding figure.

3. If $m < b$. The co-ordinates of the pole P satisfy the equation of the curve, but they render $\frac{dy}{dx}$ impossible; hence

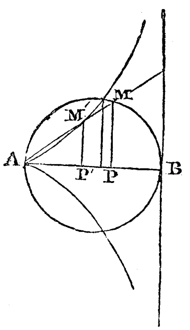


the point P is a conjugate point. The points DD′ both give $\frac{dy}{dx} = 0$, ∵ the tangents through these points are parallel to the asymptote. The figure of the conchoids is in this case represented in the preceding figure.

If $b = 0$, the conchoid becomes a circle.

If $m = 0$, it becomes a right line.

Of the cissoid of Diocles.



(412.) *Def.* A circle being described upon a given diameter (AB), and any chord (AM) being drawn from the point (A), and the ordinate MP being drawn, let AP′ = BP, and the perpendicular P′M′ being drawn to meet the chord, the locus of the point M′ is called *the cissoid.*

PROP. CCII.

(413.) *To find the equation of the cissoid.*

Let AB $= 2r$, MAP $= \omega$. By the conditions of the definition

$$\text{AM} = 2r \cos. \omega,$$

$$\text{AM}' = \text{BP} \sec. \omega = \text{PM} \tan. \omega \sec. \omega.$$

But PM $=$ AM $\sin. \omega$, ∵ AM′ $=$ AM $\tan.^2 \omega$: hence the equation sought is

$$z = 2r \tan. \omega \sin. \omega, (1.)$$

If it be related to rectangular co-ordinates, we find, by the usual substitutions,

$$y^2(2r - x) - x^3 = 0, \ (2.)$$

PROP. CCIII.

(414.) *To find the equation of the tangent to a given point on the cissoid.*

By differentiating the equation (2),

$$\frac{dy}{dx} = \frac{(3r - x)x^{\frac{1}{2}}}{(2r - x)^{\frac{3}{2}}},$$

therefore the equation of the tangent is

$$(y - y')(2r - x')^{\frac{3}{2}} - (x - x')(3r - x')x'^{\frac{1}{2}} = 0.$$

(415.) *Cor.* 1. The diameter AB is a tangent to the curve at the point A, and since the curve extends above and below the diameter, the point A is a cusp of the first kind.

(416.) *Cor.* 2. As x approaches to equality with $2r$, $\frac{dy}{dx}$ approaches to infinity; and when $x = 2r$, $\frac{dy}{dx}$ is infinite; but at the same time y is infinite, and therefore a perpendicular AB through B is an asymptote.

PROP. CCIV.

(417.) *To investigate the figure of the cissoid.*

Since for each value of x there are two equal values of y, with different signs, the branches of the curve on each side of AB, the diameter of the generating circle, are equal and similar. Since for every negative value of x, and for all positive values greater than AB, the value of y is impossible, the curve must be included between the parallels, which are perpendicular to AB through the points A and B.

Since, by differentiating twice, we find

$$\frac{d^2y}{dx^2} = \frac{3r^2}{(2r-x)^{\frac{5}{2}}x^{\frac{1}{2}}} = \frac{3r^2y^5}{x^8},$$

this having always the sign of y shows that the curve is convex towards the axis AB.

Of the lemniscata.

(418.) *Def.* The curve, which is the locus of the intersection of a tangent to an equilateral hyperbola with a perpendicular from the centre upon it, is called the *lemniscata.*

PROP. CCV.

(419.) *To find the equation of the lemniscata.*

The equation of the equilateral hyperbola, referred to its axes, is

$$y'^2 - x'^2 = -a^2.$$

The equations of the tangent, and the perpendicular to it from the centre, are

$$y'y - x'x = -a^2,$$
$$x'y + y'x = 0.$$

By these equations $y'x'$ being eliminated, the result is

$$(y^2 - x^2)a^2 + (y^2 + x^2)^2 = 0, \ (1),$$

which is the equation sought, and the locus is therefore a curve of the fourth order.

The polar equation may be found by making the necessary substitutions in the above equation, and is

$$z^2 - a^2(\cos.^2 \omega - \sin.^2 \omega) = 0,$$

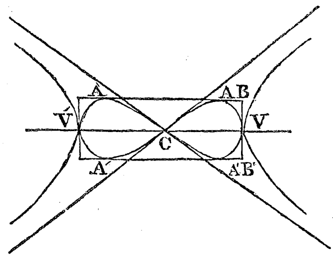
or since $\cos.^2 \omega - \sin.^2 \omega = \cos. 2\omega$,

$$z^2 - a^2 \cos. 2\omega = 0, \ (2.)$$

PROP. CCVI.

(420.) *To investigate the figure of the lemniscata.*

By the polar equation (2), when $z = 0$, $\omega = \frac{\pi}{4}$, or $\frac{3\pi}{4}$, or



$\frac{5\pi}{4}$, or $\frac{7\pi}{4}$. These values of ω show that the asymptotes of the equilateral hyperbola are tangents to the curve at the centre through which the curve must pass. Also, since z is impossible for every value of ω, except those included between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$, and between $\pm\frac{\pi}{4}$ and 0, the curve must be included between the tangents passing through the centre, as represented in the foregoing figure.

By differentiating the polar equation, we find

$$\frac{dz}{d\omega} = - z \tan. 2\omega.$$

Hence by the formula in (327),

$$\tan. tz = \cot. 2\omega, \because tz + 2\omega = \frac{\pi}{2}.$$

Hence when $\omega = 0$, $tz = \frac{\pi}{2}$, therefore the tangent to the hyperbola through the vertex is also a tangent to the lemniscata.

If the tangent be parallel to the axis $tz = \omega$, $\because \omega = \frac{\pi}{6}$, $\because$ if from the centre C, CA be drawn, making ACV one third of a right angle, the tangent to the curve at A is parallel to CV, and it is clear that the curve is included within the rectangle BB′, one side of which equals the transverse axis ($2a$), and the other $\frac{a}{\sqrt{2}}$, or the side of a square, of which the transverse axis is the diagonal.

It is obvious also that the centre is a double point.

PROP. CCVII.

(421.) *To find the area of the lemniscata.*

By (330),

$$\text{A} = \int \frac{z^2 d\omega}{2}.$$

But $z^2 d\omega = -\frac{z dz}{\tan. 2\omega}$, and since $\cos. 2\omega = \frac{z^2}{a^2}$, therefore

$$\tan. 2\omega = \frac{(a^4 - z^4)^{\frac{1}{2}}}{z^2}.$$

Hence we find

$$\text{A} = \int \frac{-z^3 dz}{2(a^4 - z^4)^{\frac{1}{2}}} = \tfrac{1}{4}(a^4 - z^4)^{\frac{1}{2}}.$$

This integral being extended to the entire curve, gives

$$\text{A} = a^2.$$

Hence the entire area is equal to the square of the semiaxis.

Of the sinusoid, &c.

(422.) *Def.* A curve, represented by the equation $y = \sin. x$, related to rectangular co-ordinates, is called the *curve of sines*, or the *sinusoid*.

PROP. CCVIII.

(423.) *To find the equation of a tangent to a given point.*

By differentiating the equation, we find

$$\frac{dy}{dx} = \frac{\cos. x}{r}.$$

r being the radius of the arc x. Hence the equation of the tangent is

$$y - y' = \frac{\cos. x'}{r}(x - x').$$

If $x = 2nr\pi$ where n is any integer number, $\cos. x = 1$. At these points the tangent makes with the axis of x an angle

$= 45^{\circ}$, and if $x = (2n + 1)r\pi$, the tangent is inclined at the angle 135° to the axis of x, these angles being measured in the positive direction.

PROP. CCIX.

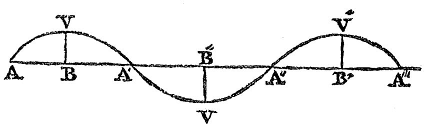
(424.) *To investigate the figure of the sinusoid.*

By differentiating the equation a second time, we find

$$\frac{d^2y}{dx^2} = -\frac{\sin. x}{r^2} = -\frac{y}{r^2}.$$

Hence the curve is always concave towards the axis of x.

If $x = nr\pi$, $y = 0$,
therefore if $AA' = r\pi$,
$AA'' = 2r\pi$,
$AA''' = 3r\pi$, &c.



the curve intersects the axis of x at the points A, A$'$, A$''$, A$'''$, &c.

For all values of x, from $x = 0$ to $x = r\pi$, y is positive; for all values from $x = r\pi$ to $x = 2r\pi$, y is negative, and so on alternately; therefore between A and A$'$ the curve lies above the axis of x, from A$'$ to A$''$ below it, from A$''$ to A$'''$ above it, &c.

The maximum positive and negative values of sin. x are $+ r$ and $- r$, of which $+ r$ corresponds to

$$x = \frac{r\pi}{2}, x = \frac{5r\pi}{2}, x = \frac{9r\pi}{2}, \text{ and } - r \text{ to } x = \frac{3\pi}{2}, x = \frac{7\pi}{2}, \text{ \&c.}$$

Hence if AA$'$, A$'$A$''$, A$''$A$'''$, be respectively bisected at B, B$'$, B$''$, &c., and perpendiculars BV, BV$'$, BV$''$, &c. erected equal to r, and a parallel V, V$''$ to AA$''$ drawn, this parallel touches the curve at the points VV$''$, &c.; the same is true of a parallel through V$'$, and the curve is included between these parallels.

If $x = nr\pi$, $\frac{d^2y}{dx^2} = 0$, hence the points A, A$'$, A$''$, &c. are points of inflection, the tangent through these points intersecting the axis of x, as has been already shown, at an angle of 45^0 degrees.

PROP. CCX.

(425.) *To find the area of the sinusoid.*

By the usual formula,

$$A = \int y dx = \int . \frac{ry\,dy}{\sqrt{r^2 - y^2}},$$

which being integrated gives

$$A = -r(r^2 - y^2)^{\frac{1}{2}} + C.$$

When $A = 0$, $y = 0$, $\therefore C = r^2$, hence

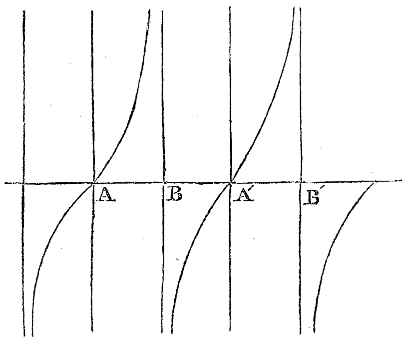
$$A = r(r - \sqrt{r^2 - y^2}).$$

If $x = AB$, $y = r$, $\therefore A = r^2$, hence the whole area AVA' is equal to twice the square of the radius of the arc x.

(426.) Other trigonometrical curves may be imagined, with equations analogous to that which we have just described. The curve $y = \cos. x$ is of the same species, since it may be expressed $y = \sin. (\frac{\pi}{2} - x)$.

PROP. CCXI.

(427.) *To investigate the figure of a curve whose equation is* y = tan. x.



If $x = nr\pi$, $y = 0$, $\therefore$ the curve must meet the axis of x at the points A, A', A'', &c., where $x = 0$, $x = r\pi$, $x = 2r\pi$, &c.

By differentiating the equation twice,

$$\frac{dy}{dx} = \frac{r^2}{\cos.^2 x},$$

$$\frac{d^2y}{dx^2} = \frac{\tan. x}{\cos.^2 x}.$$

If $x = nr\pi$, $\cos.^2 x = r^2$, $\because \frac{dy}{dx} = 1$, and $\tan. x = 0$, $\because$ $\frac{d^2y}{dx^2} = 0$. Hence the points A, A′, A″, &c. are points of inflection, the tangents through them intersecting the axis of x at an angle of 45^0.

If $x = \frac{(2n+1)r\pi}{2}$, $\frac{dy}{dx} = \frac{r^2}{0}$, and $y = \infty$. Hence if the intercepts AA′, A′A″, A″A‴, be bisected at B, B′, B″, perpendiculars through these points are asymptotes.

Since $\frac{d^2y}{dx^2}$ has always the same sign as y, the curve is convex towards the axis of x.

The figure of this curve is therefore as represented in the preceding figure.

PROP. CCXII.

(428.) *To find the area of the curve of tangents.*

By the general formula

$$A = \int \tan. x \,.\, dx.$$

By substituting for tan. x its value $\frac{r \sin. x}{\cos. x}$,

$$A = -\int \frac{r^2 \,.\, d \cos. x}{\cos. x}.$$

Hence by integrating

$$A = -r^2 \,.\, l \,.\cos. x.$$

No constant is added, because when $A = 0$, $x = 0$, $\because$ $\cos. x = 1$, $\because$ log. $\cos. x = 0$. Hence the area, included between the curve and its asymptote, is infinite.

PROP. CCXIII.

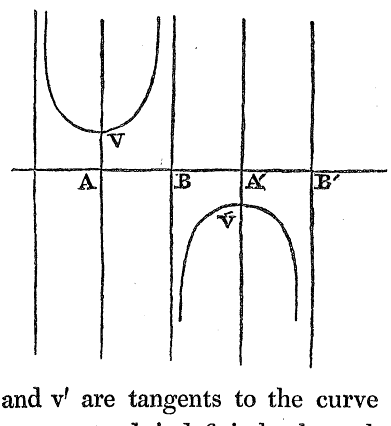
(429.) *To investigate the figure of the curve, whose equation is* y = sec. x.

By differentiating the equation twice,

$$\frac{dy}{dx} = \frac{r \cdot \sin. x}{\cos.^2 x},$$

$$\frac{d^2y}{dx^2} = \frac{r(r^2 + \sin.^2 x)}{\cos.^3 x} = r(r^2 + \sin.^2 x) \sec.^3 x.$$

Since $\frac{d^2y}{dx^2}$ has always the same sign with sec. x or y, the curve is every where convex towards the axis of x.



Sec. x is a minimum when $x = nr\pi$, which corresponds to $y = \pm r$, ∵ if AA′ $= \pi r$, AA″ $= 2\pi r$, AA‴ $= 3\pi r$, and through the points A, A′, &c. the perpendiculars AV $= r$, A′V′ $= - r$, A″V″ $= + r$, &c. be drawn parallels to AA′ through the points V and V′ are tangents to the curve at those points, and the curve extends indefinitely above the one and below the other.

When $x = \frac{(2n+1)r\pi}{2}$, y is infinite, and also $\frac{dy}{dx}$. Hence, if the intercepts between AA′, A′A″, A″A‴, be bisected at B, B′, B″, &c., perpendiculars through these points are asymptotes to the curve. The figure of this curve is therefore as represented in the preceding figure.

Of spirals.

Of the logarithmic spiral.

(430.) *Def.* The curve, whose polar equation is $z = a^{\omega}$, is called the *logarithmic spiral.*

PROP. CCXIV.

(431.) *Radii vectores which make, with the axis from which the values of ω are measured, angles in arithmetical progression, are themselves in geometrical progression.*

For let the angle under any two contiguous radii vectores be θ', then

$$z = a^{\omega},\ z' = a^{\omega+\theta'},\ z'' = a^{\omega+2\theta'},\ \&c.$$

or

$$z = a\omega,\ z' = a^{\omega}a^{\theta},\ z'' = a^{\omega}a^{2\theta},\ \&c.$$

which are in geometrical progression, a^{θ} being the common multiplier.

(432.) *Cor.* If a be the base of a system of logarithms, and z represent any number, ω will represent its logarithm, a property from which the spiral has derived its name.

PROP. CCXV.

(433.) *To find the tangent to a given point on the curve.*

The equation $z = a^{\omega}$ differentiated gives

$$mdz = zd\omega,$$

m representing the modulus of the logarithm, whose base is a. Hence by the formula (327)

$$\text{tan. } tz = m.$$

Therefore in the logarithmic spiral the radius vector is inclined to the tangent at a constant angle. Hence this curve is sometimes called the equiangular spiral.

(434.) *Cor.* The polar subtangent $= mz$.

(435.) *Def.* Similar logarithmic spirals are those in which the radius vector is equally inclined to the tangent.

PROP. CCXVI.

(436) *To find the locus of the extremity of the polar subtangent.*

Let the polar subtangent $= z'$. Hence the equation of the locus sought is

$$z' = ma^{\omega'} = \tan. \theta a^{\omega'},$$

the axis from which ω' is measured being perpendicular to that from which ω is measured.

Hence the locus is a logarithmic spiral, and since $mdz' = z'd\omega$, it is similar to the given spiral.

PROP. CCXVII.

(437.) *To find the length of an arc of the logarithmic spiral.*

By eliminating $d\omega$ from the equations

$$mdz = zd\omega,$$

$$da = (dz^2 + z^2 d\omega^2)^{\frac{1}{2}},$$

the result is

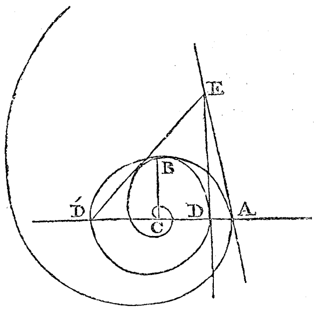
$$da = (1 + m^2)^{\frac{1}{2}} dz,$$

$$\therefore a = (1 + m^2)^{\frac{1}{2}} z + \text{c}.$$

Let the value of z, corresponding to the extremity from which the value of a is supposed to commence, be z', and supplying thus the constant, we find

$$a = (z - z') \sec. \theta.$$

Hence, if from one extremity (A) of the arc AB a tangent be drawn, and a radius vector (CB) from the other, and with the centre C, and the radius CB, the circle BD be described, and from the point D a tangent to the circle be drawn to meet the curve at E, the arc BA is equal to the right line AE.



Hence, if the pole (C) of a logarithmic spiral AB be the centre of a circle intersecting the spiral at any point, B, and a right line be drawn from the centre, intersecting the spiral and circle in A and D, and through these points tangents be drawn meeting at E, the tangent AE is equal to the arc AB of the spiral intercepted between CA and the circle.

If $z' = 0$, the value of a will be the length of the arc of the spiral continued to the pole. In this case,

$$a = z \sec.\theta.$$

Hence the intercept of the tangent between the point of contact and the polar subtangent, is equal to the arc of the spiral continued to the pole.

PROP. CCXVIII.

(438.) *To find the area included between two radii vectores of the spiral.*

By eliminating $d\omega$ from the equations,

$$dA = \tfrac{1}{2}z^2 d\omega,$$
$$mdz = zd\omega,$$

we find

$$dA = \tfrac{1}{2}mz\,dz,$$
$$A = \frac{mz^2}{4} + C.$$

If z' be the value of z when $A = 0$,

$$A = \frac{m(z^2 - z'^2)}{4} = \frac{\tan.\theta\,.\,(z^2 - z'^2)}{4}.$$

Let CB $= z'$, CA $= z$. With C as centre, and the radius CB, let a circle be described meeting CA in D, D$'$. The area BCA is equal to half the area of the triangle D$'$EA. For DA $= z - z'$, $\because$ DE $=$ tan. $\theta(z - z')$, and D$'$A $= z + z'$.

If $z' = 0$, the corresponding area will be

$$\text{A} = \frac{\tan.\, \theta\, .\, z^2}{4}.$$

In this case D and D$'$ coincide with C, and the area is half the triangle formed by the radius vector and polar subtangent.

(439.) *Cor.* 1. If $\theta = \frac{\pi}{4}$, $\because$ tan. $\theta = 1$, $\because$ $\text{A} = \frac{z^2 - z'^2}{4}$.

Hence, if a tangent be drawn from A to the circle, the area is equal to the square of half the tangent.

(440.) *Cor.* 2. In the same case the area, when $z' = 0$, is equal to the square of half the radius vector, at which the area begins.

PROP. CCXIX.

(441.) *To find the radius of curvature.*

Differentiating the equation of the spiral twice, we find

$$mdz = zd\omega,$$
$$m^2d^2z = zd\omega^2.$$

By means of these equations, that of the curve and the general equation for the radius of curvature, the quantities dz, $d\omega$, and ω, may be eliminated, and the result is

$$\text{R} = z\, .\, \text{cosec.}\, \theta.$$

(442.) *Cor.* 1. The chord of the osculating circle, which passes through the centre, is equal to twice the radius vector. For $c = 2\text{R} \sin.\, \theta = 2z$.

(443.) *Cor.* 2. The curvature of the spiral is continually increasing as it approaches the pole.

PROP. CCXX.

(444.) *To find the involute and evolute of the spiral.*

Since the pole is the point of bisection of the chord of the osculating circle, which passes through it, a line z' from it to the centre of curvature is perpendicular to z, and ∵

$$z' = \text{R} \,.\, \cos.\, \theta = z \cot.\, \theta\,;$$

hence the equation of the evolute (the values of ω being measured from a line perpendicular to that from which they are measured in the original curve), is

$$z = \cot.\, \theta \,.\, a\omega.$$

Hence the involute of the logarithmic spiral is a similar one, whose equation is

$$z'' = \tan.\, \theta\ a^{\omega},$$

the axis from which ω is measured being perpendicular to that from which it is measured in the original curve.

Of the spiral of Archimedes, &c.

(445.) *Def.* A spiral, whose equation is $z = a\omega$, is called the *spiral of Archimedes.*

(446.) *Cor.* a is the value of z, corresponding to $\omega = 1$.

PROP. CCXXI.

(447.) *If any number of values of* z *be drawn, dividing the space round the pole of the spiral into equal angles, those values will be in arithmetical progression.*

For, since a is constant, $z \propto \omega$, and therefore if ω varies arithmetically, z will also vary arithmetically.

PROP. CCXXII.

(448.) *To determine the position of the tangent.*

By differentiating the equation,

$$dz = a d\omega.$$

Hence, by the general formula (327),

$$\tan. zt = \frac{z}{a} = \omega.$$

Hence the angle zt is continually increasing as ω increases.

(449.) *Cor.* 1. If $z' =$ the polar subtangent,

$$z' = z \tan. zt = z\omega = a\omega^2.$$

(450.) *Cor.* 2. The locus of the extremity of the polar subtangent is a spiral, whose equation is

$$z' = a\omega^2,$$

ω being measured from an axis, perpendicular to that from which it is measured in the given spiral.

PROP. CCXXIII.

(451.) *To find the area of the spiral.*

By the general formula

$$A = \int . \frac{z^2 dz}{2a} = \frac{z^3}{6a} + C.$$

Let $z = z'$, when $A = 0$, ∴

$$A = \frac{z^3 - z'^3}{6a},$$

and if the area begin from the pole $z' = 0$,

$$A = \frac{z^3}{6a}.$$

(452.) The spiral of Archimedes belongs to a class of spirals included in the general equation $z = a\omega^n$, n being any positive number. The quadrature of this class of spirals can be effected; for, by the general formula,

$$\frac{z^2 d\omega}{2} = \frac{a^2 \omega^{2n} d\omega}{2}.$$

Hence, by integration

$$A = \frac{a^2 \omega^{2n+1}}{2(2n+1)} + C.$$

Substituting in this for ω its value, derived from the equation of the curve, and introducing the value of C, by z' being the value of z, where $A = 0$,

$$A = \frac{z^{\frac{2n+1}{n}} - z'^{\frac{2n+1}{n}}}{2(n+1)a^{\frac{1}{n}}}.$$

(453.) By (450) it appears that the locus of the extremity of the polar subtangent of the spiral of Archimedes is one of this class, *scil.* $z = a\omega^n$ where $n = 2$. Again, the locus of the extremity of the polar subtangent of this last spiral is $z = \frac{1}{2}a\omega^3$; and, in general, the locus of the extremity of the polar subtangent of $z = a\omega^n$, is

$$z' = \frac{a}{n} \cdot \omega^{n+1}.$$

For by differentiating

$$dz = na\omega^{n-1}d\omega.$$

Hence, by the general formula,

$$\tan. zt = \frac{\omega}{n}.$$

If therefore the polar subtangent be z', $z' = z, \tan. zt$, $\therefore$

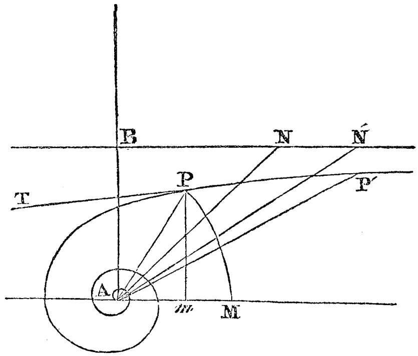
$$z' = \frac{a}{n} \cdot \omega^{n+1},$$

which is the equation of the locus of its extremity, the values of ω being measured from an axis at right angles to that from which it is measured in the equation $z = a\omega^n$.

In this class of spirals, the angle zt is continually approximating to 90^0 as the curve recedes from its pole, but never becomes actually equal to 90^0.

Of the hyperbolic spiral, &c.

(454.) *Def.* The spiral, whose equation is $z\omega = a$, is called *the hyperbolic spiral.*



(455.) *Cor.* Since $z\omega$ is the arc of a circle, whose radius is z, subtending the angle ω, it follows that this spiral may be conceived to be generated by taking any portion (AM) from the pole, and with the radius AM describing a circular arc MP always equal to a, the point P will be always in the spiral.

PROP. CCXXIV.

(456.) *If through the pole of the spiral* AB = a *be drawn perpendicular to the fixed axis,* a, *parallel to* AM *through* B, *is an asymptote to the spiral.*

Let Pm be a perpendicular from a point of the spiral on the fixed axis. P$m = z \sin. \omega$. Hence

$$\text{P}m = a \cdot \frac{\sin. \omega}{\omega}.$$

Now, as ω is diminished without limit, the limit of $\frac{\sin. \omega}{\omega}$ is unity, therefore the limit of Pm is a, *scil.* AB. Hence the curve is continually approaching the parallel through B, but never meets it.

PROP. CCXXV.

(457.) *To find the tangent to any point in the hyperbolic spiral.*

By differentiating the equation

$$dz = -\frac{z^2 d\omega}{a}.$$

Hence, by the general formula

$$\tan. zt = \frac{a}{z}.$$

(458.) *Cor.* 1. Hence follows a geometrical method of drawing a tangent to this curve.

From the point B on the asymptote take BN = AP, and draw AN; then PT, making the angle APT equal to ANB, and PT will be a tangent.

(459.) *Cor.* 2. Hence, as the spiral approaches the pole A, the angle zt approaches 90°.

PROP. CCXXVI.

(460.) *To find the polar subtangent of the hyperbolic spiral.*

Let z' be the polar subtangent,

$$z' = z \tan. zt = a.$$

Hence the subtangent in this spiral is constant.

(461.) *Cor.* 1. The locus of the extremity of the polar subtangent in this spiral is a circle, whose radius is a, and whose centre is the pole.

(462.) *Cor.* 2. If the polar subtangent of a spiral be constant, it must be the hyperbolic spiral; for, let z' be the polar subtangent,

$$z' = z \tan. zt = -\frac{z^2 d\omega}{dz}.$$

Hence we find

$$z'z^{-2}dz = -d\omega,$$

and by integrating

$$-z'z^{-1} = -\omega,$$

$$z' = z\omega,$$

which is the hyperbolic spiral.

PROP. CCXXVII.

(463.) *To find the area included by two values of* Z.

By (457),

$$z^2 d\omega = -adz.$$

Hence by the general formula (330), we find, after integration,

$$\text{A} = -\frac{az}{2} + \text{C}.$$

Let $z = z'$ when $\text{A} = 0$,

$$\text{A} = \frac{(z' - z)a}{2}.$$

If the area be measured from the centre, $z = 0$, ∴

$$\text{A} = \frac{z'a}{2}.$$

Hence, if BN = AP, and BN′ = AP′, APP′ = ANN′, and the area continued from P to the centre, is equal to the triangle ABN.

(464.) The hyperbolic spiral is one of a class of spirals included in the equation $z = a\omega^{-n}$. One of the most remarkable of this class is the *lituus*, whose equation is $z = a\omega^{-\frac{1}{2}}$, or $z^2\omega = a^2$.

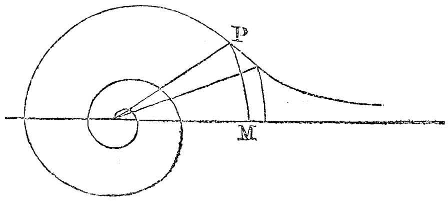
PROP. CCXXVIII.

(465.) *If, with any value of* Z *in the lituus as radius, a circular sector be described, whose angle is* ω, *the area of this sector is invariable.*

For, $z\omega$ being the arc of the sector, its area is $\frac{1}{2}z^2\omega$, which is, by the equation of the lituus, equal to $\frac{1}{2}a^2$.

PROP. CCXXIX.

(466.) *The axis from which the values of* ω *are measured is an asymptote.*



For, by the last proposition, the arc $\text{PM} = z\omega = \frac{a^2}{z}$, which continually diminishes as z

increases and as ω diminishes, and the condition $\omega = 0$ gives $z = \infty$, and PM $= 0$.

PROP. CCXXX.

(467.) *To find the position of a tangent to the lituus.*

By differentiating the equation,

$$\frac{dz}{d\omega} = -\frac{z}{2\omega}.$$

Hence by the general formula,

$$\tan. zt = \frac{2a^2}{z^2} = 2\omega.$$

Hence in this spiral zt continually approaches 90° as the curve approaches its pole.

(468.) *Cor.* 1. Hence the polar subtangent z' may be found,

$$z' = z \tan. zt = 2a\omega^{\frac{1}{2}}.$$

(469.) *Cor.* 2. The locus of the extremity of the polar subtangent is a spiral, whose equation is

$$z'^2 = 4a^2\omega,$$

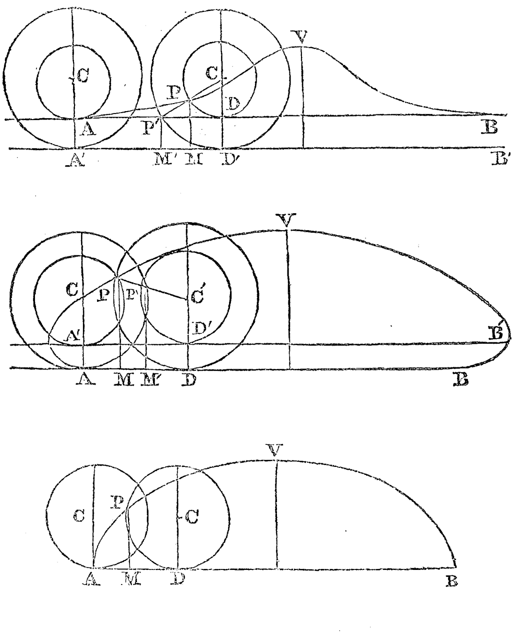
which is called the parabolic spiral, and is one of the class mentioned in (450).

(470.) *Cor.* 3. The triangle contained by the polar subtangent and z is equal to a^2, and is therefore constant.

Of Cycloids.

(471.) *Def.* The curve, traced out by a point (P) in the plane of a circle, which rolls in a given plane upon a right line given in position, is called a *cycloid.*

If the generating point be within the circle, the curve is called the prolate cycloid: if without it, the curtate cycloid; and if on it, the common cycloid.



PROP. CCXXXI.

(472.) *To find the equation of a cycloid.*

Let A′B′ be the right line on which the generating circle is supposed to roll. Let A be the generating point when the radius CAA′, passing through it, is perpendicular to the right line A′B′, and through A let a parallel AB to A′B′ be drawn. Let P be the position of the generating point after the circle has rolled over any portion A′D′, and let CP be produced to meet the circle at P′. By the definition A′D′ = D′P′, ∴

AD = D′P′. Let the distance of the generating point from the centre C be r, and let the circle with this radius be described. Let the angle DCP, related to the radius unity, be A, and the radius of the generating circle mr,
∵ $P'D' = mrA = AD$, $MD = r\sin.A$. If AB and AC be taken as axes of co-ordinates, the preceding conclusions are expressed in the equations

$$x = r(mA - \sin.A),\ (1),$$
$$y = r(1 - \cos.A),\ (2).$$

Eliminating A from these equations, we find

$$y + r\cos.\frac{x + \sqrt{2ry - y^2}}{mr} - r = 0,\ (3).$$

If $m > 1$, this is the equation of the prolate cycloid; of the curtate, if $m < 1$; and of the common cycloid, if $m = 1$.

(473.) *Cor.* 1. To find the point where the cycloid meets the axis of x (AB), let $y = 0$, ∵ $\cos.\dfrac{x}{mr} = 1$, ∵ $x = 0$, $x = 2\pi mr$, $x = 4\pi mr$, &c.; and since $2\pi mr$ is equal to the circumference of the generating circle, it is evident that the curve meets the line AB after every revolution of that circle, and the intercept AB between two points, where it meets it, is called the base of the cycloid, and is equal to the circumference of the generating circle.

(474.) *Cor.* 2. The ordinate to the middle point of the base may be found by making $A = \pi$ in (2), which gives $y = 2r$. This ordinate is called the axis of the cycloid, and, as is manifest from the same equation, is the greatest ordinate.

(475.) *Cor.* 3. If the origin be removed to the middle point of the base by substituting $x + \pi mr$ for x in the equation (1), and the angle A measured from the vertex V by substituting $\pi + A$ for A in (1) and (2), the results are

$$x = r(mA + \sin.A),\ (4),$$
$$y = r(1 + \cos.A),\ (5),$$

from which A being eliminated,

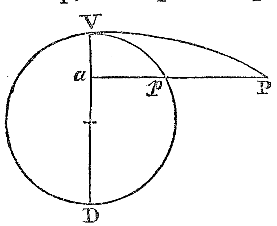
$$y - r\cos.\frac{x - \sqrt{2ry - y^2}}{mr} - r = 0, \ (6).$$

(476.) *Cor.* 4. If the origin be removed to the vertex V, by substituting $y + 2r$ for y in the last equation, we find

$$y - r\cos.\frac{x - \sqrt{-2ry - y^2}}{mr} + r = 0, \ (7).$$

PROP. CCXXXII.

(477.) *A circle* (VpD) *being described on the axis as diameter, and a perpendicular from any point* (a) *of the axis being drawn to meet the cycloid at* P, *and the circle at* p, *then* $\text{P}p = m \,.\, p\text{V}$.



The origin being assumed at the centre of the base, the equation (6) gives

$$\cos.^{-1}.\frac{y-r}{r} = \frac{x - \sqrt{2ry - y^2}}{mr}.$$

But by (5)

$$\cos.^{-1}\frac{y - r}{r} = \frac{\text{V}p}{r},$$

And

$$\frac{\sqrt{2ry - y^2}}{mr} = \sin.\ \frac{\text{V}p}{r}.$$

Hence,

$$\text{V}p = \frac{\text{P}a - pa}{m} = \frac{\text{P}p}{m},$$

$$\therefore \text{P}p = m \,.\, \text{V}p.$$

In the common cycloid therefore $\text{P}p = \text{V}p$.

(478.) *Cor.* Hence, if the ordinate to the diameter of a circle be produced, until the produced part bear a given ratio to the arc intercepted between the ordinate and the extremity of the diameter, the locus of the extremity of the produced part is a prolate cycloid, if the ratio be of major inequality; a curtate, if of minor inequality; and a common cycloid if it be a ratio of equality.

PROP. CCXXXIII.

(479.) *To find the equation of a tangent to a given point on a cycloid.*

By differentiating the equation (3) of the curve, we find

$$\frac{dy}{dx} = \frac{(2ry - y^2)^{\frac{1}{2}}}{mr - r + y}.$$

Hence the equation of the tangent sought is

$$(y - y')(mr - r + y') - (x - x')(2ry' - y'^2)^{\frac{1}{2}} = 0.$$

For the common cycloid this equation becomes

$$(y - y')y'^{\frac{1}{2}} - (x - x')(2r - y')^{\frac{1}{2}} = 0,$$

since in this case $mr - r = 0$.

PROP. CCXXXIV.

(480.) *To investigate the figure of the cycloid.*

By differentiating the equation a second time,

$$\frac{d^2y}{dx^2} = \frac{r(mr - r - my)}{(mr - r + y)^3}.$$

1. If the curve be the prolate cycloid.

At the vertex V, $y = 2r$, $\because \frac{d^2y}{dx^2} < 0$, $\because$ at this point the curve is concave towards the base.

The value of $\frac{d^2y}{dx^2}$ continues negative, until $y = \frac{m-1}{m}r$, for which value $\frac{d^2y}{dx^2} = 0$; the point therefore whose ordinate is $\frac{m-1}{m}.r$, is a point of inflection. After passing through this value $\frac{d^2y}{dx^2}$ becomes positive, and then the curve is convex towards the base. When $y=0, \frac{dy}{dx} = 0$, $\because$ the base touches the curve. Hence the figure of the prolate cycloid is as represented in the first figure of page 216.

2. If the curve be the curtate cycloid.

In this case, as before, at the vertex, the curve is concave towards the base, and the value of $\frac{d^2y}{dx^2}$ continues negative from this until it becomes infinite, which it does when $y = r(1 - m)$, that is, at the point where y is equal to the distance of the generating point from the circumference of the generating circle. The same value of y also renders $\frac{dy}{dx}$ infinite, and therefore at this point the tangent is perpendicular to the base.

If $y = 0$, $\frac{dy}{dx} = 0$, therefore the base touches the curve. Hence the figure of the curtate cycloid is as represented in the second figure of page 216.

3. If the curve be the common cycloid.

The value of $\frac{d^2y}{dx^2}$ is always negative, except for $y = 0$, which renders it infinite. Hence the curve is always concave towards the base and at the points, where it meets the base, has cusps of the first kind.

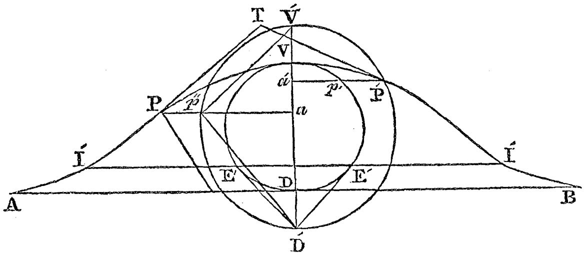
The figure of this curve is represented in the third figure of page 216.

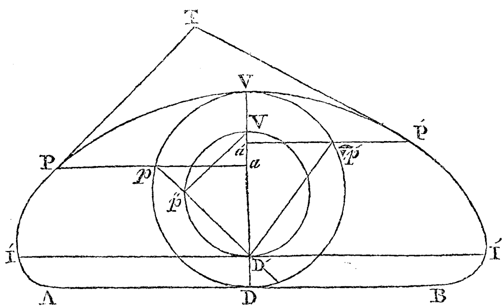
PROP. CCXXXV.

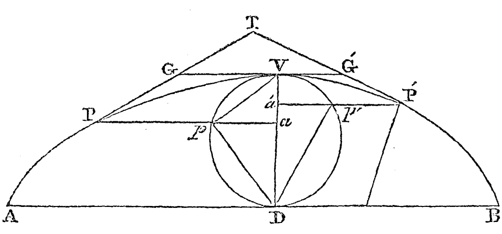
(481.) *To draw geometrically a tangent to a given point in a cycloid.*

1. If the curve be the prolate cycloid.

Let AB be the base, DV the axis, and D′V′ the diameter of the generating circle.







Let PT be a tangent at the point P. By (479),

$$\tan. \text{TP}a = \frac{(2ry - y^2)^{\frac{1}{2}}}{mr - r + y}.$$

Now $pa = (2ry - y^2)^{\frac{1}{2}}$, $a\text{D}' = mr - r + y$, therefore if $p\text{D}'$ be drawn, $\text{TP}a = p\text{D}'a$; therefore if $\text{D}'p$ be produced to meet the generating circle at p'', and $p''\text{V}'$ be drawn, $p''\text{V}'$ is parallel to PT: hence the manner of drawing PT is obvious.

2. If the curve be the curtate cycloid,

As before, $pa = (2ry - y^2)^{\frac{1}{2}}$, $a\text{D}' = mr - r + y$, ∵ $p\text{D}'a = \text{TP}a$, ∵ $p''\text{V}'$ is parallel to PT.

3. If the curve be the common cycloid.

In this case p and p'' coincide, ∵ the tangent is parallel to pV.

(482.) *Cor.* 1. In the prolate cycloid, if a tangent be drawn from D′ to the circle, described upon VD, and from the point of contact E a parallel to the base be drawn, meeting the cycloid in I, the points I are the points of inflection.

(483.) *Cor.* 2. In the curtate cycloid, if a parallel to the base be drawn through the point D′, meeting the cycloid at I, the points I are those at which the tangent is perpendicular to the base.

(484.) *Cor.* 3. The normal of the cycloid for the point P, is equal to that part of pD′, intercepted between p and the base of the cycloid in all the cycloids. In the common cycloid the normal is equal to pD.

(485.) *Cor.* 4. If tangents be drawn at any two points P, P′, of a cycloid, and the parallels Pp, P′p', to the base be drawn, the angle PTP′ under the tangents is equal to the angle in the segment of the generating circle, intercepted between the line D′p and D′p', (produced if necessary.) In the common cycloid, this angle is the angle contained in the segment pVp'.

(486.) *Cor.* 5. If, in the common cycloid, a parallel GG′ to the base be drawn through the vertex, the part of it intercepted between the tangents PT, P′T, is equal to the arc pVp'.

PROP. CCXXXVI.

(487.) *To find the area of the cycloid.*

By differentiating (1) in (472), and multiplying the result by (2),

$$ydx = r^2\left\{ m d\text{A} - (1+m)d\,.\,\sin.\,\text{A} + \cos.\,d.\sin.\text{A} \right\},$$

which being integrated, and the integral taken between the limits $A = 0$ and $A = 2\pi$, and observing that

$$\int \cos. A d \sin. A = \pi,$$

$$\therefore \int y dx = (2m + 1) r^2 \pi.$$

Hence the area of the cycloid is $(2m + 1)$ times the area of the circle described upon the axis.

The area of the common cycloid is three times that of the generating circle.

PROP. CCXXXVII.

(488.) *To find the length of an arc of the common cycloid.*

By the general formula for the rectification of curves,

$$a = \int \sqrt{dy^2 + dx^2} + C.$$

In the common cycloid $dx^2 = \dfrac{y dy^2}{2r - y}$, therefore

$$dy^2 + dx^2 = \frac{2r}{2r - y} \cdot dy^2.$$

Hence, by integrating,

$$a = \int \frac{\sqrt{2r}}{\sqrt{2r - y}} dy = 2 \sqrt{2r(2r - y)},$$

the arc being measured from the vertex, no constant need be added; for when $a = 0$, $2r - y = 0$.

Since $VD = 2r$, and $Va = 2r - y$, $\therefore VD \cdot Va = 2r(2r - y)$, but $VD \cdot Va = pV^2$, $\therefore Pa = 2pV$.

(489.) *Cor.* Hence $VB = 2VD$, $\therefore AVB = 4VD$, that is, the circumference of the common cycloid is equal to four times the diameter of the generating circle.

PROP. CCXXXVIII.

(490.) *To find the evolute of the common cycloid.*

The values of the first and second differentials, found in (479), (480), being substituted in the general formulæ for the co-ordinates of the centre of the osculating circle (334), give

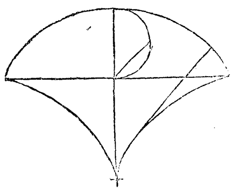
$$y - y' = 2y,$$
$$x - x' = -2\sqrt{2ry - y^2}.$$

Hence we find

$$y = -y',\ x = x' - 2\sqrt{-2ry' - y'^2},$$

which being substituted in the equation of the cycloid, give

$$y' - r\cos.\frac{x' - \sqrt{-2ry' - y'^2}}{r} + r = 0,$$



which is the equation of a cycloid, whose generating circle is equal to that of the given one, and whose vertex coincides with the extremity of the base, lying, however, below the base.

(491.) *Cor.* The involute of a cycloid is an equal cycloid, the extremity of whose base coincides with the vertex of the given one.

PROP. CCXXXIX.

(492.) *To find the radius of curvature for any point in a common cycloid.*

The values of the differentials, already found, being substituted in the general expression for the radius of curvature, found in (335), give

$$R^2 = 4ry.$$

Hence the radius of curvature is equal to double the chord pD, or to twice the normal.

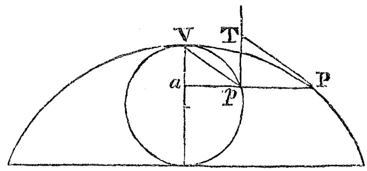
(493.) *Cor.* 1. Hence, at the extremities of the base the radius of curvature vanishes, and therefore the curvature at these points is greater than that of any circle.

(494.) *Cor.* 2. At the vertex the radius of curvature is equal to twice the axis.

(495.) *Cor.* 3. The base is the locus of the point of bisection of the tangents to the evolute from points in the curve.

PROP. CCXL.

(496.) *A parallel to the base of the common cycloid being drawn, intersecting it, and the circle described upon the axis in* PP, *to find the locus of* T, *the point of intersection of tangents to the curve and circle at these points.*

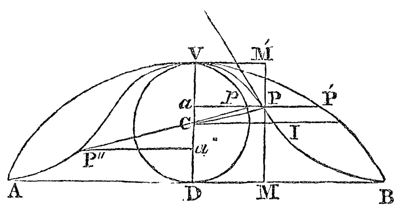


Since, by (481), Vp is parallel to TP, VpT $= p$TP, and Vpa = TPp, but Vpa = VpT,

∵ Tp = pP = pV.

Hence the locus of the point T is the involute of the generating circle described upon the axis.

Of the companion of the cycloid.



(497.) *Def.* If an ordinate (ap) to the diameter of a circle be produced, until it is equal to the arc (pV) of the circle intercepted between it and the extremity V of the diameter, the locus of its extremity P is called the *companion of the cycloid.*

PROP. CCXLI.

(498.) *To find the equation of the companion of the cycloid.*

Let the radius CV of the generating circle be r, the angle VCp = A, Vp = rA. If D be taken as origin, DM $= x$, and PM $= y$, ∵

$$y = r(1 + \cos. A),\ (1),$$
$$x = rA,\ (2.)$$

Eliminating A from these equations, we find

$$y - r \cos. \frac{x}{r} - r = 0, \ (3),$$

which is the equation sought.

(499.) *Cor.* 1. The base of the curve is equal to the circumference of the generating circle.

(500.) *Cor.* 2. If a common cycloid be described on the same axis, it will have also the same base, and aP being produced to meet it, ap = PP$'$.

(501.) *Cor.* 3. If the origin be at the vertex, the equation is $y + r \cos. \frac{x}{r} - r = 0$.

PROP. CCXLII.

(502.) *To find the equation of a tangent to the curve.*

By differentiating the equation, we find

$$\frac{dy}{dx} = - \frac{(2ry - y^2)^{\frac{1}{2}}}{r}.$$

Hence the equation sought is

$$r(y - y') + (2ry' - y'^2)^{\frac{1}{2}} (x - x') = 0.$$

PROP. CCXLIII.

(503.) *To investigate the figure of the curve.*

Let the equation be differentiated a second time, and the result is

$$\frac{d^2y}{dx^2} = \frac{r - y}{r^2},$$

which being negative for all values of y between $y = 2r$ and $y = r$, shows that, if through the centre CI be drawn parallel to the base, the curve from V to I is concave towards the base. If $y = r$, $\frac{d^2y}{dx^2} = 0$. Hence the point I is a point of inflection, and from I to B the value of $\frac{d^2y}{dx^2}$ is positive; therefore the curve is convex towards the base, and for $y=0$, $\frac{dy}{dx}=0$, which shows that the curve touches the base at A and B. Hence the figure of the curve is as represented in the preceding figure.

PROP. CCXLIV.

(504.) *To find the area of the curve.*

The equation (2) being differentiated, and the result multiplied by (1), we find

$$ydx = r^2(d\text{A} + \cos.\ \text{A}d\text{A}),$$

which by integration, gives

$$\int ydx = r^2(\text{A} + \sin.\ \text{A}),$$

no constant being added, as the area is supposed to begin when $\text{A} = 0$. Now $r^2\text{A}$ is equal to twice the area of the sector pCV, and $r^2 \sin.\ \text{A}$ is twice the area of the triangle pCV; therefore the area VPMD is equal to twice the sum of the sector and triangle.

If a tangent be drawn through V meeting MP produced in M′, the area VM′MD is equal to $2r^2\text{A}$ ∵ $\text{VM}'\text{P} = r^2\text{A} - r^2 \sin.\ \text{A}$, ∴ the area VM′P equals twice the difference between the sector V$\text{C}p$ and the triangle V$\text{C}p$, which is twice the segment Vp.

The whole area of the curve is equal to twice that of the generating circle.

It is plain that the semicircle VpD bisects the area DVPB, and also that the semicycloidal area DVP′B is trisected by the semicircle and the curve VPB.

If right lines be drawn connecting the vertex with the extremities of the base, the area of the curve is equal to that of the triangle AVB; and hence the segments of the curve cut off by these lines are equal. It is also plain from the last proposition, that these lines intersect the curve at the points of inflection.

From what has been said, it may also be proved that if $ca = \mathrm{C}a'$, the area $\mathrm{P}''\mathrm{VP}$ is equal to the rectangle under pa and the axis.

(505.) All the *cycloidal curves* which have been treated of are embraced in the general equation

$$y + r \cos. \frac{x + n\sqrt{2ry - y^2}}{mr} - r = 0.$$

If $n = m = 1$, the curve is the common cycloid.

If $n = 1$ and $m > 1$, the curve is the prolate cycloid.

If $n = 1$ and $m < 1$, the curve is the curtate cycloid.

If $n = 0$ and $m = 1$, the curve is the companion of the cycloid.

As the other cycloidal curves do not possess any particular interest, it is sufficient merely to have stated their equations.

Of epitrochoids, epicycloids, &c.

(506.) *Def.* The curve traced by a point in the plane of a circle, which is supposed to roll upon the periphery of a given circle, and in the same plane with it, is called *an epitrochoid.* If the generating point be upon the periphery of the generating circle, the curve is called *an epicycloid.*

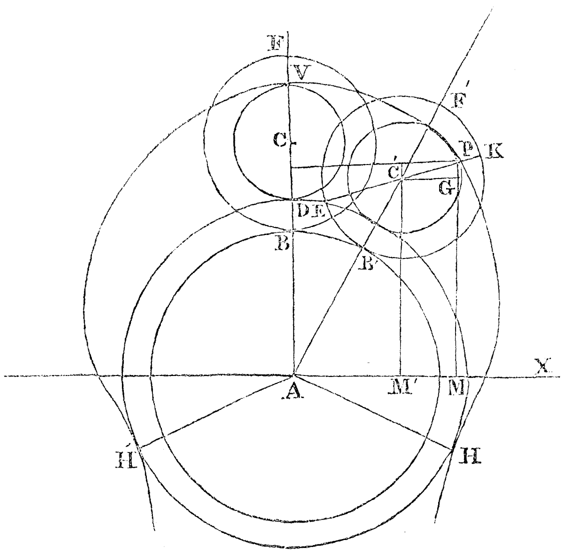
If the generating circle be supposed to roll upon the concave part of the given circle, it is called *an hypotrochoid.*

If in this case the generating point be upon the circumference, the curve is called *an hypocycloid.*

PROP. CCXLV.

(507.) *To find the equation of an epitrochoid.*

Let A be the centre and AB the radius of the base, C the centre and CB the radius of the generating circle, and let BDC be the position of the generating circle when the line connecting the centres A, C, passes through the generating point at V. Let C′ be the centre of the generating circle in any other position, and P the generating point; let AB and AX perpendicular to it be assumed as axes of co-ordinates. Let CAC′ $= \phi$ and PC′ be produced to E.



By the manner in which the curve is generated BB′ = B′E. If AB $= r$, BC $= r'$, C′P $= d$, BB′ $= r\phi =$ B′E; but

B′C′E $= \frac{\text{B'E}}{r'} = \frac{r\phi}{r'} =$ FC′P. Let C′G be parallel and PM per-

pendicular to AX, ∵ PC′G $= \frac{\pi}{2} - \frac{r+r'}{r'}\phi$, ∵

$$\text{C'G} = d \,.\, \sin. \frac{r+r'}{r'}\phi, \text{ and PG} = d \cos. \frac{r+r'}{r'}\phi,$$

GM $= (r + r') \cos. \phi$, AM′ $= (r + r') \sin. \phi$.

Hence the equations of the epitrochoid are,

$$\left.\begin{aligned} y &= (r + r')\cos. \phi + c \cos. \frac{r+r'}{r'}\phi \\ x &= (r + r') \sin. \phi + c \sin. \frac{r+r'}{r'}\phi \end{aligned}\right\}(1).$$

If the curve be the hypotrochoid, r' is negative, and these equations become

$$\left.\begin{aligned} y &= (r - r')\cos.\varphi + d\cos.\frac{r-r'}{r'}\varphi \\ x &= (r - r')\sin.\varphi - d\sin.\frac{r-r'}{r'}\varphi \end{aligned}\right\} (2).$$

If the curve be the epicycloid, c $= r'$, and the equations are

$$\left.\begin{aligned} y &= (r + r')\cos.\varphi + r'\cos.\frac{r+r'}{r'}\varphi \\ x &= (r + r')\sin.\varphi + r'\sin.\frac{r+r'}{r'}\varphi \end{aligned}\right\} (3).$$

If the curve be the hypocycloid, the equations are

$$\left.\begin{aligned} y &= (r - r')\cos.\varphi + r'\cos.\frac{r-r'}{r'}\varphi \\ x &= (r - r')\sin.\varphi - r'\sin.\frac{r-r'}{r'}\varphi \end{aligned}\right\} (4).$$

(508.) *Cor.* 1. If with the centre A and the radius AD a circle be described, and H, H$'$ be the points where the epitrochoid meets the circumference. To find the angle DAH, let the equations (1) be squared and added, and since for the points H and H$'$, $y^2 + x^2 = \text{AD}^2 = (r + r' - d)^2$, ∵

$$(r + r')^2 + d^2 + 2d(r + r')\left(\cos.\varphi\cos.\frac{r+r'}{r'}\varphi\right.$$

$$\left.+ \sin.\varphi\sin.\frac{r+r'}{r'}\varphi\right) = (r + r' - d)^2.$$

But by trigonometry,

$$\cos.\varphi\cos.\frac{r+r'}{r'}\varphi + \sin.\varphi\sin.\frac{r+r'}{r'}\varphi = \cos.\frac{r}{r'}\varphi;$$

hence after reduction, $\cos.\frac{r}{r'}\varphi = -1$,

$$\therefore \frac{r}{r'}\varphi = \pi, \therefore \varphi = \frac{r'\pi}{r}.$$

The same result applies to the hypotrochoid.

(509.) *Cor.* 2. If in the equations (4) we substitute $\frac{r+e}{2}$ for r' we find

$$y = \frac{r-e}{2} \cos. \phi + \frac{r+e}{2} \cos. \frac{r-e}{r+e}\phi;$$

$$x = \frac{r-e}{2} \sin. \phi - \frac{r+e}{2} \sin. \frac{r-e}{r+e}\phi,$$

and if in the same equations we substitute $\frac{r-e}{2}$ for r' and $\frac{r+e}{r-e}\theta = \theta'$, the result is

$$y = \frac{r-e}{2} \cos. \phi' + \frac{r+e}{2} \sin. \frac{r-e}{r+e}\phi'.$$

$$x = \frac{r-e}{2} \sin. \phi' - \frac{r+e}{2} \sin. \frac{r-e}{r+e}\phi'.$$

These equations being the same as the preceding, show that the generating circles, whose radii are $\frac{r-e}{2}$ and $\frac{r+e}{2}$, give the same hypocycloids.

PROP. CCXLVI.

(510.) *To find the equation of a tangent to an epitrochoid.*

The equations (1) being differentiated, the result after division is

$$\frac{dy}{dx} = -\frac{r' \sin. \phi + d \sin. \frac{(r+r')}{r'}\phi}{r' \cos. \phi + d \cos. \frac{r+r'}{r'}\phi}.$$

The equation of the tangent to the epitrochoid is therefore

$$(y - y')\,(r' \cos. \phi + d \cos. \frac{r+r'}{r'}\phi)$$

$$+\ (x - x')\,(r' \sin. \phi + d \sin. \frac{r+r'}{r'}\phi) = 0,$$

which for the hypotrochoid is

$$(y - y')\left(r' \cos. \phi - d \cos. \frac{r - r'}{r'} \phi\right)$$

$$+ (x - x')\left(r' \sin. \phi + d \sin. \frac{r - r'}{r'} \phi\right) = 0.$$

The equations of the tangent to the epicycloid and hypocycloid may be found from these by making $c = r'$ and observing that

$$-\frac{\sin. \phi \pm \sin. \frac{r \pm r'}{r'} \phi}{\cos. \phi + \cos. \frac{r \pm r'}{r'} \phi} = \mp \tan. \frac{r \pm 2r'}{2r'} . \phi.$$

Hence the equation of the tangent to the epicycloid is

$$(y - y') + \tan. \frac{r + 2r'}{2r'} . \phi(x - x') = 0,$$

and that of the hypocycloid is

$$(y - y') - \tan. \frac{r - 2r'}{2r'} \phi(x - x') = 0.$$

(511.) *Cor.* 1. Hence for the epicycloid the angle PTM $= \pi - \frac{r + 2r'}{2r'} \phi$, and for the hypocycloid the corresponding angle $= \frac{r - 2r'}{2r'} \phi$.

(512.) *Cor.* 2. In the epitrochoid and hypotrochoid if $\phi = 0$, the equation of the tangent becomes $(y - y') = 0$, therefore at the point V, where the curve meets AY, the tangent is perpendicular to AY. In the epicycloid and hypocycloid, in this case $\tan. \frac{r + 2r'}{2r'} \phi = 0$.

PROP. CCXLVII.

(513.) *To find the length of an arc of an epicycloid.*

By differentiating the equations (3),

$$dy = -(r + r')\,(\sin.\,\phi + \sin.\,\frac{r+r'}{r'}\phi)d\phi,$$

$$dx = (r + r')\,(\cos.\,\phi + \cos.\,\frac{r+r'}{r'}\phi)d\phi\,;$$

but by trigonometry

$$\sin.\,\phi + \sin.\,\frac{r+r'}{r'}\phi = 2\,\sin.\,\frac{r+2r'}{2r'}\phi\,\cos.\,\frac{r}{2r'}\phi,$$

$$\cos.\,\phi + \cos.\,\frac{r+r'}{r'}\phi = 2\cos.\,\frac{r+2r'}{2r'}\phi\,\cos.\,\frac{r}{2r'}\phi.$$

After making these substitutions, squaring and adding the above equations, we find

$$\sqrt{dy^2 + dx^2} = 2(r + r')\,\cos.\,\frac{r}{2r'}\phi\,.\,d\phi.$$

Hence by integration,

$$\int\sqrt{dy^2 + dx^2} = \frac{4r'(r+r')}{r}\,\sin.\,\frac{r}{2r'}\,.\,\phi.$$

No constant is necessary, the arc being supposed to begin from the point where $\phi = 0$.

For the hypocycloid the expression becomes

$$\int\sqrt{dy^2 + dx^2} = \frac{4r'(r-r')}{r}\,\sin.\,\frac{r}{2r'}\phi.$$

(514.) *Cor.* If $\frac{r\phi}{r'} = \text{FC}'\text{P} = \beta$,

$$\text{VP} = \frac{4r'(r\pm r')}{r}\,\sin.\,\tfrac{1}{2}\beta.$$

If $\beta = \pi$, the generating point coincides with H, and we find

$$\text{VPH} = \frac{4r'(r\pm r')}{r}.$$

PROP. CCXLVIII.

(515.) *To find the evolute of an epicycloid.*

By twice differentiating the equations we find

$$\frac{dy}{dx} = - \tan. \frac{r+2r'}{2r'}\phi,$$

$$\frac{d^2y}{dx^2} = - \frac{r+2r'}{4r'(r+r')\cos.^3 \frac{r+2r'}{2r'}\phi \cos. \frac{r}{2r'}\phi}.$$

Substituting these in the general formulæ for the co-ordinates $y'x'$ for the centre of the osculating circle,

$$y' = y - \frac{4r'(r+r') \cos. \frac{r+2r'}{2r'}\phi.\cos.\frac{r}{2r'}\phi}{r+2r'},$$

$$x' = x - \frac{4r'(r+r') \sin. \frac{r+2r'}{2r'}\phi \cos. \frac{r}{2r'}\phi}{r+2r'}.$$

But by trigonometry,

$$2 \cos. \frac{r+2r'}{2r'}\phi \cos. \frac{r}{2r'}\phi = \cos. \frac{r+r'}{r'}\phi + \cos. \phi,$$

$$2 \sin. \frac{r+2r'}{2r'}\phi . \cos. \frac{r}{2r'}\phi = \sin. \phi + \sin. \frac{r+r'}{r'}\phi.$$

And by the equations of the curve itself,

$$y = (r + r') \cos. \phi + r' \cos. \frac{r+r'}{r'}\phi,$$

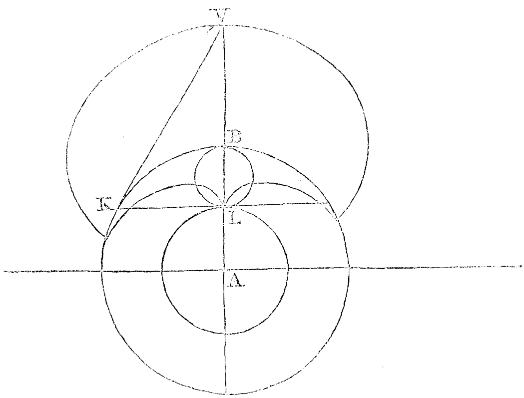
$$x = (r + r') \sin. \phi + r' \sin. \frac{r+r'}{r'}\phi.$$

By these substitutions, the equations of the evolute are

$$y' = \frac{r(r+r')}{r+2r'} \cos. \phi - \frac{rr'}{r+2r'} \cos. \frac{r+r'}{r'}\phi,$$

$$x' = \frac{r(r+r')}{r+2r'} \sin. \phi - \frac{rr'}{r+2r'} \sin. \frac{r+r'}{r'}\phi,$$

which are the equations of an epicycloid, the radius of whose base is $\frac{r^2}{r+2r'}$, and the radius of whose generating circle is $\frac{rr'}{r+2r'}$, and since these are in the ratio of r to r', the evolute is similar to the epicycloid. It is obvious also, that the centre of the base of the epicycloid is also the centre of the base of its evolute.



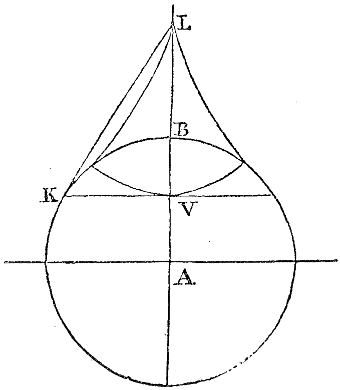
To construct the base of the evolute geometrically, let the circle whose radius equals $r + 2r'$ meet the epicycloid at V, and draw AV: from V let a tangent to the base be drawn, and from the point of contact let KL be drawn perpendicular to AV, the circle described with the radius AC is the base of the evolute for AL $= \frac{r^2}{r+2r'}$. Also LB$=$AB$-$AL$= \frac{2rr'}{r+2r'}$, therefore LB is the diameter of the generating circle of the evolute, which is represented in the preceding figure.

For the hypocycloid the result is by the same process,

$$y' = \frac{(r-r')r}{r'-2r'} \cos. \phi + \frac{r'r}{r-2r'} \cos. \frac{r-r'}{r'} \phi,$$

$$x' = \frac{(r-r')r}{r-2r'} \sin. \phi + \frac{rr'}{r-2r'} \sin. \frac{r-r'}{r'} \phi.$$

Hence the evolute of an hypocycloid is an hypocycloid, the radius of whose base is $\frac{r^2}{r-2r'}$, and the radius of whose generating circle is $\frac{rr'}{r-2r'}$, and since these are in the ratio of r to r', the evolute is similar to the hypocycloid. It is obvious also that their bases are concentrical.



Let AV $= r - 2r'$, and V the point where a circle with the radius $r - 2r'$ meets the hypocycloid; let VK be drawn perpendicular to AV, and KL touching the base at K, AL is the radius of the base of the evolute, and since

BL $=$ AL $-$ AB $= \frac{r^2}{r-2r'} - r = \frac{2rr'}{r-2r'}$, BL is the diameter of the generating circle of the evolute, which is represented in the figure.

Of the cardioide.

(516.) The *epicycloid*, the radii of whose base and generating circle are equal, is called *the cardioide*.

(517.) *Cor.* The hypocycloid corresponding to the cardioide is the base itself.

PROP. CCXLIX.

(518.) *To find the equation of the cardioide.*

For this curve the equations (3), after changing x into y and y into x, become

$$x = r\,(2 \cos.\, \phi + \cos.\, 2\phi);$$
$$y = r\,(2 \sin.\, \phi + \sin.\, 2\phi),$$

from which by eliminating φ, we find

$$(y^2 + x^2 - r^2)^2 - 4r^2(y^2 + (x + r)^2) = 0.$$

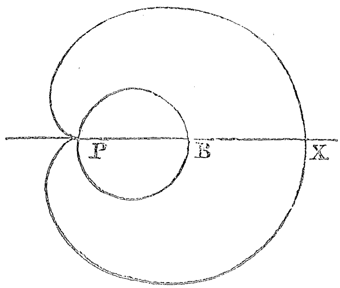
If the origin be removed to the point where the curve meets the base, the equation becomes

$$(y^2 + x^2 - 2rx)^2 - 4r^2(y^2 + x^2) = 0.$$

The polar equation is therefore

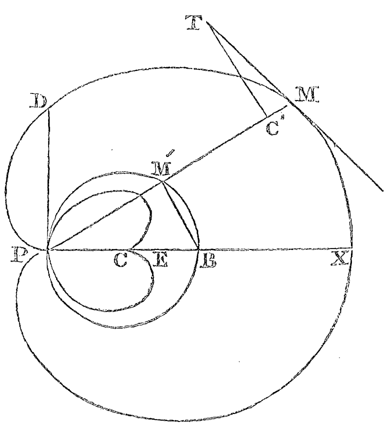
$$z = 2r(1 + \cos. \omega).$$

The point P being the pole, and PX the axis from which ω is measured; the curve being placed as in the annexed figure.



PROP. CCL.

(519.) *If a line* (PM) *be drawn from the pole to the curve, the part* M′M *intercepted between the curve and circle is equal to the diameter of the circle.*



For $PM' = 2r \cos. \omega$; but by the polar equation,

$$PM - PM' = 2r.$$

PROP. CCLI.

(520.) *To find the equation of a tangent to the cardioide.*

The polar equation being differentiated, gives

$$dz = -2r \sin. \omega . d\omega.$$

Hence by the general formula,

$$\tan. tz = \frac{z}{2r \cdot \sin. \omega}.$$

(521.) *Cor.* 1. Hence follows a geometrical construction for drawing a tangent, $BM' = 2r \sin. \omega$, therefore

$$\tan. tz = \frac{PM}{BM'}.$$

Let MC' be assumed on the radius vector equal to BM', and a perpendicular $C'T$ drawn equal to PM, TM will be the tangent to the point M.

(522.) *Cor.* 2. The tangent at X is perpendicular to PX.

(523.) *Cor.* 3. PX is a tangent to the curve at P, and P is therefore a cusp of the first kind.

(524.) *Cor.* 4. If a perpendicular to PX be drawn through P meeting the curve in D, the tangent at D is inclined to PD at 45°.

PROP. CCLII.

(525.) *To find the area of the cardioide.*

By squaring both sides of the polar equation,

$$z^2 = 4r^2 (1 + \cos. \omega)^2,$$

and multiplying both sides by $d\omega$, and integrating

$$\int \frac{z^2 d\omega}{2} = 2r^2\omega + 4r^2 \sin. \omega + 2r^2 \int \cos. \omega d \sin. \omega.$$

Taking this integral between the limits $\omega = 0$ and $\omega = 2\pi$, we find the entire area A, ∴

$$A = 4r^2\pi + 2r^2 \int \cos. \omega d \sin. \omega;$$

but the last term is manifestly twice the area of the circle,

$$\therefore A = 6r^2\pi;$$

that is, the area of the curve is six times the area of the generating circle.

PROP. CCLIII.

(526.) *To find the length of the arc of a cardioide.*

By substituting in the general formula for rectification the particular values of the terms, in this case

$$\int(z^2 d\omega^2 + dz^2)^{\frac{1}{2}} = 2r\int\{(1 + \cos.\ \omega)^2 + \sin.^2\ \omega\}^{\frac{1}{2}} d\omega;$$

but by trigonometry,

$$(1 + \cos.\ \omega)^2 + \sin.^2\ \omega = 2(1 + \cos.\ \omega),$$

$$1 + \cos.\ \omega = 2\cos.^2\ \tfrac{1}{2}\omega.$$

Hence we find,

$$\int(z^2 d\omega^2 + dz^2)^{\frac{1}{2}} = 4r\int\cos.\ \tfrac{1}{2}\omega d\omega.$$

which by integration is

$$\int(z^2 d\omega^2 + dz^2)^{\frac{1}{2}} = 8r\sin.\ \tfrac{1}{2}\omega.$$

And if this be assumed between the limits $\omega = 0$ and $\omega = \pi$, we find the length of half the curve to be $8r$, and therefore that of the entire curve $16r$.

PROP. CCLIV.

(527.) *To find the evolute of the cardioide.*

By (515) the radius of the base of the evolute is $\frac{r}{3}$, which is also the radius of its generating circle. Hence if CE $= \frac{1}{3}$CB, the cardioide, whose base is the circle with the radius CE, is the evolute sought.

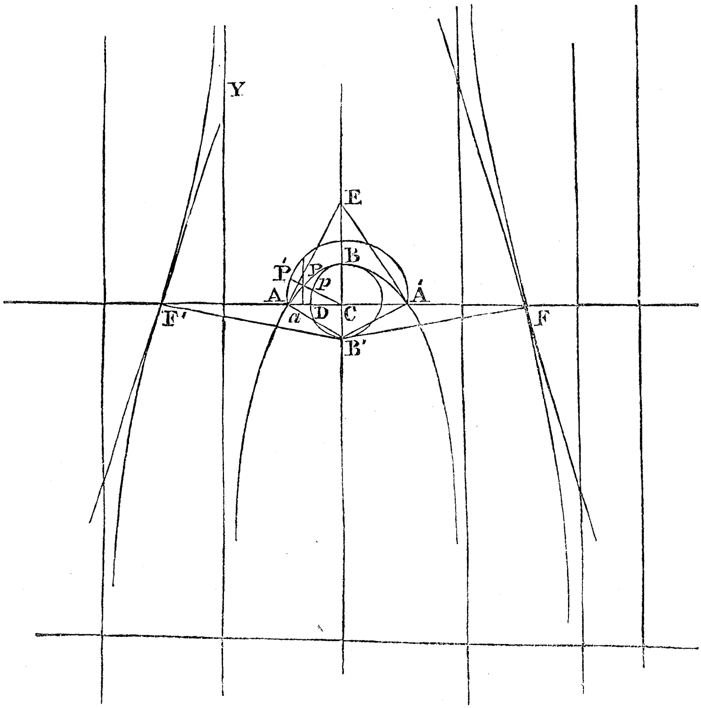
528. *Cor.* The involute of a cardioide is a cardioide the radius of whose base is three times that of the base of the given curve.

Of the quadratrix of Dinostratus.

Def. A right line being supposed to revolve with an uniform angular motion round a fixed point, and an indefinite right line at the same moving uniformly parallel to itself meets the former, the locus of their intersection is called the Quadratrix of Dinostratus.

PROP. CCLV.

(529.) *To find the equation of the quadratrix, and determine its figure.*



Let C be the fixed centre round which the revolving radius turns. Let CA and AY be the positions of it, and the parallel where they intersect at right angles, and let these be the axes of co-ordinates. Let CP and *ay* be their position after the revolving line has described the angle PCA. Let CA $= r$. A$a = x$, aP $= y$. By the conditions of the question,

$$x : r :: \text{PCA} : \frac{\pi}{2}.$$

The angle PCA being expressed in relation to the radius unity; the equation of the curve is therefore

$$y = (r - x) \tan. \frac{\pi x}{2r}.$$

If $x = 0$, $y = 0$, ∵ the curve passes through A.

As x increases from 0 to r, y continually increases, and as x passes from r to $2r$, y continually diminishes, and when $x = 2r$. $y = 0$.

The value of y corresponding to $x = r$, assumes the form $\frac{0}{0}$, its real value will be found by differentiating both numerator and denominator; by this we find it to be $\frac{r}{\frac{1}{2}\pi}$.

In passing through 0, y changes its sign and becomes negative, and continues so as x passes from $2r$ to $3r$, since in that case the factors of y have different signs; and when $x = 3r$, y is infinite. Hence a perpendicular to the axis of x at this point is an asymptote.

Similar observations apply to the negative values of x intercepted between 0 and $-r$, and therefore a perpendicular to the axis of x intersecting it at distance $= r$ on the negative side of the origin is another asymptote. The values of x between $x = 3r$ and $x = 4r$ give positive values for y, for this case the factors of y have like signs. For $x = 4r$, $y = 0$, and at this point the curve intersects the axis of x; and from $x = 4r$ to $x = 5r$ the values of y are negative; and for $x = 5r$ the value of y is infinite, which points out another asymptote. By continuing this reasoning, it appears that there exists on either side of the origin an infinite series of asymptotes, and that the figure of the curve is as represented in the figure.

PROP. CCLVI.

(530.) *If with the centre* C *and the radius* CA *a circle be described, and the line* CP *produced until it meets this circle at* P′, *then* AP′ : Aa : : $\frac{1}{2}\pi$: 1.

For AP′ $= r \cdot \frac{\pi x}{2r} = \frac{\pi x}{2}$ ∵ AP′ : Aa : : $\frac{1}{2}\pi$: 1.

PROP. CCLVII.

(531.) *The ordinate* CB : CA : : 1 : $\frac{1}{2}\pi$.

For, by (529), CB $= \frac{r}{\frac{1}{2}\pi}$ ∵ CB : r : : 1 : $\frac{1}{2}\pi$.

(532.) *Cor.* 1. Hence CB is a third proportional to the quadrant AP′B′ and AC. For CB : r : : r : $\frac{1}{2}r\pi$ = CP′B′.

(533.) *Cor.* 2. CB : 2CA = AA′ : : the diameter of a circle : its circumference.

(534.) *Cor.* 3. The area of the circle on AA′ : $4r^2$: : $\frac{1}{2}r$: CB. Hence, if this curve could be described geometrically, the quadrature of the circle would be effected, and from this property the curve has derived its name.

PROP. CCLVIII.

(535.) *If, with* C *as centre, and* CB *as radius, a circle be described, the arc* Dp = Aa.

For Dp = CB $\times \frac{\pi x}{2r}$, CB $= \frac{2r}{\pi}$, ∵ D$p = x$.

(536.) *Cor.* The quadrant DpB = AC.

PROP. CCLIX.

(537.) *To find the equation of a tangent to a given point in the quadratrix.*

By differentiating the equation of the curve, the result is

$$\frac{dy}{dx} = \sec.^2 \frac{\pi x}{2r} \cdot \frac{\pi}{2}(1 - \frac{x}{r}) - \tan. \frac{\pi x}{2r}.$$

Hence the equation of a tangent through the point $y'x'$ is

$$y - y' = \left\{ \sec.^2 \frac{\pi x'}{2r} \cdot \frac{\pi}{2}(1 - \frac{x'}{r}) - \tan. \frac{\pi x'}{2r} \right\} (x - x').$$

If $x' = 0$. The equation of the tangent is

$$y = \frac{\pi}{2} \cdot x;$$

and if $x' = 2r$, it is

$$y = -\frac{\pi}{2}(x - 2r.)$$

Hence, if CE be assumed equal to the quadrant, AB′, AE, and A′E, are tangents at the points A, A′, which may be effected by drawing B′A, and drawing B′E perpendicular to it. For

B′C : CA : CE : : $1 : \frac{\pi}{2}$.

Also if $x' = 2nr$. The equation of the tangent is

$$y = \frac{\pi}{2}(1 - 2n)(x - 2nr).$$

At the point F the tangent of the inclination of the tangent to the axis of x is $-\frac{3\pi}{2}$.

And in like manner the tangent of the inclination at F′ is $+\frac{3\pi}{2}$.

The position of the tangents at these points is determined by drawing B′F, B′F′, perpendiculars to which are the tangents at these points.

The successive hyperbolic branches of the curve therefore intersect the axis AX at angles continually approaching to a right angle, and the angles at which branches equidistant from C on each side intersect it, are supplemental angles.

The subtangents, corresponding to the successive points where the curve intersects AX, and measured upon AY, are obviously the quadrant AB multiplied by 1, 3, 5, 7, &c.

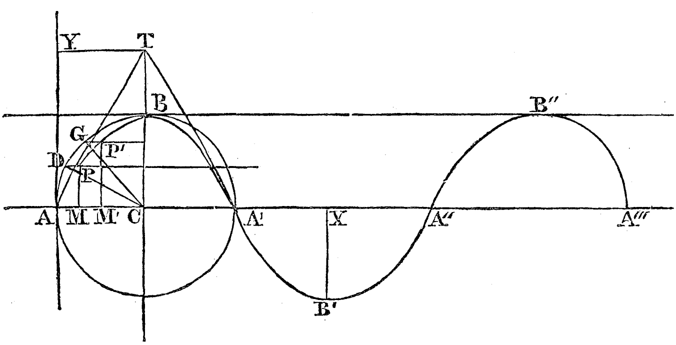
PROP. CCLX.

(538.) *To divide an angle in any number of equal parts by the quadratrix.*

Let PCA′ be the angle, let A′*a* be divided into the required number of equal parts, and lines drawn from C to the corresponding points of the curve divide the angle into the required parts.

Of the quadratrix of Tschirnhausen.

(539.) *Def.* Suppose a right line AY, touching a given circle at A, to move uniformly parallel to itself, until it coincides with CB; at the same time, suppose the line AA′ to move parallel to itself, so that its intersection with the circle moves uniformly from A to B, while the former line moves from A to C. The point P of intersection of these two lines traces a curve, called the *quadratrix of Tschirnhausen.*



PROP. CCLXI.

(540). *To find the equation of this quadratrix.*

Let AY and AX be the axes of co-ordinates. Let AC $= r$. By the definition, AD : AB : : AM : AC, or

$$\text{AD} : \frac{r\pi}{2} :: x : r \because \text{AD} = \frac{\pi x}{2}.$$

Hence the equation of the curve is

$$y = r \sin. \frac{\pi x}{2r}.$$

PROP. CCLXII.

(541.) *To find the equation of a tangent through a given point.*

By differentiating the equation

$$\frac{dy}{dx} = \frac{\pi}{2} \cos. \frac{\pi x}{2r} = \frac{\pi \sqrt{r^2 - y^2}}{2r}.$$

The equation of the tangent is therefore

$$y - y' = \frac{\pi}{2} \cos. \frac{\pi x'}{2r} . (x - x'),$$

or

$$y - y' = \frac{\pi \sqrt{r^2 - y'^2}}{2r} (x - x').$$

PROP. CCLXIII.

(542.) *To investigate the figure of this quadratrix.*

If $x = 2nr$, $y = 0$. The curve therefore meets the axis of x at intervals, equal to the diameter of the circle, and continues so to intersect it *ad infinitum.*

The equations of the tangents to the points $x = 0$, $x = 4r$, $x = 8r$, &c. &c. are

$$y = \frac{\pi}{2}(x - x'),$$

and those to the points $x = 2r$, $x = 6r$, $x = 10r$, &c. are

$$y = -\frac{\pi}{2}(x - x').$$

Hence the subtangent CT = the quadrant AB. If

$x = (2n + 1)r$, $\frac{dy}{dx} = 0$, therefore at the points B, B′, B″, &c. the tangents are parallel to AA′.

By differentiating a second time,

$$\frac{d^2y}{dx^2} = -\frac{\pi^2}{4r} \sin. \frac{\pi x}{2r}.$$

This is $= 0$, if $x = 2nr$, hence A, A′, A″, &c. are points of inflection; and since $\frac{d^2y}{dx^2}$ has always the sign opposite to that of y, the curve is always concave towards the axis of x.

PROP. CCLXIV.

(543.) *To find the area of the quadratrix.*

By the formula for quadrature

$$A = \int y dx = r \int \sin. \frac{\pi x}{2r} . dx,$$

which integrated gives

$$A = -\frac{2r^2}{\pi} \cos. \frac{\pi x}{2r} + C.$$

To determine C, when $A = 0$, $x = 0$, ∵ $\cos. \frac{\pi x}{2r} = 1$;

hence $C = \frac{2r^2}{\pi}$, ∵

$$A = \frac{2r^2}{\pi}(1 - \cos. \frac{\pi x}{2r}).$$

The area ACB is found by assuming $x = r$, and is ∵

$$A = \frac{2r^2}{\pi}.$$

Hence the square of the radius is a mean proportional between the area APBC and that of the semicircle.

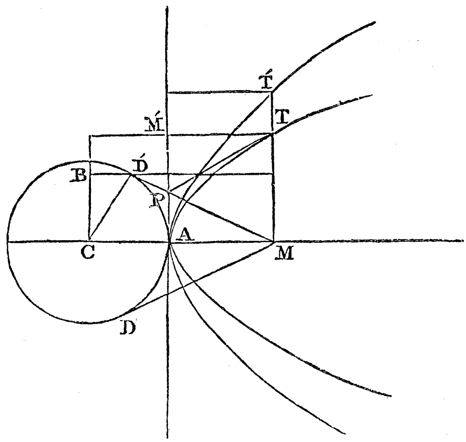
PROP. CCLXV.

(544.) *To divide an angle into any required number of equal parts by the quadratrix.*

Let the angle be ACG, and from the point P' the perpendicular $P'M'$ being drawn to AA', let AM' be divided into the required number of equal parts, and the corresponding ordinates being drawn, parallels to AA' through their extremities, divide the arc of the circle into the required parts, as is evident from the genesis of the curve.

Of the catenary.

(545.) *Def.* A curve such that the arc, intercepted between two tangents, one of which passes through the vertex, is proportional to the tangent of the angle at which they are inclined, is called *the catenary*.



Thus if

$$AT = s, \text{ and } APT = \phi, \; s \propto \tan. \phi.$$

PROP. CCLXVI.

(546). *To find the equation of the catenary.*

By the definition

$$\frac{dy}{dx} = \frac{s'}{s}, \; (1),$$

s' being a constant magnitude, and is called the parameter. Hence it follows, that

$$\frac{dy^2 + dx^2}{dx^2} = \frac{s'^2 + s^2}{s^2}.$$

But since $dy^2 + dx^2 = ds^2$, ∵

$$dx = \frac{sds}{\sqrt{s'^2 + s^2}},$$

which by integration gives

$$x = \sqrt{s'^2 + s^2} - s', \ (2),$$

which is the equation of the curve expressed by x and s as variables.

By equation (1) it follows in like manner that

$$\frac{dy^2 + dx^2}{dy^2} = \frac{s'^2 + s^2}{s'^2};$$

whence we find

$$dy = \frac{s'ds}{\sqrt{s'^2 + s^2}},$$

which by integrating, gives

$$y = s'l. \frac{s + \sqrt{s'^2 + s^2}}{s'} \quad (3).$$

By solving equation (3) for s, we find

$$s = \tfrac{1}{2}s' \left\{ e^{\frac{y}{s'}} - e^{-\frac{y}{s'}} \right\} \quad (4),$$

which is the equation of the curve between the variables s and y.

By eliminating s by (2) and (3), the result is

$$y = s'.l. \frac{x + s' + \sqrt{x^2 + 2s'x}}{s'},$$

which solved for x gives

$$x = \tfrac{1}{2}s' \left\{ e^{\frac{y}{s'}} + e^{-\frac{y}{s'}} \right\} - s' \quad (5),$$

in which e is the base of the hyperbolic logarithms, and which is the equation of the curve between the variable co-ordinates xy.

PROP. CCLXVII.

(547.) *To draw a tangent to the catenary.*

Let the point of contact be $y'x'$; by (1) the equation of the tangent is

$$y - y' = \frac{s'}{s}(x - x')$$

and by (2),

$$s = \sqrt{x^2 + 2s'x};$$

hence the equation sought is

$$y - y' = \frac{s'}{\sqrt{x'^2 + 2s'x'}}(x - x').$$

This equation points out the geometrical construction for drawing a tangent. Let AC $= s'$, and with C as centre and AC as radius, describe a circle, and draw MD touching this circle, MD $= \sqrt{x'^2 + 2s'x'}$; therefore the tangent TP is parallel to MD.

Hence as T recedes from A, the tangent continually approaches to parallelism with the axis.

PROP. CCLXVIII.

(548.) *To find the length of an arc of the catenary measured from the vertex.*

By equation (2),

$$s = \sqrt{x^2 + 2s'x};$$

hence the arc AT = MD.

(549.) *Cor.* 1. If with C as centre and CA as semiaxis an equilateral hyperbola be described, its ordinate MT$'$ is equal to the corresponding arc AT of the catenary.

(550.) *Cor.* 2. TT$'$ = AT − TM.

PROP. CCLXIX.

(551.) *To find the radius of curvature to the catenary.*

By substituting for s its value as a function of x and s', and differentiating equation (1), we find

$$\frac{d^2y}{dx^2} = -\frac{s'(s'+x)}{s^3},$$

and by making the proper substitutions in the general formula for the radius of curvature, we find

$$\text{R} = \frac{(s'+x)^2}{s'}.$$

(552.) *Cor.* 1. Hence a parabola and catenary having a common vertex and common vertical tangent, will have at that point the same osculating circle when they have equal parameters. Hence the catenary near its vertex is nearly coincident with a parabola.

PROP. CCLXX.

(553.) *To find the evolute of the catenary.*

Let $y'x'$ be the centre of the osculating circle, and yx the corresponding point on the curve by (334),

$$y - y' = -\frac{dy^2 + dx^2}{d^2y},$$

$$x - x' = \frac{dy^2 + dx^2}{d^2y} \cdot \frac{dy}{dx}.$$

Now by what has been already established,

$$dy^2 + dx^2 = \frac{(s'+x)^2}{s^2} \cdot dx^2,$$

$$d^2y = -\frac{s'(s'+x)}{s^3} \cdot dx^2,$$

$$dy = \frac{s'}{s}dx.$$

By these substitutions,

$$y - y' = \frac{s(s' + x')}{s'},$$

$$2x = x' - s'.$$

By these and the equations

$$s^2 = x^2 + 2xs',$$

$$y = s'l\,.\,\frac{s + \sqrt{s'^2 + s^2}}{s'};$$

eliminating s, x, and y, the result solved for y' is

$$y' = s'l\,.\,\frac{s' + x' + \sqrt{(s' + x')^2 - 4s'^2}}{2s'} - \frac{(s' + x')\sqrt{(s' + x')^2 - 4s'^2}}{2s'},$$

the equation of the evolute sought.

This equation will assume a more simple form by changing the origin to the point C. In this case $s' + x'$ becomes x', and the equation of the evolute is

$$y' = s'l\,\frac{x' + \sqrt{x'^2 - 4s'^2}}{2s'} - \frac{x'\sqrt{x'^2 - 4s'^2}}{2s'}.$$

PROP. CCLXXI.

(554.) *To find the area of the catenary.*

By the general formula for quadratures, $\int xdy$ = the area ATM. By equation (5), prop. CCLXVI.

$$\int xdy = s'\int\left\{\tfrac{1}{2}e^{\frac{y}{s'}}dy + \tfrac{1}{2}e^{-\frac{y}{s'}}dy - dy\right\}$$

which by integration is

$$\int xdy = s'\left\{\tfrac{1}{2}s'(e^{\frac{y}{s'}} - e^{-\frac{y}{s'}}) - y\right\}$$

which by equation (4) is

$$\int xdy = s'(s - y).$$

No constant is added, because the area vanishes with y and s.

The area ATM′ is therefore equal to the rectangle under the parameter, and the difference between the arc and the ordinate.

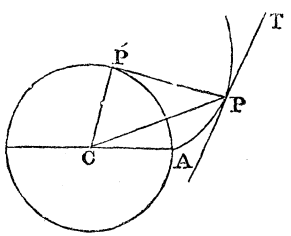
The area ATM is $y(s' + x) - s's$. Hence, if through T and C parallels to CM and AP be drawn, and from M the tangent MD′ be drawn to the circle, and through D′ a parallel to CM be drawn, the rectangle BT is equal to the space ATM.

Of the involute of the circle.

PROP. CCLXXII.

(555.) *To find the equation of the involute.*

An arc of the circle AP′ being supposed to be always measured from the fixed point A, and through its extremity P′ a tangent P′P drawn equal to the arc AP′, the locus of the point P is the involute.



Let CP $= r$, CA $= a$, PCA $= \omega$,

$\because$ P′CP $= \cos.^{-1}\dfrac{a}{r}$. Hence

PP′ $= a\omega + a\cos.^{-1}\dfrac{a}{r}$, and

therefore the equation of the curve is

$$\omega = \frac{\sqrt{r^2 - a^2}}{a} - \cos.^{-1}\frac{a}{r}.$$

(556.) *Cor.* It is obvious that the area of the triangle CP′P is equal to that of the sector ACP′.

PROP. CCLXXIII.

(557.) *To apply a tangent to the involute.*

Let PT be the tangent, and CPT $= \theta$;

$$\tan.\,\theta = -\frac{rd\omega}{dr}.$$

By differentiating the equation of the curve, we find

$$\frac{dr}{d\omega} = \frac{ra}{\sqrt{r^2 - a^2}};$$

and therefore,

$$\tan.\ \theta = -\frac{\sqrt{r^2 - a^2}}{a}.$$

Hence the angle TPC is supplemental to P′CP, and therefore the tangent is parallel to the radius CP′.

(558.) *Cor.* The radius CA touches the curve at A.

PROP. CCLXXIV.

(559.) *Of the quadrature of the involute.*

By the general formula for the quadrature of curves, if A be the area of the sector PCA,

$$A = \int \frac{r^2 d\omega}{2},$$

which, if the value for $d\omega$ already found be substituted first, becomes

$$A = \int \frac{r\sqrt{r^2 - a^2}\,dr}{2a},$$

which by integration is

$$A = \frac{(r^2 - a^2)^{\frac{3}{2}}}{6a},$$

which is the area of the sector.

(560.) *Cor.* 1. Hence the area is equal to the third power of the arc AP′ divided by the radius, or may be otherwise expressed, thus; let P′CA $= \phi$,

$$A = \frac{a^2\phi^3}{6}.$$

PROP. CCLXXV.

(561.) *Of the rectification of the involute.*

Let A′ be an arc of the curve measured from A,

$$A' = \int (r^2 d\omega^2 + dr^2)^{\frac{1}{2}}.$$

By substituting the value of $d\omega$ and integrating, we find

$$A' = \frac{r^2}{2a};$$

the arc of the curve therefore is a third proportional to the diameter of the circle, and the radius vector of the curve.

PROP. CCLXXVI.

To find the polar subtangent.

Let P be the polar subtangent. By the general formula,

$$P = -\frac{r^2 d\omega}{dr} = -\frac{r\sqrt{r^2 - a^2}}{a}.$$

(562.) *Cor.* 1. The intercept of the tangent between the point of contact and the polar subtangent is therefore a third proportional to the radius of the circle and the radius vector of the curve; for let this intercept be T,

$$T^2 = P^2 + r^2 = \frac{r^4}{a^2}$$

$$\therefore T = \frac{r^2}{a}.$$

(563.) *Cor.* 2. By the last cor. and prop. (CCLXXV), it appears that the arc of the curve is equal to half the tangent.

(564.) *Cor.* 3. If $r' =$ a perpendicular on the tangent from the point of contact, $r' = r \sin. \theta = \sqrt{r^2 - a^2}$. This perpendicular therefore equals the arc AP′ of the circle.

(565.) *Cor.* 4. The intercept of the tangent between the perpendicular r' and the point of contact is always equal to the radius of the circle.

PROP. CCLXXVII.

(566.) *To find the locus of the extremity of the perpendicular from the pole upon the tangent.*

By cor. 4 of the last proposition, if Cp = r',

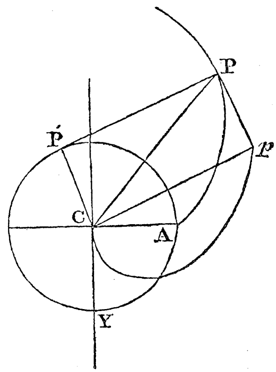
$$r' = \sqrt{r^2 - a^2}.$$

Let ACp = φ. Since

$$\text{P}'\text{CP} = \cos.^{-1}\frac{a}{r},$$

$$\frac{\pi}{2} + \varphi = \omega + \cos.^{-1}\frac{a}{r}.$$

By means of these equations and that of the curve, r and ω being eliminated, the result is



$$r' = a(\frac{\pi}{2} + \varphi).$$

If CY be drawn at right angles to CA, and

YCp = $\omega' = \frac{\pi}{2} + \varphi$, the equation of the locus sought is

$$r' = a\omega'.$$

The locus is therefore the spiral of Archimedes.

Of the tractrix and equitangential curves.

(567.) *Def.* The *tractrix* is a curve whose characteristic property is, that the locus of a point on the tangent, at a given distance from the point of contact, is a right line; and this line is called *the directrix* of the curve.

PROP. CCLXXVIII.

(568.) *To find the equation of the tractrix.*

Let the intercept of the tangent between the directrix and the point of contact be a. By the general formula for the subtangent,

$$-\frac{ydx}{dy} = \sqrt{a^2 - y^2} \quad (1),$$

which by integration, gives

$$= al \,.\, \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} \quad (2),$$

which is therefore the equation of the curve, and which may be otherwise expressed thus,

$$a + \sqrt{a^2 - y^2} = ye^{\frac{x + \sqrt{a^2 - y^2}}{a}} \quad (3).$$

PROP. CCLXXIX.

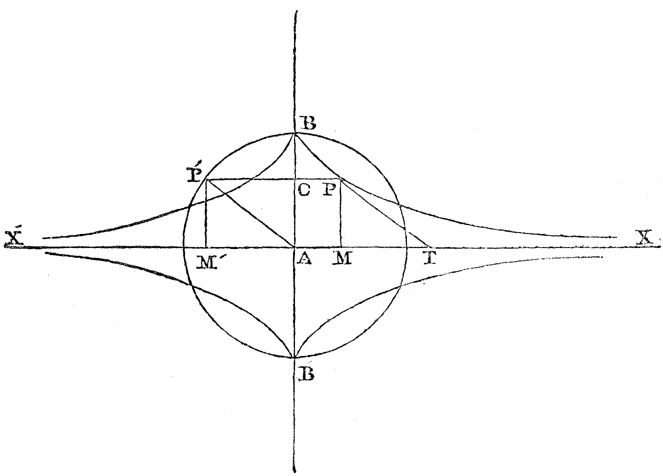
(569.) *To find the equation of a tangent through a given point.*

Let the given point be $y'x'$. By (1),

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}} \quad (4).$$

The equation of the tangent is therefore

$$y - y' = -\frac{y'}{\sqrt{a^2 - y'^2}}(x - x') \quad (5).$$



The geometrical construction for applying a tangent to this curve is obviously pointed out by this equation. With the centre A and the radius a = AB let a circle be described; through any point P of the curve let the ordinate PM be drawn, and PP′ parallel to XX′, and meeting the circle in P′, and let P′A be drawn; a line PT parallel to P′A is a tangent to the curve at P. For tan. P′AM′ $= \frac{y'}{\sqrt{a^2-y'^2}} =$ tan. PTM.

PROP. CCLXXX.

(570.) *To investigate the figure of the tractrix.*

By (3), when $x = 0$, $y = \pm a$, therefore if AB $= + a$, AB′ $= - a$, the curve meets the axis of y at the points B, B′; and in (5), if $y' = \pm a$, and $x' = 0$, the equation becomes $x = 0$, which shows that the axis of y touches the curve at the points B, B′.

By differentiating (4), the result is

$$d^2y = \frac{a^2}{(a^2 - y^2)^2} \cdot y \cdot dx^2.$$

Therefore d^2y and y have always the same sign, and therefore the curve is every where convex towards the directrix.

By (2) it appears that for each value of y there are two equal and opposite values of x, and for each value of x there are two equal and opposite values of y. Therefore the four branches of the curve, included in the four right angles round the origin, are perfectly equal and similar, and such as if placed upon each other would coincide. It also appears by this equation that, as x increases without limit, y diminishes without limit, and therefore the directrix is an asymptote. It also appears, from what has been said, that the points B, B′, are *cusps* of the first kind.

PROP. CCLXXXI.

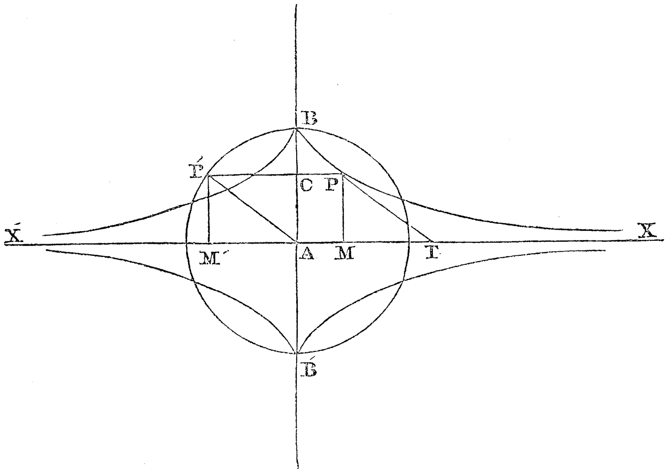
(571.) *The quadrature of the tractrix.*

By (1)

$$ydx = - \sqrt{a^2 - y^2} \cdot dy.$$

One side of this equation is the differential of the area ABPM, and since $-\sqrt{a^2 - y^2} = $ AM′, the other side is the differential of the area BP′C, and therefore taking the integrals BPMA = BP′C.

Also, since the triangle P′AM′ = PTM, the area BPTA is equal to the sector BAP′.



It follows also that the whole area included by the four branches of the tractrix is equal to the area of the circle.

PROP. CCLXXXII.

(572.) *The rectification of the tractrix.*

By (1) we find

$$\sqrt{dy^2 + dx^2} = -\frac{ady}{y},$$

where the negative sign is used, because the arc increases as y diminishes, and which integrated gives

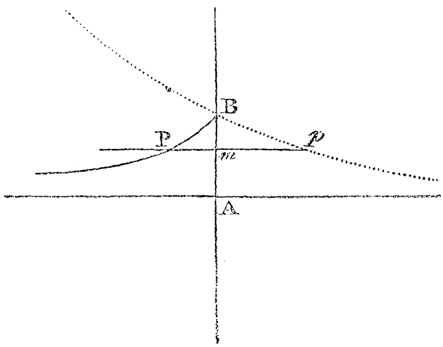
$$\int \sqrt{dy^2 + dx^2} = -\, aly + \text{C}.$$

To determine C, let the arc A be supposed to begin at B, so that when the arc A $= 0$, $y = a$; hence

$-ala + c = 0$, $\because c = ala$, hence we find

$$A = al\frac{a}{y}.$$

Hence it appears that if with the axis of the tractrix, and $-a$ as subtangent, a logarithmic be drawn, the line $pm = BP$.



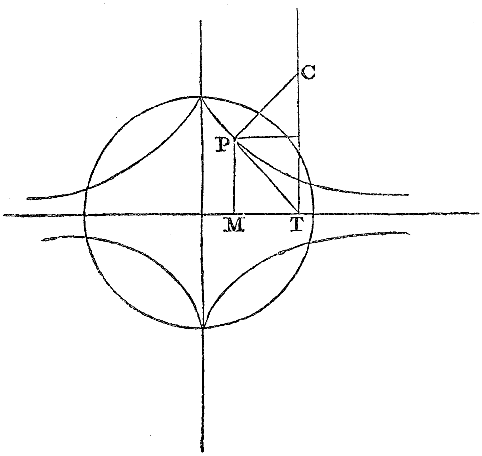
PROP. CCLXXXIII.

(573.) *To find the radius of curvature of the tractrix.*

By substituting in the general formula for the radius of curvature the values of the first and second differential coefficients, we find

$$r = -\frac{a\sqrt{a^2 - y^2}}{y}.$$

Hence by geometrical construction the radius and centre of the osculating circle may be found thus: let PC be perpendicular to the tangent at P, and produced to meet a perpendicular to the directrix at T, the intercept PC is the radius, and C the centre of the osculating circle; for PM : PT :: MT : PC, by the similar triangles.



PROP. CCLXXXIV.

(574.) *To find the evolute of the tractrix.*

Let the co-ordinates of the centre of the osculating circle be $y'x'$. By substituting in the general formulæ for the values of these the particular values of the differential co-efficients, the results are

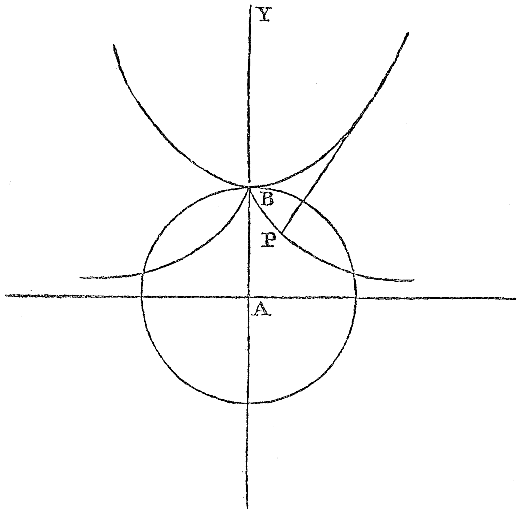
$$y' = \frac{a^2}{y},$$

$$x' = x + \frac{a\sqrt{y'^2 - a^2}}{y'}.$$

By eliminating y and x by means of these equations, and that of the curve, the result is

$$x' = al\frac{y' + \sqrt{y'^2 - a^2}}{a},$$

which is the equation of the evolute. The evolute is therefore a catenary, whose parameter is $a =$ AB, whose vertex is at B, and whose axis is AY.



Hence, if a string applied to a catenary have its extremity at the vertex, and be wound off, its extremity P will describe the tractrix.

(575.) *Def.* The locus of a point p upon the tangent of the tractrix at a given distance b from the extremity T of the tangent is called the *syntractrix.*

PROP. CCLXXXV.

(576.) *To find the equation of the syntractrix.*

Let the co-ordinates of a point on the tractrix be $y'x'$, and those of the corresponding point of the syntractrix xy. The conditions of the definition furnish the equations

$$ay = by',$$

$$(x - x')a = (a - b)\sqrt{a^2 - y'^2}.$$

By means of these equations, and that of the tractrix, y' and x' being eliminated, the result is

$$x = al \cdot \frac{b + \sqrt{b^2 - y^2}}{y} - \sqrt{b^2 - y^2}.$$

which is the equation of the syntractrix.

PROP. CCLXXXVI.

(577.) *To find the equation of a tangent to the syntractrix.*

By differentiating the equation, we find

$$\frac{dy}{dx} = -\frac{y\sqrt{b^2 - y^2}}{ab - y^2};$$

hence the equation of a tangent through the point $y'x'$ is

$$y - y' = -\frac{y'\sqrt{b^2 - y'^2}}{ab - y'^2}(x - x').$$

PROP. CCLXXXVII.

(578.) *To investigate the figure of the syntractrix.*

1°. Let $b < a$.

By the equation of the curve it appears that when $x = 0$, $y = \pm b$, and by that of the tangent that the tangent to this point is parallel to the directrix. It also follows, as in the tractrix, that the directrix is an asymptote, and that the branches or portions of the curve included in each of the four right angles round the origin are symmetrical.

By differentiating the equation a second time, we find

$$\frac{d^2y}{dx^2} = by \cdot \frac{ab^2 + y^2(b - 2a)}{(ab - y^2)^3}.$$

This equals 0 when

$$y = b\sqrt{\frac{a}{2a - b}},$$

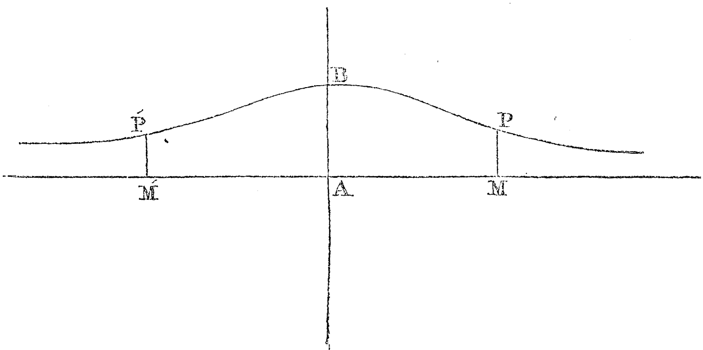
and the corresponding value of x is

$$x = al\frac{\sqrt{2a - b} + \sqrt{a - b}}{\sqrt{a}} - b\sqrt{\frac{a - b}{2a - b}}.$$

Since, by supposition, $b < a$, these values are real. There is therefore a point of contrary flexure at the point whose co-ordinates are the values of y and x, found as above.

Let AB $= b$, and AM, AM$'$ be the values of x, which give $y = b\sqrt{\frac{a}{2a - b}}$; for all values of x between $x = 0$ and

$x = \text{AM}$, or AM', the second differential of y is negative, and therefore the intercept of the curve between the points PP' is concave towards the axis of x, and beyond these limits on

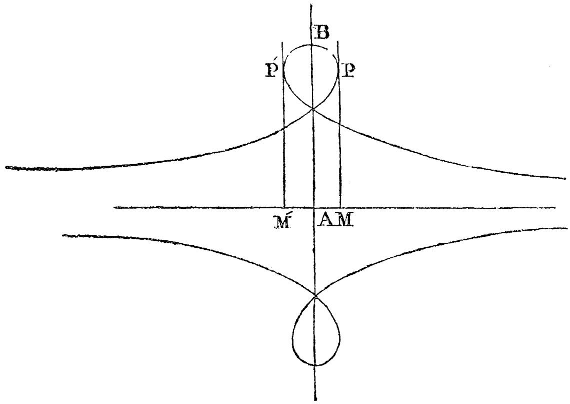


each side it is convex towards the axis of x. The form of the curve when $b < a$ is therefore represented above.

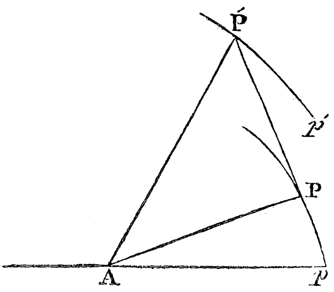
2°. Let $b > a$.

In this case, as before, when $y = b$ and $x = 0$, the tangent is parallel to the axis of x, and between $x = 0$ and that value of x which gives $ab = y^2$, the first differential coefficient is increasing, and becomes infinite under this last condition, which shows that the tangent is approaching to parallelism with the axis of x, and at this point becomes parallel to it. Also, between these points the second differential of y is negative, and therefore the curve is concave towards the axis of x.

Let the points P, P′, be those at which the tangent is parallel to the axis of y. The portion PBP' is concave towards the axis. At the points P, P′, the second differential coefficient passes through infinity, and therefore changes its sign, and becomes positive, and remains so, and therefore every other part of the curve is convex towards the axis. The same reasoning applies to the part of the curve on the other side of the axis.



(579.) Equitangential curves in general are those, the intercept of whose tangent between the point of contact and any given line of any proposed species is of a given magnitude, and in general that line, of what kind soever it may be, which is the locus of the extremity of the tangent, is called the *directrix*. The consideration of these curves presents two classes, problems to the analyst. 1°. *Given the nature of the directrix, the magnitude of the tangent and its position at any given point of the directrix, to find the curve of traction.* 2°. *Given the nature of the curve of traction, and the magnitude of the tangent, to find the directrix.* Those problems which come under the latter class are much more easily investigated than the former, the solution of which, except in a few instances, involves difficulties almost insurmountable. There are one or two instances, however, in which the solution is obvious enough. Thus, if the directrix be a circle, and the tangent equal to



half the chord with which it coincides in any position, the curve of traction is a concentrical circle, the square of whose radius is equal to the difference of the squares of the tangent and radius of the directrix. It appears also (566), that the involute of the circle is the curve of traction of which the directrix is the spiral of Archimedes.

SECTION XX.

The nature and properties of the roots of equations illustrated by the geometry of curves.

(580.) Every algebraical equation of a degree expressed by m, that is, where m is the index of the highest dimension of the unknown quantity, after the equation is cleared of surds, is necessarily included in the general formula

$$x^m + \text{A}x^{m-1} + \text{B}x^{m-2} + \text{C}x^{m-3} \ldots\ldots\ldots \text{T}x + \text{V} = 0.$$

Any value whatever being assigned to x, let the corresponding value which the first member of this equation assumes be y, and the result is

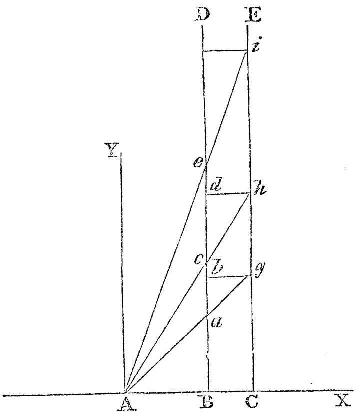
$$y = x^m + \text{A}x^{m-1} + \text{B}x^{m-2} + \text{C}x^{m-3} \ldots\ldots\ldots \text{T}x + \text{V}.$$

If this equation, related to rectangular co-ordinates, be supposed to represent a curve, the examination of the course of the locus will point out several important theorems concerning the roots of the proposed equation. But before we proceed to this investigation, we shall give a method for constructing the values of y, corresponding to any assumed value of x, and thereby constructing the curve which represents the equation.

Since the equation, in the form in which it is given, is not homogeneous, let n be the linear unit, and this being introduced in such a manner as to render all the terms linear, the equation becomes

$$y = \frac{x^m}{n^{m-1}} + \frac{\text{A}x^{m-1}}{n^{m-1}} + \frac{\text{B} \,.\, x^{m-2}}{n^{m-2}} \ldots\ldots\ldots \text{T}x + \text{V} = 0,$$

the coefficients A, B, C.........T, V, being supposed linear.



Let AB $= n$, AC $= x$, and let BD and CE be parallels to AY: let B$a = n$, and through a draw Aa meeting CE at g, and draw gb parallel to AX. Let bc = A, and draw Ac, meeting CE in h, and draw hd parallel to AX as before, and take de = B; and, as before, draw Ae meeting the line CE in i, and continue this process to Tx, and finally from the point where the last of the lines, drawn from A, meets the line CE, take a portion on it equal to V, and the extremity of this is that of y.

For since AB : AC : : Ba : Cg, ∴ C$g = x$.

Also AB : AC : : BC : Ch, ∴ Ch = BC . $\frac{x}{n}$

$$= (x + \text{A}) \,.\, \frac{x}{n} = \frac{x^2}{n} + \frac{\text{A}x}{n};$$

And AB : AC : : Be : Ci, ∴ Ci = Be . $\frac{x}{n}$;

$$\text{but B}e = \text{B}d + de = \frac{x^2}{n} + \frac{\text{A}x}{n} + \text{B},$$

$$\therefore \text{C}i = \frac{x^3}{n^2} + \frac{\text{A}x^2}{n^2} + \frac{\text{B}x}{n}.$$

And it is plain that by a similar process the successive intercepts between C and the lines drawn from A are found by adding to each former intercept the next coefficient, and multiplying the result by $\frac{x}{n}$, ∵ the successive intercepts will be

$$x,$$

$$\frac{x^2}{n}+\frac{\text{A}x}{n},$$

$$\frac{x^3}{n^2}+\frac{\text{A}x^2}{n^2}+\frac{\text{B}x}{n},$$

$$\frac{x^4}{n^3}+\frac{\text{A}x^3}{n^3}+\frac{bx^2}{n^2}+\frac{\text{C}x}{n},$$

$$\frac{x^5}{n^4}+\frac{\text{A}x^4}{n^4}+\frac{\text{B}x^3}{n^3}+\frac{\text{C}x^2}{n^2}+\frac{\text{D}x}{n},$$

. . . .

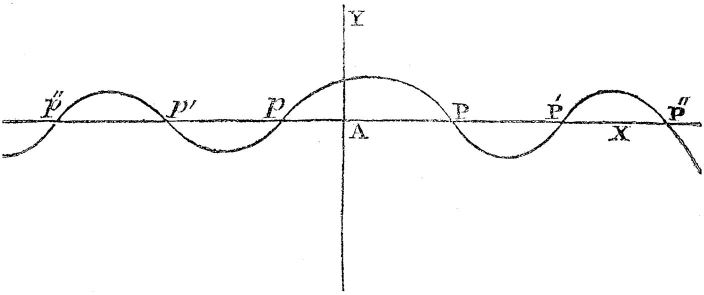
. . . .

$$\frac{x^m}{n^{m-1}}+\frac{\text{A}x^{m-1}}{n^{m-1}}+\frac{\text{B}x^{m-2}}{n^{m-2}}\ \dots\dots\ \frac{\text{T}x}{n}.$$

Adding then to the last intercept the line V, the result is the value of y; for any negative coefficient, the line represeting it is to be taken in an opposite direction: thus, if A were negative, bc should be taken from b towards A.

Being thus able to construct the values of y, corresponding to every value of x, the curve can be constructed by points. The values of x, corresponding to the points P, P', P'', p, p', p'', when the curve meets the axis AX, are the roots of the proposed equation.

Since in general the curve cannot pass from one side to the other of the axis of x without intersecting it, it necessarily follows that, between two points of the curve, situated at opposite sides of the axis AX, the curve must at least intersect that axis once, and may intersect it an odd number of times, that is, between two values of x, which give values of y with opposite signs, there must be an odd number of intersections of the curve with the axis of x, and there must at least be one point of intersection between them. Hence *if two numbers, substituted for* X *in any equation, produce results with opposite signs, there must be an odd number of real roots between them, and at least there must be one.*



Between two points of the curve, situate at the same side of the axis of x, there must be either an even number of intersections with that axis or none. That is, if two values of x give corresponding values of y, affected with the same sign, between those values of x, there must either be no intersection of the curve with the axis AX, or an even number. Hence, *if two numbers substituted for* X *in any equation give results affected with the same sign, there must be either no real root between them or an even number of real roots.*

The intercepts PP', P'P'', &c. pp', $p'p''$, &c. between two consecutive points of intersection of the curve with the axis of x, are the successive differences between the consecutive roots of the proposed equation. If two values of x be assumed, whose difference is less than the least of these differences, there cannot be more than one point of intersection between them; for if there were two points of intersection between them, the intercept between those two points would be necessarily less than the difference of the assumed values of x, which is contrary to hypothesis. Hence, if two such values of x give values of y with different signs, there will be one and only one point of intersection between them; and if they give values of y with the same sign, there will be no point of intersection between them. Therefore, *if two numbers, whose difference is less than the least difference of two consecutive roots of an equation, substituted for* X *in the equation, give results affected with different signs, one and only one real root lies between them; and if they give results affected with the same sign, no real root lies between them.*

When any of the intercepts PP′, *pp′*, &c. equal nothing, the curve touches the line AX at that point. The intercept which vanishes, being the difference of two values of x, which give $y = 0$, these values must be equal, and therefore a point of contact with the axis of X is the indication of equal roots of the proposed equation. If one of the intercepts PP′ vanishes, the curve at each side of the point of contact lies at the same side of the line AX, and therefore two values of x, which intercept between them a point of contact of this kind, give values of y affected with the same sign. Hence, *if two numbers, which include between them two real and equal roots, be substituted for* X *in any equation, they will give results affected with the same sign.*

If two consecutive intercepts PP′, P′P″, both vanish, the curve also touches the axis of x at that point; but the parts of the curve on each side of the point of contact lie at different sides of the axis of x, and therefore the point of contact is a point of inflection. It appears, as before, that in this case three points of intersection unite in one, and that the values on each side of the point of contact give values of y with opposite signs. Hence, if *two numbers which include between them three equal and real roots of an equation be substituted for* X *they will give results with different signs.*

In general, if an odd number of consecutive intercepts PP′, P′P″, &c. vanish, and therefore an even number of points of intersection unite in one, the curve will touch the axis of x, and the parts of the curve on each side of the point of contact will lie at the same side of the axis of x. It follows from this, that if *two numbers substituted for* X *in any equation include between them an even number of real and equal roots, they will give results with the same sign.*

If an even number of consecutive intercepts PP', $P'P''$, &c. vanish, and therefore an odd number of points of intersection unite in one, the curve also touches the axis of x; but in this case the parts of the curve at each side of the point of contact lie at different sides of the axis of x, and therefore that point is a point of inflection. The values of x on each side of the point in this case give values of y affected with contrary signs. Hence it follows, that two numbers which include between them an odd number of real and equal roots of an equation, substituted in it for x, will give results affected with opposite signs.

The point of contact corresponding to four real and equal roots is called a point of simple undulation. If it corresponds to six real and equal roots, it is said to be a point of double undulation.

A point of contact corresponding to three real and equal roots is called a point of simple inflection; if it corresponds to five real and equal roots, triple inflection, &c. The character of such points is merely algebraical, there being no visible geometrical distinction.

As in algebraical curves, the number of intersections are in general the same as the index of the highest dimension of x, when the equation is cleared of fractional indices, that number is finite. The entire of the curve, therefore, which extends beyond the most distant point of intersection on the positive side of the origin must lie at the same side of the axis. And the same inference is applicable for the same reasons to that part of the curve which lies beyond the most distant point of intersection on the negative side of the origin; and therefore all values of x greater than that of the most distant point of intersection give values of y continually affected with the same sign. Hence, if *numbers greater than the greatest root of an equation, whether positive or negative, be substituted for* x, *they will continually give results with the same sign.*

In any algebraical equation, a value of a may be assigned to x so great, that the first term shall exceed the sum of all the others, and its excess above the others will continually increase with the increase of x. The sign of y will therefore ultimately be the same as that of the highest dimension of x, and will continue to be so as x is increased without limit. Consequently, if the highest dimension of x be even, and therefore its sign necessarily positive, whether x itself be positive or negative, it follows that the sign of y is ultimately positive on both sides of the origin, and that therefore the two parts of the curve which extend beyond the last points of intersection on each side of the origin both lie above the axis of x, and that therefore either no point of intersection or an even number of such points must be included between them. Hence it follows that *every equation in which the index of the highest power of* x *is even, has either no real root or an even number of them; and since the number of roots altogether is the same as the highest index, it follows that there must be either an even number of impossible roots or none.*

If the index of the highest power of x be odd, the first term will be positive or negative according as x is positive or negative, and therefore if continually increasing positive values be assigned to x, the value of y will be ultimately positive, and continue so as the positive values of x increase without limit: and if continually increasing negative values be assigned to x, the value of y will be ultimately negative, and will continue so as the negative values of x increase without limit. These conclusions obviously follow from the same principle as the former scil., that such a value may be assigned to x as will render its highest power greater than all the remaining terms of the equation together. The parts of the curve lying beyond the most distant points of intersection on each side of the origin therefore lie at different sides of the axis of x; that beyond the most distant point of intersection on the positive side of the origin lying on the positive side, and that beyond the most distant point of intersection on the negative side of the origin lying on the negative side of the axis of x; therefore the number of intersections of the curve with the axis of x is odd, and it follows therefore that *every equation* in which the index of the highest power of x is odd, must have at least one real root, and in general has an odd number of real roots; and since the entire number of roots, being that of the index of the highest dimension of x, is odd, the number of impossible roots is even, if there be any such.

It follows therefore in general, that the number of impossible roots, if there be any, must be even, and there can therefore never be less than two.

The absolute term v is the value of y corresponding to $x = 0$, and is therefore the distance between the origin and the point where the curve meets the axis of y, and therefore the curve intersects that line, above or below the origin, according as v is positive or negative. If $v = 0$, the curve meets the axis of y at the origin.

Hence, *if any equation wants the last term, one of its roots must be equal to nothing.* Also since, in case the index of the highest dimension of the unknown quantity is even, the curve ultimately extends above the axis of x on both sides of the origin; if the absolute term be negative, and therefore it intersect the axis of y below the axis of x, it must necessarily intersect the curve at least once on each side of the origin. Hence, if *in any equation whose dimension is even the absolute term be negative, it will have at least two real roots, one positive and the other negative.*

By a change of origin on the axis of y, that is, by moving the axis of x parallel to itself, any of the intercepts PP$'$, P$'$P$''$, &c. may be made to vanish, and by a further change, the axis of x will cease to meet the curve at those points; thus by changing the axis of x without altering its inclination, two points of intersection will first approach each other, then meet, and finally disappear altogether. Also by the same change of the axis of x, it may meet the curve in other points, where before the change it did not meet it, first touching it, and then intersecting it. This change in the axis of x is effected by increasing or diminishing the values of y by any given quantity, which is equivalent to a change in the magnitude of the absolute term V. Hence it follows, that *by a change in the absolute term, any two real and unequal roots may be first made to become real and equal, and afterwards impossible;* and vice versa, *any two impossible roots may, by a similar change, be made to become first real and equal, and afterwards real and unequal.* It appears therefore that the minima values of y indicate the impossible roots of the equation.

To determine the maxima and minima of y, or the points of the curve at which the tangent is parallel to the axis of x, let the equation be differentiated, and its differential equated with zero, the result of which is

$$mx^{m-1}+(m-1)\text{A}x^{m-2}+(m-2)\text{B}x^{m-3}+\ \ldots\ \text{T}=0 \quad (\text{A}).$$

If the consecutive roots of this equation substituted in the proposed equation give results with opposite signs, the points at which the tangent is parallel to AX lie alternately at the positive and negative sides of AX. Between every pair of such successive values of x the curve must intersect the axis of x, and intersect it but once, because if it intersected it more than once, there would necessarily be another point at which the tangent would be parallel to the axis of x between the two assumed values of x, which is contrary to hypothesis. Hence it appears, that if all the roots of the proposed equation be real, all the roots of the equation (A) are also real, and correspond to maxima values of y; and between every two consective roots of the equation (A), a root of the proposed equation must be contained; the roots of this equation are called *limits* of the roots of the proposed equation, and the equation is called the *equation of limits* of the proposed.

If three consecutive points of the curve, at which the tangent is parallel to the axis of x, be situate on the same side of that line, and the value of y for the second is less than those for the first and third, there must be two impossible roots of the proposed equation intercepted between those values of x, which correspond to the first and third values of y; this is plain from what has been already said. And hence it follows, that if three successive roots of the equation (A) substituted in the proposed give results with the same sign, the second being less than the first and third, there will be two impossible roots of the proposed equation included between the first and third values of x.

If an even number of successive points of the curve at which the tangent is parallel to the axis of x be situated at the same side of that line, half their number will be points at which y is a minimum, and since every such point indicates two impossible roots, it follows that if an even number of consecutive roots of the equation of limits substituted in the proposed equation give results with the same sign, the proposed equation will have as many impossible roots at least.

Since for every minimum value of y there are two impossible roots, the number of impossible roots must be double the number of such values. To investigate this, let the equation (A) be differentiated, and the result is

$$m\,.\,(m-1)\,.\,x^{m-2}+(m-1)\,(m-2)\text{A}x^{m-3}$$
$$+\,(m-2)\,(m-3)\text{B}x^{m-4}\,\ldots\,+\text{S}=0.$$

Such roots of the equation (A) as being substituted in this and the proposed equation, give results affected with the same sign, correspond to minima values of y; and for every such value there are two impossible roots of the proposed equation.

(581.) Before the methods of approximation to the values of the roots of equations which are now used were known, they were frequently represented by geometrical constructions. This method of representing them is now, however, used only as an illustration, and as it is not inelegant, we shall here explain the principles on which it is founded.

Let the equation whose roots it is proposed to represent be $F(x) = 0$, and let any part of the first member be $F'(x)$, and let the equation $F'(x) = F''(y)$ be assumed, in which the form of the second member is arbitrary, and let this value of $F'(x)$ be substituted in the proposed equation, the result will be an equation $F'''(yx) = 0$, between y and x. It is obvious from this process, that if y be eliminated from the equations $F'(x) = F''(y)$ and $F'''(xy) = 0$, the result will be the proposed equation; and it follows therefore, that if two curves be constructed which are the loci of the equations $F'(x) = F''(y)$ and $F''(xy) = 0$, the values of x corresponding to their points of intersection are the roots of the proposed equation. The investigation which we have just given on the nature of the roots of equations may be looked on as an example of the application of the principle, since the equations of the two loci, whose intersection gave the roots of the equation sought, were

$$y = 0,$$

$$y = x^m + Ax^{m-1} + Bx^{m-2} \ldots\ldots Tx + V.$$

But this is evidently useless, as it requires the solution of the equation itself to construct the second locus. We shall however proceed to apply the principle to some examples which will render it clearer than any abstract explanation could make it.

PROP. CCLXXXVIII.

(582.) *To represent by geometrical construction the roots of a quadratic equation.*

Let the proposed equation be

$$x^2 + 2\text{A}x + \text{B}^2 = 0.$$

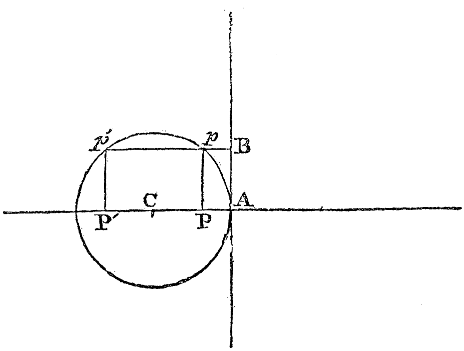
Let one of the loci whose intersection give the roots of the equation be a right line parallel to the axis of x, represented by the equation

$$y = \text{B}.$$

This substitution being made in the given equation, gives, when $\text{B}^2 > 0$,

$$y^2 + x^2 + 2\text{A}x = 0$$

for the equation of the other locus; this is the equation of a circle whose radius is A, and whose centre is on the axis of x at a distance from the origin equal to $-\text{A}$. Let $\text{AC} = -\text{A}$, and with the centre C and the radius AC let the circle be described; let $\text{AB} = \text{B}$, and let the parallel $\text{B}p'$ be drawn; the values of x scil. AP, AP′, corresponding to the points p, p', where this parallel meets the circle, are the roots of the proposed equation.



The centre lies at the positive side of the origin if $\text{A} < 0$, and at the negative side if $\text{A} > 0$; therefore in the one case both roots are positive, and in the other negative.

$\text{AP} + \text{AP}' = 2\text{AC}$, *i. e.* the sum of the roots taken with their proper signs is equal to the co-efficient of x in the given equation.

$AP \times AP' = Pp^2 = B^2$, *i. e.* the product of the roots is equal to the absolute term.

If $AB = AC$, p and p' coincide, and $AP = AP'$, *i. e.* if $B = A$ the roots of the equation are equal.

If $AB > AC$, the parallel does not meet the circle, $\because$ if $B > A$ and $B^2 > 0$, the roots are impossible.

If $B^2 < 0$, the second equation is

$$y^2 - x^2 + 2Ax = 0,$$

which is the equation of the equilateral hyperbola, the transverse axis of which coincides with the axis of x, and the origin being at the vertex.

But the equation can be constructed in all cases by the circle alone. In general let the equation of the right line parallel to the axis of x be

$$y = \sqrt{m^2 + B^2}.$$

Making this substitution the equation becomes

$$y^2 + x^2 + 2Ax - m^2 = 0;$$

m being arbitrary, it may be supposed $= 0$, if $B^2 > 0$, which reduces this case to the former. But if $B^2 < 0$, in order that y may be real in the equation of the parallel, m cannot be less than B; in this case let $m = B$. The parallel will in that case coincide with the axis of x itself, and the equation of the circle is

$$y^2 + x^2 + 2Ax - B^2 = 0.$$

This is the equation of a circle whose radius is $\sqrt{A^2 + B^2}$, and whose centre is at a distance from the origin equal to $-A$. Hence let $AC = -A$, and the circle be described with the radius $\sqrt{A^2 + B^2}$, and the roots of the equation are AP, AP'.

The equation may be constructed by the intersection of the right line with any line of the second degree as well as the circle. Thus let the equation of the first locus be

$$y = x + \frac{B^2}{2A},$$

which is the equation of a right line intersecting the axis of y at a distance from the origin expressed by $\frac{B^2}{2A}$ and inclined to the axes of co-ordinates at angles of 45^0. The equation of the other locus will be

$$x^2 + 2Ay = 0,$$

which is the equation of a parabola whose axis is the axis of y, whose vertex is the origin, and whose parameter is $-2A$. The intersection of this with the right line gives the roots of the proposed equation.

PROP. CCLXXXIX.

(583.) *To represent by geometrical construction the roots of a cubic equation.*

Let the proposed equation be

$$x^3 + Ax^2 + B^2x + C^3 = 0.$$

Let the equation of one of the curves whose intersection is to determine the roots of this equation be

$$x^2 + Ax = By.$$

By substituting y in the proposed equation the result is

$$Bxy + B^2x + C^3 = 0.$$

The former of these equations represents a parabola, the equation of whose axis is $x = -\frac{A}{2}$, which is therefore parallel to the axis of y. The value of y which gives the vertex is $-\frac{A^2}{4B}$, and the principal parameter is B.

The latter equation represents an hyperbola, the axis of y is one asymptote, and the equation of the other is $y = -B$, which is therefore parallel to the axis of x; and since the asymptotes are rectangular, the hyperbola is equilateral.

Let a be its semi-axis, its equation related to the asymptotes is

$$yx = \frac{a^2}{2}.$$

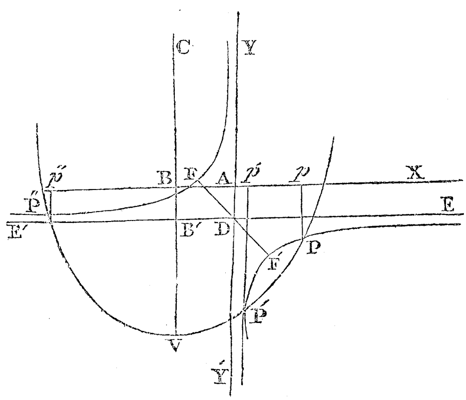
Hence $\frac{a^2}{2} = -\frac{\text{C}^3}{\text{B}}$, $\therefore a = \sqrt{-\frac{2\text{C}^3}{\text{B}}}$.

Let AB $= -\frac{\text{A}}{2}$, and through B draw a parallel to AY;

and let BV $= -\frac{\text{A}^2}{4\text{B}}$. On the axis VC with the vertex V, and a parameter equal to B, let a parabola be described. Let AD $= -$ B, and through D draw a parallel to AX. Let FF′ be drawn bisecting the angle D, and take

$$\text{DF} = \sqrt{-\frac{2\text{C}^3}{\text{B}}};$$

describe an hyperbola whose vertices are F, F′, and whose asymptotes are YY′ and EE′. The roots of the proposed equation are Ap, Ap', Ap'', the values of x corresponding to the points of intersection P, P′, P″, of the parabola and hyperbola.



If the equation of the first locus be

$$\text{A}xy + \text{C}^3 = 0,$$

the other will be by substituting for c^3, and dividing the result by x,

$$x^2 + Ax - Ay + B^2 = 0.$$

The former, related to rectangular co-ordinates, is the equation of an equilateral hyperbola, the axes of co-ordinates being the asymptotes, and its semiaxis equal to

$$\sqrt{-\frac{2c^3}{A}}.$$

The latter equation represents a parabola, the equation of whose axis is $x = -\frac{A}{2}$, and the co-ordinates of whose vertex are $x = -\frac{A}{2}$, $y = \frac{B^2}{A} - \frac{A}{4}$.

Through the origin D let FF′ be drawn bisecting the angle D, and let $DF = \sqrt{-\frac{2c^3}{A}} = DF'$, with the line FF′ as transverse axis, and the points F, F′, as vertices, let an equilateral hyperbola be described. Also let $DB' = -\frac{A}{2}$, and

$$B'V = \frac{B^2}{A} - \frac{A}{4};$$

with the vertex V, the axis VC, and the parameter A, let a parabola be described; the points of intersection of this with the hyperbola will give the roots required.

It is always possible by a transformation to remove the second term of the equation. Suppose this done with the proposed equation, and that it is reduced to the form

$$x^3 + B^2x + c^3 = 0.$$

Let it be multiplied by $x = 0$, by which it becomes

$$x^4 + B^2x^2 + c^3x = 0.$$

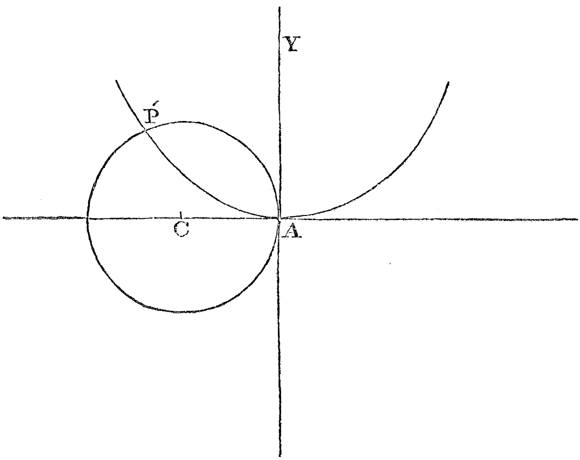
If the equation of one of the curves whose intersections determine the roots be

$$x^2 = By.$$

That of the other will be, when $B^2 > 0$,

$$y^2 + x^2 + \frac{C^3}{B^2}x = 0.$$

The former is the equation of a parabola whose vertex is at the origin, whose axis is AY, and whose parameter is B.

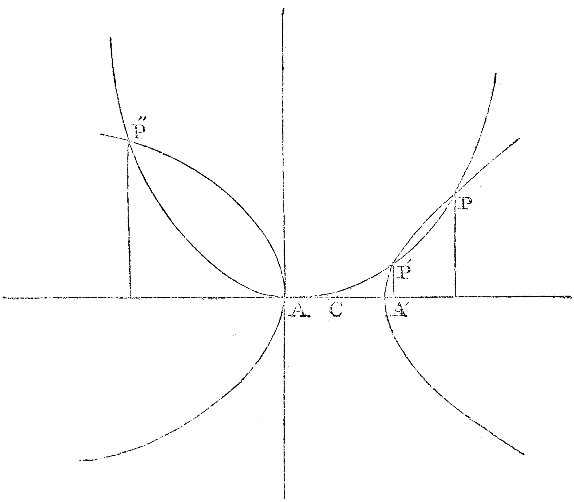


The latter is a circle passing through the origin with its centre on the axis of x at the distance $-\frac{C^3}{2B^2}$. Therefore let a parabola be described with the vertex A, the axis AY, and the principal parameter B; and a circle with the radius $CA = -\frac{C^3}{2B^2}$. The point of intersection P gives the root of the proposed equation. The point A gives the root $x = 0$, which was introduced by multiplying by x. The other two roots must in this case be impossible. The circle will lie on the negative or positive side of the origin, according as C^3 is positive or negative; and therefore the real root will in the one case be negative, and in the other positive.

If $B^2 < 0$, the second equation becomes

$$y^2 - x^2 + \frac{c^3}{b^2}x = 0.$$

This is the equation of an equilateral hyperbola whose vertex is the origin, and whose transverse axis is the axis of x.



Let $AC = \frac{c^3}{2b^2} = CA'$, and with the vertices AA' let an equilateral hyperbola be described, the points P, P', P'', give the roots of the proposed equation.

The centre of the hyperbola lies on the positive or negative side of A, as $c^3 > 0$ or < 0. If C lie on the positive side of A, there must be one point of intersection on the negative side; and if C lie on the negative side, there must be one point of intersection on the positive side. Hence the equation must have one real root having the sign contrary to that of c^3.

Supposing both parts of the hyperbola to intersect the parabola, the roots will be real and unequal; and two will be positive and one negative if $c^3 > 0$, and two negative and one positive if $c^3 < 0$.

If the parabola and hyperbola touch, there will be two equal roots.

If one of the branches of the hyperbola does not meet the parabola, two of the roots of the equation are impossible.

PROP. CCXC.

(584.) *To construct the roots of an equation of the fourth degree.*

Since the second term may always be removed by a transformation, the equation can always be brought to the form

$$x^4 + B^2x^2 + C^3x + D^4 = 0.$$

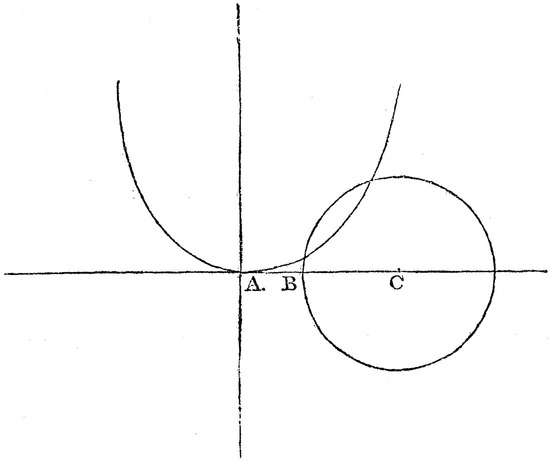
Let one of the curves be the parabola represented by the equation $$x^2 = By,$$
which being substituted for x^4, gives for the equation of the other when $B^2 > 0$,

$$y^2 + x^2 + \frac{C^3}{B^2}x + \frac{D^4}{B^2} = 0.$$

And when $B^2 < 0$,

$$y^2 - x^2 + \frac{C^3}{B^2}x + \frac{D^4}{B^2} = 0.$$

1. Let $B^2 > 0$. The parabola being constructed as in the last proposition, let a circle be described with its centre at C on the axis of x at the distance $AC = -\frac{C^3}{2B^2}$, and with the radius $CB = \sqrt{\frac{C^6}{4B^4} - \frac{D^4}{B^2}}$. As this circle, from its position, cannot intersect the parabola in more than two points, there can be only two real roots to the equation in this case.



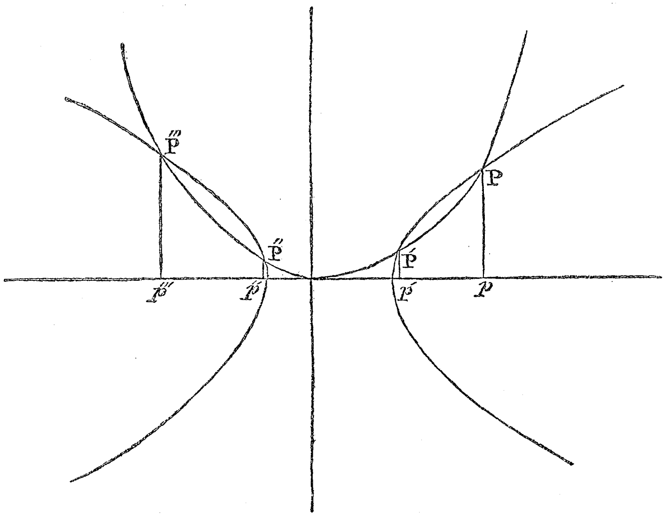
If the circle touches the parabola, the two roots are equal; and if it does not meet it, they are impossible.

If CB > CA, the circle must intersect the parabola, therefore in this case the roots must be real; they will also have in this case different signs: this necessarily happens when the last term in the given equation is negative.

2. Let $B^2 < 0$. The equation of the second curve is in this case that of an equilateral hyperbola, its centre is on the axis of x at the distance $-\frac{C^3}{2B^2}$ from the origin, and its semiaxis is $\sqrt{\frac{C^6}{4B^4} - \frac{D^4}{B^2}}$.

This curve being constructed as in the annexed figure, gives the roots required.

The observations made in the other cases as to the real, equal, and imaginary roots apply here also. It is evident that since the hyperbola may intersect the parabola in four points, all the roots may be real. Also it is plain that in this case one of them at least must be negative, and two at least positive.



Since one or both branches of the hyperbola may touch the parabola, there may be one or two pairs of equal roots, and since neither branch may meet it, all the roots may be impossible.

PROP. CCXCI.

(585.) *To find a cube which shall bear a given ratio to a given cube.*

This problem is in effect to construct the equation

$$x^3 - ma^3 = 0.$$

Let it be multiplied by x, and we find

$$x^4 - ma^3x = 0.$$

Let one of the curves whose intersection is to determine the roots be the parabola

$$x^2 = ay.$$

This being substituted in the above equation, gives

$$y^2 = max.$$

Hence the root is determined by the intersection of two parabolas having a common vertex, and their axes at right angles, and whose parameters are in the given ratio.

PROP. CCXCII.

(586.) *To find two mean proportionals between two given lines.*

Let the given lines be a, b, and the sought means y and x. Hence $a : y : x : b$, and therefore

$$y^2 = ax,$$
$$x^2 = by.$$

Hence, if two parabolas be described having a common vertex and their axes at right angles, and whose parameters are equal to the two given lines, the co-ordinates of their point of intersection related to their axes, as axes of co-ordinates, are the sought means.

PROP. CCXCIII.

(587.) *To trisect an angle.*

Let A be the given angle. By trigonometry,

$$\cos.^3 \tfrac{1}{3}\text{A} - \tfrac{3}{4}\cos. \tfrac{1}{3}\text{A} - \tfrac{1}{4}\cos. \text{A} = 0;$$

which by supplying the radius r, and representing $\cos. \frac{1}{3}\text{A}$ by x, becomes

$$4x^3 - 3r^2x - r^2 \cos. \text{A} = 0;$$

which multiplied by x, gives

$$4x^4 - 3r^2x^2 - r^2 \cos. \text{A} \cdot x = 0.$$

Let the equation of one of the curves be

$$2x^2 = ry,$$

and the other by substitution will be

$$2y^2 - 3ry - 2 \cos. \text{A}x = 0.$$

The former is the equation of a parabola, the axis of which is the axis of y, the origin the vertex, and the principal parameter equal to $\frac{1}{2}r$.

The latter is also a parabola, the equation of its axis is $y = \frac{3}{4}r$, and the co-ordinates of its vertex are $y = \frac{3}{4}r$, $x = -\frac{9r^2}{16\cos. A}$, and its principal parameter is cos. A.

These parabolas being described, their points of intersection give the roots of the equation. The intersection at the origin gives the root $x = 0$, which was introduced by the multiplication by x.

The equation having more than one real root, it might appear that there were more values than one for the third of the given angle. But upon examining the process, it will be seen that the question really solved was not to find an angle equal to the third of a given angle, but to find the cosine of an angle which is the third of an angle whose cosine is given. Since then the arcs

$$A,$$
$$2\pi - A, \quad 2\pi + A,$$
$$4\pi - A, \quad 4\pi + A,$$
$$6\pi - A, \quad 6\pi + A.$$

And in general all arcs which come under the general formula $2m\pi \pm A$ have the same cosine, the question really solved is to find the cosine of the third of any of these arcs. And here again another apparent difficulty arises. If the number of arcs involved in the question be unlimited, shall there not be an unlimited number of values for the cosine of the third parts of these? To account for this it should be considered that in general the arc $\frac{2m}{3}\pi \pm \frac{A}{3}$ must have the same cosine as some one of the three arcs,

$$\frac{A}{3}, \tfrac{2}{3} \cdot \pi - \tfrac{1}{3}A, \tfrac{4}{3}\pi - \tfrac{1}{3}A;$$

for the number $\frac{m}{3}$ must be either of these forms, n, $n + \frac{1}{3}$, or $n + \frac{2}{3}$, where n is an integer. If it have the form n, that is, if 3 measure m, then

$$\frac{2m}{3}\pi \pm \tfrac{1}{3}A = 2n\pi \pm A; \text{ therefore}$$

$$\cos. \frac{2m}{3}\pi \pm \tfrac{1}{3}A = \cos. (2n\pi \pm \tfrac{1}{3}A) = \cos. \tfrac{1}{3}A.$$

If it have the form $n + \frac{1}{3}$;

$$\frac{2m}{3}\pi \pm \tfrac{1}{3}A = 2n\pi + \tfrac{2}{3}\pi \pm \tfrac{1}{3}A; \text{ therefore}$$

$$\cos. \frac{2m}{3}\pi \pm \tfrac{1}{3}A = \cos. (2n\pi + \tfrac{2}{3}\pi \pm \tfrac{1}{3}A) = \cos. \tfrac{1}{3}(2\pi \pm A).$$

If it have the form $n + \frac{2}{3}$;

$$2 \,.\, \frac{m}{3}\pi \pm \tfrac{1}{3}A = 2n\pi + \tfrac{4}{3}\pi \pm \tfrac{1}{3}A; \text{ therefore}$$

$$\cos. (\frac{2m}{3}\pi \pm \tfrac{1}{3}A) = \cos. (2n\pi + \tfrac{4}{3}p \pm \tfrac{1}{3}A) = \cos. \tfrac{1}{3}(4\pi \pm A).$$

And hence it follows that the $\cos. (\frac{2m\pi}{3} \pm A)$, whatever be the value of m, must be equal to one or other of the quantities

$$\cos. \tfrac{1}{3}A,$$
$$\cos. \tfrac{1}{3}(2\pi - A),$$
$$\cos. \tfrac{1}{3}(4\pi - A),$$

which correspond to the three roots of the cubic equation already found.

PROP. CCXCIV.

(588.) *To resolve the formula $x^m \pm a^m$ into its simple factors by geometrical construction.*

Let $x = a(\cos. \phi + \sqrt{-1} \,.\, \sin. \phi)$, and since by trigonometry,

$$(\cos. \phi + \sqrt{-1} \sin. \phi)^m = \cos. m\phi + \sqrt{-1} \sin. m\phi;$$

it follows that

$$x^m = a^m (\cos. m\phi + \sqrt{-1} \sin m\phi).$$

Hence subtracting a^m from both members,

$$x^m - a^m = a^m (\cos. m\phi + \sqrt{-1} \sin. m\phi - 1.)$$

The question then is to find the factors of

$$\cos. m\phi + \sqrt{-1} \sin. m\phi - 1,$$

which will be found by investigating the values of φ which satisfy the equation,

$$\cos. m\phi + \sqrt{-1} \sin. m\phi - 1 = 0.$$

This condition can only be fulfilled by the real and impossible parts being separately equal to nothing, which gives

$$\cos. m\varphi - 1 = 0,$$

$$\sqrt{-1} \sin. m\varphi = 0.$$

And hence $\cos. m\varphi = 1$, $\because m\phi = 2n\pi$, $\because \varphi = n\frac{2\pi}{m}$.

Hence the factors sought are found by supposing n in the formula

$$x - a \left\{ \cos. n\frac{2\pi}{m} + \sqrt{-1} \sin. n\frac{2\pi}{m} \right\},$$

successively to assume the values, 0, 1, 2, 3, ... $m - 1$, which give

$$x - a,$$

$$x - a \left\{ \cos. \frac{2\pi}{m} + \sqrt{-1} \sin. \frac{2\pi}{m} \right\}$$

$$x - a \left\{ \cos. \frac{4\pi}{m} + \sqrt{-1} \sin. \frac{4\pi}{m} \right\}$$

$$x - a \left\{ \cos. \frac{6\pi}{m} + \sqrt{-1} \sin. \frac{6\pi}{m} \right\}$$

. . . .

. . . .

$$x - a \left\{ \cos. (m-2)\frac{2\pi}{m} + \sqrt{-1} \sin. (m-2)\frac{2\pi}{m} \right\}$$

$$x - a \left\{ \cos. (m-1)\frac{2\pi}{m} + \sqrt{-1} \sin. (m-1)\frac{2\pi}{m} \right\}$$

After this factor the values recur; for if $n = m$,

$$\cos.\ n\frac{2\pi}{m} = \cos.\ 2\pi = 1, \text{ and } \sin.\ n\frac{2\pi}{m} = \sin.\ 2\pi = 0,$$

which gives $x - a$, which is the same as the first factor, and in like manner every succeeding factor would be only a repetition of the former one. These therefore are the simple factors of $x^m - a^m$. Their forms may be somewhat modified by observing that

$$\cos.\ (m-1)\frac{2\pi}{m} = \cos.\ \frac{2\pi}{m},$$

$$\sin.\ (m-1)\frac{2\pi}{m} = -\sin.\ \frac{2\pi}{m},$$

$$\cos.\ (m-2)\frac{2\pi}{m} = \cos.\ \frac{4}{m},$$

$$\sin.\ (m-2)\frac{2\pi}{m} = -\sin.\ \frac{4\pi}{m},$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

and therefore omitting the factor $x - a$, the series of remaining factors will be

$$\left.\begin{array}{l}
x - a\left\{\cos.\ \frac{2\pi}{m} + \sqrt{-1}\sin.\ \frac{2\pi}{m}\right\} \\
x - a\left\{\cos.\ \frac{4\pi}{m} + \sqrt{-1}\sin.\ \frac{4\pi}{m}\right\} \\
x - a\left\{\cos.\ \frac{6\pi}{m} + \sqrt{-1}\sin.\ \frac{6\pi}{m}\right\} \\
\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots \\
x - a\left\{\cos.\ \frac{6\pi}{m} - \sqrt{-1}\sin.\ \frac{6\pi}{m}\right\} \\
x - a\left\{\cos.\ \frac{4\pi}{m} - \sqrt{-1}\sin.\ \frac{4\pi}{m}\right\} \\
x - a\left\{\cos.\ \frac{2\pi}{m} - \sqrt{-1}\sin.\ \frac{2\pi}{m}\right\}
\end{array}\right\} \text{(B)}$$

If m be odd, and therefore $(m - 1)$ even, we find, by multiplying the extreme terms of this series and every pair of terms equidistant from them, this series of real quadratic factors,

$$\left.\begin{array}{l}
x^2 - 2a \cos. \dfrac{2\pi}{m} . x + a^2 \\
x^2 - 2a \cos. \dfrac{4\pi}{m} . x + a^2 \\
x^2 - 2a \cos. \dfrac{6\pi}{m} . x + a^2 \\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
x^2 - 2a \cos. \dfrac{\pi}{m} . x + a^2
\end{array}\right\} (C)$$

Therefore in this case the real factors of $x^m - a^m$ are $(x - a)$ and the above series of quadratic factors; all the simple factors except $(x - a)$ being impossible.

If m be even, and therefore $(m - 1)$ odd, after multiplying the extreme terms of the series (B), and also every pair of terms equidistant from them, a solitary term will remain in the middle. The coefficient of $\frac{2\pi}{m}$ in this term will be $\frac{m}{2}$, and therefore the term will be

$$x - a\{\cos. \pi + \sqrt{-1} \sin. \pi\} = x + a.$$

Hence in this case $x^m - a^m$ has two real simple factors $x - a$ and $x + a$; all the other simple factors being impossible. It has also the real quadratic factors expressed in the series (C). These results may be thus expressed.

1°. If m be odd,

$$x^m - a^m = (x - a)\,(x^2 - 2a \cos. \frac{2\pi}{m} x + a^2)$$

$$(x^2 - 2a \cos. 2\,\frac{2\pi}{m} . x + a^2)\,(x^2 - 2a \cos. 3\,\frac{2\pi}{m} . x + a^2) \ldots$$

$$(x^2 - 2a \cos. \frac{m-1}{2} . \frac{2\pi}{m} . x + a^2).$$

2°. If m be even,

$$x^m - a^m = (x - a)(x + a)\left(x^2 - 2a\cos.\frac{2\pi}{m} + a^2\right)$$

$$\left(x^2 - 2a\cos.2\frac{2\pi}{m}.x + a^2\right)\left(x^2 - 2a\cos.3\frac{2\pi}{m}.x + a^2\right)\ldots$$

$$\left(x^2 - 2a\cos.\frac{m-2}{2}.\frac{2\pi}{m} + a^2\right).$$

To represent these factors geometrically, let a circle be described with the radius CA $= a$, and let CP $= x$, and let the circumference be divided into $2m$ equal parts at the points A, A_1, A_2, ... A_{2m-1}, and let PA $= z_0$, PA_1 $= z_1$, PA_2 $= z_2$, ... PA_{2m-1} $= z_{2m-1}$. Hence,

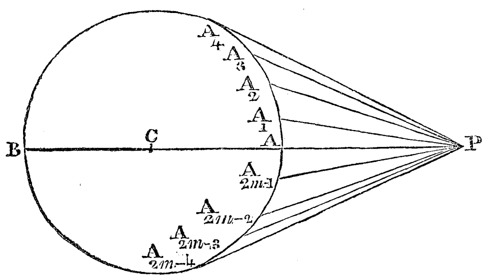
$$z_0 = x - a,$$

$$z_2^2 = x^2 - 2a\cos.\frac{2\pi}{m}.x + a^2,$$

$$z_4^2 = x^2 - 2a\cos.\frac{4\pi}{m}.x + a^2,$$

. . . .

. . . .



If m be even, one of the points of division will coincide with B, so PB $= x + a = z_{\frac{1}{2}m}$. Since $z_1 = z_{2m-1}$, $z_2 = z_{2m-2}$, $z_3 = z_{2m-3}$, ... we find

$$z_2 z_{2m-2} = x^2 - 2a\cos.\frac{2\pi}{m}.x + a^2,$$

$$z_4 z_{2m-4} = x^2 - 2a\cos.\frac{4\pi}{m}.x + a^2,$$

. . . .

. . . .

Hence in general,

$$x^m - a^m = z_0, z_2, z_4, z_6 \ldots z_{2m-2}.$$

To find the factors of $x^m + a^m$, it is necessary to proceed in a similar manner, which will give

$$x^m + a^m = a^m (\cos. m\phi + \sqrt{-1} \sin. m\phi + 1).$$

Therefore,

$$\cos. m\phi + 1 = 0,$$

$$\sin. m\phi = 0,$$

which gives $\phi = \frac{(2n+1)\pi}{m}$, the result of which is

$$x = a\,(\cos. \frac{(2n+1)\pi}{m} + \sqrt{-1} \sin. \frac{(2n+1)\pi}{m}),$$

n being supposed successively to assume the values, 0, 1, 2, . . . $m-1$ as before, the simple factors will be

$$x - a\,(\cos. \frac{\pi}{m} + \sqrt{-1} \sin. \frac{\pi}{m}),$$

$$x - a\,(\cos. \frac{3\pi}{m} + \sqrt{-1} \sin. \frac{3\pi}{m}),$$

$$x - a\,(\cos. \frac{5\pi}{m} + \sqrt{-1} \sin. \frac{5\pi}{m}),$$

. . . .

. . . .

or beginning with $n = m - 1$, &c. the series will be

$$x - a\,(\cos. \frac{\pi}{m} - \sqrt{-1} \sin. \frac{\pi}{m}),$$

$$x - a\,(\cos. \frac{3\pi}{m} - \sqrt{-1} \sin. \frac{3\pi}{m}),$$

$$x - a\,(\cos. \frac{5\pi}{m} - \sqrt{-1} \sin. \frac{5\pi}{m}),$$

. . . .

. . . .

And by uniting the extreme factors and those equidistant from them, we find

$$x^m + a^m = (x^2 - 2a\cos.\frac{\pi}{m}.x + a^2)$$

$$(x^2 - 2a\cos.\frac{3\pi}{m}.x + a^2)\,(x^2 - 2a\cos.\frac{5\pi}{m}.x + a^2)\ldots.$$

The last factor being simply $x + a$ when m is odd, and $x^2 - 2a\cos.\frac{(m-1)\pi}{m}.x + a^2$ when m is even; the number of real quadratic factors being in the former case $\frac{m-1}{2}$, and in the latter $\frac{m}{2}$.

As before, let the circumference of the circle with the radius a be divided into $2m$ equal parts, and the lines drawn from a point P at the distance x from the centre to the successive points of division being denominated as before,

$$z_1^2 = x^2 - 2a\cos.\frac{\pi}{m}.x + a^2,$$

$$z_3^2 = x^2 - 2a\cos.\frac{3\pi}{m}x + a^2,$$

$$z_5^2 = x^2 - 2a\cos.\frac{5\pi}{m}x + a^2.$$

. . . .

. . . .

And since $z_1 = z_{m-1}$, $z_2 = z_{m-2}$, &c.

$$x^m + a^m = z_1 . z_3 . z_5 . \ldots.$$

(589.) *Cor.* 1. The formula,

$$x = a\,(\cos.2n.\frac{\pi}{m} + \sqrt{-1}\sin.2n\frac{\pi}{m}),$$

is a general formula for the mth roots of a^m.

(590.) *Cor.* 2. The mth roots of unity are expressed by the formula,

$$\cos.2n\frac{\pi}{m} + \sqrt{-1}\sin.2n\frac{\pi}{m}.$$

(591.) *Cor.* 3. If $a = 1$, this proposition gives the resolution of $x^m - 1$ into its simple factors.

SECTION XXI.

Of the general properties of algebraic curves.

(592.) As every equation between two variables may be conceived to generate a curve, the variety of curves are as infinite as the variety of the equations by which they are represented. The classification of curves therefore should be conformable to that of equations; and as the first and principal division of equations is into algebraical and transcendental, the curves represented by them have been similarly divided and similarly denominated. An equation between two variables (yx) is called an algebraical equation when it is reducible to a finite series of terms involving only factors of the variables (yx) with integral and positive exponents. An equation, which is not reducible to such a series, or which, when reduced to a series of such terms, will consist of an infinite number of terms, is called a transcendental equation. Accordingly, the two principal classes of curves are algebraic and transcendental. Thus the lemniscata, whose equation is

$$y^4 + 2y^2x^2 + x^4 + a^2y^2 - a^2x^2 = 0,$$

is an algebraic curve. The logarithmic and the cycloid whose equations are

$$y = a^x,$$

$$y - r - r\cos. \frac{x - \sqrt{2ry - y^2}}{r} = 0,$$

are transcendental curves, for they would, if resolved to a series of integral and positive powers of y and x, consist of an infinite number of terms. From the nature of transcendental equations, it is impossible to form any regular classification of the curves they represent. They possess no generic properties, and the peculiar properties of each curve may be investigated by the rules already established. This, however, is not the case with algebraic curves. The means of their classification are obvious; they possess general properties which may be discovered from the nature and properties of general algebraic equations, as well as those distinctive and peculiar properties which characterise each subordinate species, and are derivable from its particular equation.

In a classification of equations, with a view to a corresponding classification of the curves represented by them, we should use such a means of distinction, as that equations coming under different classes may not represent the same curve.

Thus, for example, if the equations were classed according to the number of their terms, the equations,

$$y^2 + x^2 = r^2,$$
$$y^2 + x^2 - 2rx = 0,$$

would come under different classes, and yet they represent equal circles. Such a distinction between the classes of equations must therefore be adopted as will prevent the possibility of the same curve coming under two different classes. We shall find this distinction by investigating how the transformation of co-ordinates affects an equation; for as this never affects the curve represented by the equation, any quality in the equation which is changed by this operation cannot be used as a distinction of classes; and, on the other hand, any quality which the transformation does not affect, is a fit one for the purpose. The formulæ expressing the co-ordinates of a point relatively to one system of axes, as a function of those relatively to another being of the first degree, cannot make any change in the degree of the equation in which they are introduced. They may change the values of its co-efficients or its number of terms, but can never change its degree. Hence it is that algebraical equations between two variables are classed according to their dimensions, and the lines represented are accordingly denominated lines of the first or second or *m*th degree. The degree of an equation is marked by the sum of the exponents of the variables in that term in which it is highest. Thus,

$$x^2y + bx^2 + cy^2 + dy + ex + f = 0,$$

is an equation of the third degree, and represents a curve of the third degree.

(593.) Newton, in his classification of lines, made a distinction which, however, is now nearly abandoned. Considering that equations of the first degree represent only right lines, and those of the superior degrees curves, he designates the *order* of a *line* by the degree of its equation. According to him, equations of the *second degree* represent lines of the *second order*, those of the *third degree*, lines of the *third order*, &c. He divides *curves* into *kinds* or *genera*, and denominates *lines* of the *second order*, *curves* of the *first kind*, *lines* of the *third order*, *curves* of the *second kind*, &c. This distinction is, however, now out of use, and we say *lines* or *curves* of the second *degree* or *order* indifferently.

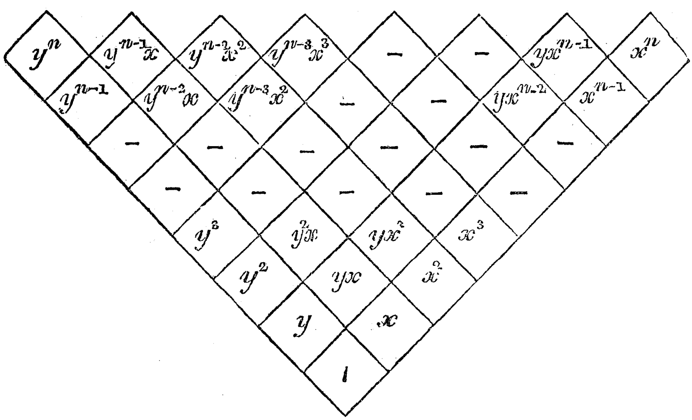
(594.) The manner in which the equation of a curve indicates the different peculiarities of its course has been already fully explained. By the principles which have been established, the different sinuosities and inflexions of any line will appear as plainly by its equation to the eye of the analyst, as if they were traced on a diagram, and actually exhibited to the senses. And, indeed, more plainly, as several peculiarities of a curve may be developed in the discussion of its equation, which would escape the utmost sagacity of the descriptive geometer. These principles enable us to follow the course of any particular curve by its equation; but there are some general properties of algebraic curves, arising from different principles, which have not yet been noticed.

(595.) A general algebraic equation of the nth order is one which includes terms in which the variables are involved in every variety of dimensions not exceeding n. A method of determining the terms of a general equation of any proposed order has been given by Newton.

He supposes the space included within a right angle, whose sides are horizontal and vertical, to be subdivided into squares by parallels to the sides. In the first horizontal row of squares the successive powers of x, scil. $1, x, x^2, x^3$, &c. are inserted, and in the first vertical column the powers of y, scil. $1, y, y^2$, &c. are inserted. Let such dimensions of x and y be inserted in the other squares, that each horizontal row shall contain the same dimensions of x, and each vertical column the same dimensions of y, so that the whole will stand thus:

1	x	x^2	x^3	x^4	x^5	x^6	-	-	x^{n-1}	x^n
y	yx	yx^2	yx^3	yx^4	yx^5	yx^6	-	-	yx^{n-1}	yx^n
y^2	y^2x	y^2x^2	y^2x^3	y^2x^4	y^2x^5	y^2x^6	-	-	y^2x^{n-1}	y^2x^n
y^3	y^3x	y^3x^2	y^3x^3	y^3x^4	y^3x^5	y^3x^6	-	-	y^3x^{n-1}	y^3x^n
y^4	y^4x	y^4x^2	y^4x^3	y^4x^4	y^4x^5	y^4x^6	-	-	y^4x^{n-1}	y^4x^n
y^5	y^5x	y^5x^2	y^5x^3	y^5x^4	y^5x^5	y^5x^6	-	-	y^5x^{n-1}	y^5x^n
y^6	y^6x	y^6x^2	y^6x^3	y^6x^4	y^6x^5	y^6x^6	-	-	y^6x^{n-1}	y^6x^n
-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-
y^{n-1}	$y^{n-1}x$	$y^{n-1}x^2$	$y^{n-1}x^3$	$y^{n-1}x^4$	$y^{n-1}x^5$	$y^{n-1}x^6$	-	-	$y^{n-1}x^{n-1}$	$y^{n-1}x^n$
y^n	y^nx	y^nx^2	y^nx^3	y^nx^4	y^nx^5	y^nx^6	-	-	y^nx^{n-1}	y^nx^n

Thus each vertical column consists of regularly increasing powers of y multiplied by the same power of x, and each horizontal row consists of regularly increasing powers of x multiplied by the same power of y. By reading the vertical columns successively, and supplying the co-efficients, the terms of a general equation ordered according to the dimensions of x will be found; and by reading the horizontal rows, the terms of an equation arranged by the dimensions of y will be found. By reading it diagonally, the terms of an equation arranged by the dimensions of both variables are obtained. This method is called the *analytical parallelogram*. As an improvement on this, De Gua proposed converting the parallelogram into a triangle, thus:



which is called the *analytical triangle*. This, when read horizontally, the co-efficients being supplied, will give a general algebraic equation arranged by the dimensions of both variables; and when read parallel to either side, will give one arranged by the dimensions of either variable. The first two horizontal rows give the general equation of the first degree, the first three that of the second degree, and the first $(n + 1)$ horizontal rows give the general equation of the nth degree.

(596.) An obvious conclusion from this arrangement is, that the number of terms in a general equation of the nth degree is the sum of an arithmetical series, whose first term and common difference are each unity, and whose number of terms is $n + 1$. Hence the number of terms in the equation is $\frac{(n+1)(n+2)}{2}$.

The entire number of constant quantities in any equation is the same as its number of terms. But this number may always be diminished by one by dividing the whole equation by any one co-efficient, and from this it appears, that if two equations of the same degree have their corresponding co-efficients proportional, they will be in effect identical; for by dividing each by the co-efficient of the same term, the new co-efficients will become equal. The number of determinate and distinct co-efficients in a general algebraic equation of any degree is therefore one less than the number of terms, and therefore the number of determinate co-efficients in a general equation of the nth degree is $\frac{n(n+3)}{2}$.

(597.) In the classification by the degrees of the equations it should be observed, that although the angle of ordination does not affect the class of a curve nor its generic properties, yet that different curves of the same class may be generated by the same equation related to systems of axes of different inclinations. Thus, for example, the equation

$$y^2 + x^2 - r^2 = 0$$

always represents an ellipse, whatever be the angle of ordination; but the eccentricity of the ellipse represented by it will be a function of the angle of ordination. If the co-ordinates be rectangular, the locus is a circle; if otherwise, the axes of co-ordinates coincide with the equal conjugate diameters of the ellipse represented by the equation. Thus it appears, that the peculiar properties of the individual curve represented by any equation are affected by the axes of co-ordinates by which the equation is constructed; but the class to which the curve belongs is still the same.

(598.) It should also be observed, that it does not follow that every equation represents a curve of the order designated by its degree. If an equation of the nth degree can be resolved into two or more rational factors of inferior degrees, it no longer represents a curve of the order expressed by its dimensions. In this case the equation is equivalent to two or more equations of inferior degrees, and instead of representing one curve of the degree expressed by its dimensions, represents a number of curves of inferior degrees, whose equations are expressed by the rational factors into which the given equation can be resolved. Examples of this occurred in the discussion of the general equation of the second degree. It was there shown that in some cases the equation represented two right lines, and in that case the equation is the product of two equations of the first degree. Thus, under the conditions

$$B^2 - 4AC > 0,$$

$$AE^2 + CD^2 + B^2F - BDE - 4ACF = 0,$$

the equation was found to represent two right lines expressed by the equations,

$$y = -\frac{Bx + D + (x - x')\sqrt{B^2 - 4AC}}{2A},$$

$$y = -\frac{(Bx + D) - (x - x')\sqrt{B^2 - 4AC}}{2A}.$$

In which $x' = -\frac{BD - 2AE}{B^2 - 4AC}$. If these equations be arranged thus,

$$(2Ay + Bx + D) - (x - x')\sqrt{B^2 - 4AC} = 0,$$

$$(2Ay + Bx + D) + (x - x')\sqrt{B^2 - 4AC} = 0;$$

and multiplied together, and the result divided by $4A$, and arranged by the dimensions of x and y, it will become

$$\text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x + \frac{\text{BDE} - \text{AE}^2 - \text{CD}^2}{\text{B}^2 - 4\text{AC}} = 0;$$

but by the given condition,

$$\frac{\text{BDE} - \text{AE}^2 - \text{CD}^2}{\text{B}^2 - 4\text{AC}} = \text{F},$$

which reduces the equation to the form of the general equation of the second degree.

(599.) In order therefore that an equation of any proposed degree should represent a curve of the same degree, it should not be capable of being resolved into rational factors of inferior degrees. If the equation can be resolved into two or more such factors, it will really involve two or more equations, each of which will represent a peculiar curve. Such equations then do not represent one curve but several, which have no other connexion than that their equations are multiplied together. A system of different lines thus represented by one equation is called a complex curve. An equation of the second degree may represent a complex line composed of two right lines. One of the third degree may represent a complex line composed of three right lines, or of one right line and a line of the second degree. And in general, an equation of the nth degree may represent a system of n right lines, or a line of the second degree, and $n - 2$ right lines, or any number of lines of inferior degrees, the sum of whose exponents does not exceed n. It should be observed also, that in some cases the factors of the equation may be impossible: such factors represent no loci.

It may also happen that two or more factors may be identical, or that all the factors may be identical. In the former case, the sum of the exponents of the different lines which the equation represents will be less than the exponent of the degree of the equation itself. In the latter case, the equation being a complete power, will only represent one line, whose equation will be the root of that power.

The student will probably form clearer ideas of these general principles from an example. An equation of the fifth degree, not resolvable into rational factors of inferior degrees, will represent one continued line of the fifth order. If it be resolvable into two equations of the first and fourth degree, it will represent a right line and a line of the fourth order. If it be resolvable into factors of the second and third degree, it represents two lines, one of the second, the other of the third degree.

If it be resolvable into three factors, two of the first degree and one of the third, it represents two right lines and a line of the third degree. If two be of the second degree and one of the first, it represents two lines of the second degree and a right line.

If of the three factors two be identical, scil. those of the first degree, it will represent a right line and a line of the third degree. If the two identical factors be of the second degree, it will represent a line of the second degree, and a right line.

From these observations it appears that every general equation of any order embraces under it all curves whatever, whether simple or complex, of inferior orders. Thus, a general equation of the nth order embraces under it every combination of right lines from one to n; every combination of right lines with a line of the pth order from one to p, with a curve of the $(n-p)$th order, and in general every combination of lines, the sum of whose exponents does not exceed n.

PROP. CCXCV.

(600.) *To determine the number of points through which an algebraic curve of the* n*th degree may be drawn.*

Let the co-ordinates of the several points be supposed to be substituted successively for the variables in the general equation of the proposed curve. There will by these means be as many equations as there are points. In order to determine the equation of the curve, it is necessary to determine its several co-efficients, the number of which has been already proved to be $\frac{n(n+3)}{2}$. To determine these will require as many independent equations. If there are therefore $\frac{n(n+3)}{2}$ given, these are sufficient to determine all the constants, and therefore to determine the curve. A curve therefore of the nth order may always be drawn through $\frac{n(n+3)}{2}$ given points.

If there should be a less number of given points, they will be insufficient to determine all the co-efficients, and therefore an infinite number of curves of the proposed order may be made to pass through them.

It should be observed that the equation of the proposed order, determined by the given points through which the line is required to pass, may not represent one continued line of that order. The values of its co-efficients, determined by those points, may be such as to render the equation resolvable into rational factors; in which case, as we have seen, it is not a line of the required order which is drawn through the given points, but several lines of inferior orders. Considerations purely geometrical plainly indicate this; for if n be the number of points, they may all be on the same right line, and in that case the sought equation will be a complete nth power, whose root being extracted, gives the equation of the right line.

The solution of the problem to determine the equation of a line of the nth order passing through $\frac{n(n+3)}{2}$ given points can never be impossible, as the several co-efficients are determined by simple equations.

The practical solution of the question in particular cases may be simplified by assuming axes of co-ordinates passing through four or more of the given points; but in this case, what is gained in simplicity is lost in symmetry, for the resulting values of the sought quantities are never symmetrical when one or more points are assumed to have any peculiar position with regard to the axes of co-ordinates.

PROP. CCXCVI.

(601.) *To find the greatest number of points in which a right line can meet an algebraic curve.*

As the lines assumed as axes of co-ordinates are entirely arbitrary, it is always possible to assume them so that the equation of the curve shall be a complete equation of the nth degree with all its terms. In this case, if $y = 0$, the resulting equation is of the form

$$\text{A}x^n + \text{B}x^{n-1} + \text{C}x^{n-2} \dots\dots \text{M}x + \text{N} = 0.$$

Each real root of this equation determines a point where the axis of x meets the curve. The number of real roots cannot exceed n, and it therefore follows that the number of points where the axis of x meets the curve cannot exceed n. As some of the roots may be impossible, there may be a less number of points of intersection than n, or there may even be no point where the axis of x meets the curve if n be even; as in that case all the roots may be impossible.

Hence we find that *every algebraic curve may be intersected by a right line in as many points as there are units in the exponent of its order, but not in more.*

Since a transformation of the origin to any other point on the axis of y cannot affect those terms which are independent of y, it follows that the greatest number of points in which a parallel to the axis of x can meet the curve is expressed by the exponent of the highest power of x, which is not multiplied by a power of y.

Similar conclusions may be made with respect to the axis of y.

PROP. CCXCVII.

(602.) *To determine the greatest number of points in which two algebraic curves, the exponents of whose orders are* m *and* n, *can intersect.*

Let it be supposed that such lines are assumed as axes of co-ordinates, that neither of them shall be parallel to a line joining any two points of intersection, and that therefore there shall be distinct values of y and x for each particular point of intersection. Suppose y eliminated by means of the equations of the two curves, the resulting equation will give the values of x for the several points of intersection, and from the manner in which the position of the axes of co-ordinates has been assumed, there will be one point of intersection for every real value of x. Since the equations from which y has been eliminated are of the mth and nth degree, the resulting equation in terms of x only will be of the mnth degree, and therefore the greatest number of real roots it can have is equal to the product of the exponents of the orders of the two equations. Hence we find, that two algebraic curves may intersect each other in a number of points equal to the product of the exponents of their orders, but not in more.

In the actual investigation of the points of intersection of two curves, it should be observed, that it cannot be inferred that there are only as many points of intersection as there are real roots in the equation found by eliminating y. For it may happen that a right line passing through two or more points of intersection is parallel to the axis of y, and in this case for such points there is the same value of x. To find all the points of intersection, therefore, each real value of x given by the elimination of y should be substituted in the equations of each of the lines, and the corresponding values of y found; such of these as are real and unequal, give points of intersection.

If y occurs only in the first degree in one of the proposed equations, it is evident that for each real value of x found by the elimination of y, there can be but one value of y in that equation in which it occurs in the first degree, which must be real, and therefore in this case there are as many, and only as many, intersections as there are real values of x.

A similar conclusion obviously applies when x occurs only in the first degree.

(603.) *Cor.* 1. If $m > n$, and it be required that the line of the mth order shall pass through a number of given points expressed by $\frac{m(m+3)}{2}$, whenever a number of these points greater than mn are upon the line of the nth order, the line of the mth order passing through the required points must be a complex line composed of the line of the nth order and other lines. For if not, the two lines would intersect in a greater number of points than mn.

(604.) *Cor.* 2. Two lines of the mth order can only intersect in a number of points expressed by m^2.

(605.) *Cor.* 3. If m^2 be not less than $\frac{m(m+3)}{2}$, or in other words, if m be greater than 2, it follows that two curves of the mth order can intersect in a greater number of points than $\frac{m(m+3)}{2}$, that is, in a greater number of points than are in general sufficient to determine the curve. In this case, some of the co-efficients arising from the given points assume the form $\frac{0}{0}$, and are therefore indeterminate. *Hence a number of points not exceeding* m^2 *may be so placed, that an infinite number of curves of the* m*th order may pass through them.* This is generally true; for if $m < 3$ it is true, because then $m^2 < \frac{m(m+3)}{2}$, and therefore a number of points expressed by m^2 is insufficient to determine the curve; and if $m > 2$, it is true for the reason above stated.

PROP. CCXCVIII.

(606.) *Two right lines intersect each other and a curve of the* n*th order, to investigate the relation between the continued products of the intercepts of each between their common point of intersection and the points where they respectively meet the curve.*

These right lines themselves being assumed as axes of co-ordinates, and y and x being successively supposed $= 0$ in the equation of the curve, the resulting equations will have the forms,

$$\text{A}x^n + \text{B}x^{n-1} + \text{C}x^{n-2} \ldots\ldots \text{M}x + \text{N} = 0,$$

$$\text{A}'y^n + \text{B}'y^{n-1} + \text{C}'y^{n-2} \ldots\ldots \text{M}'y + \text{N} = 0.$$

The continued products of the roots of these equations are respectively $\frac{\text{N}}{\text{A}}$ and $\frac{\text{N}}{\text{A}'}$, which are as $\text{A}' : \text{A}$; that is to say, the ratio of these products is the reciprocal of that of the co-efficients of the highest powers of x and y respectively, which enter the equations whose roots are their factors. Now, as no change of origin can affect the values of A or A′, this ratio remains the same for all systems of secants parallel to those assumed; and hence follows the general theorem, that, *If two right lines parallel to two right lines given in position intersect a curve of the* n*th order, the continued products of their segments intercepted between their point of intersection and the curve will be in a constant ratio.*

The particular application of this theorem, and its consequences to lines of the second degree, has been shown in (138).

As a further example, we shall consider the application of this theorem to lines of the third order. The general equation of the third degree is

$$Ay^3 + Bxy^2 + Cx^2y + Dx^3 + Ey^2 + Fyx + Gx^2 + Hx + Iy + K = 0.$$

If y and x be supposed successively $= 0$, the resulting equations are

$$Dx^3 + Gx^2 + Hx + K = 0,$$
$$Ay^3 + Ey^2 + Iy + K = 0.$$

The products of the roots of these equations are $\frac{K}{D}$ and $\frac{K}{A}$, and therefore they are as A : D, and as a change of origin makes no change in A or D, they remain in the same ratio for all axes of co-ordinates parallel to those assumed. Hence if two right lines in given directions intersect a line of the third degree in three points, the solids contained by the three segments of each line shall be in a constant ratio.

If by a change of origin two of the roots of either equation become equal, the line, instead of intersecting the curve in three points, touches it in one, and intersects it in the other. In this case, instead of the solid contained by the three segments, we consider that whose base is the square of the tangent, and whose altitude is the secant.

If two roots of each equation become equal, we consider the two solids, whose bases are the squares of the tangents, and whose altitudes are the secants.

If the three roots of either equation become equal, they indicate a point of contact formed by the union of three points of the curve, and which is therefore a point of inflection. In this case, the solid considered is the cube of the tangent.

If $K = 0$, one of the roots of each equation $= 0$, which shows that the origin is on the curve.

If $K = 0$ and $H = 0$, two roots are $= 0$, which shows that the axis of x touches the curve at the origin; and if $K = 0$, $H = 0$, and $G = 0$, the three roots are $= 0$, which shows that the origin is a point of inflection.

If $D = 0$, the first equation has but two roots, and the axis of x and its parallels cannot therefore meet the curve in more than two points. Hence the conclusion is, that the solid contained by the three segments of the one line shall vary as the rectangle under the two segments of the other; or that the solid contained by the three segments of the one shall bear a given ratio to a solid whose base is the rectangle under the segments of the other, and whose altitude is given.

If $D = 0$ and $G = 0$, the first equation will have but one root, and therefore the axis of x and its parallels meet the curve in but one point. In this case the solid contained by the three segments of the one line shall vary as the segment of the other, or shall bear a given ratio to a solid whose base is given, and whose altitude is the other line.

Similar conclusions, *mutatis mutandis*, can be applied to the second equation. We shall not pursue this example further. The student can with facility examine all its applications by proceeding as above, and as we have proceeded with the equation of the second degree in (138).

(607.) If the general equation of the nth degree be ar-

ranged by the dimensions of y, it will be of the form
$\text{A}y^n + (\text{B}x + \text{C})y^{n-1} + (\text{D}x^2 + \text{E}x + \text{F})y^{n-2} + \ldots . \text{N} = 0.$
For the same value of x the sum of all the values of y is therefore

$$-\frac{\text{B}x+\text{C}}{\text{A}},$$

and as the number of values of y for any value of x is in general n, it follows, that if a right line be drawn represented by the equation

$$n\text{A}y + \text{B}x + \text{C} = 0,$$

it will possess this property, that if it be made the axis of x, the sum of all the values of y on one side of it will equal the sum of all the values of y on the other side. This property points out an extension of the signification of a *diameter*, which may in general be understood to be *a line intersecting a system of parallel chords in such a manner, that the sums of the intercepts between it and the several points of the curve on each side are equal.*

As the variety of lines which may be assumed as axes of co-ordinates is infinite, so every curve may have an infinite number of diameters.

For the same value of x the sum of the products of every two values of y is

$$\frac{\text{D}x^2+\text{E}x+\text{F}}{\text{A}},$$

and the number of such products is $\frac{n.(n-1)}{1.2}$. Hence a line of the second degree represented by the equation

$$\frac{n.(n-1)}{1.2}\text{A}y^2 + (n-1)(\text{B}x+\text{C})y + \text{D}x^2 + \text{E}x + \text{F} = 0,$$

will have the same diameter as the curve, and also the rectangle under the coincident values of y will be equal to the nth part of the sum of the rectangles under every pair of corresponding values of y in the proposed line. And it follows, that the sum of the positive rectangles under the intercepts between this line of the second degree and the proposed line is equal to the sum of the negative rectangles.

In like manner a curve of the third order whose equation is

$$\frac{n.(n-1).(n-2)}{1.2.3}\text{A}y^3 + \frac{(n-1)\,(n-2)}{1.2}(\text{B}x + \text{C})y^2$$

$$+ \frac{(n-2)}{1}(\text{D}x^2 + \text{E}x + \text{F})y^3 + \text{G}x^3 + \text{H}x^2 + \text{I}x + \text{N} = 0,$$

will have similar properties, that is, will have the same diameter, and the product of every three coincident values of y will be equal to an $\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{1\,.\,2\,.\,3}$th part of the sum of the products of every three coincident values of y in the given line, and therefore the sum of the products of every three intercepts between this and the proposed line measured positively, is equal to the sum of every three measured negatively.

Curves thus related to any algebraic curve are called *curvilinear diameters.* And from what has been shown above, it appears that a curve can have a curvilinear diameter of any order inferior to its own.

A rectilinear diameter, which bisects its ordinates, is called an *absolute diameter.* Thus all diameters of lines of the second degree are absolute diameters.

In order that a curve should admit of an absolute diameter, it is necessary that a transformation of co-ordinates which would make all the terms involving odd powers of one of the variables disappear should be possible, and as this is not always the case, curves of orders exceeding the second may not have any absolute diameter.

A *counter-diameter* is a line which, being assumed as axis of x, will, for equal and opposite values of x, give equal and opposite values of y. Thus an axis is both a diameter and a counter-diameter.

The axis of x being a counter-diameter, and the axis of y properly placed, the equation ought to be fulfilled after changing x into $-x$, and y into $-y$. This always happens when the even rows in a descending order in the analytical triangle, beginning from the highest row, are wanted in the equation. For if the degree of the equation be even, this change leaves the signs of all the terms unaltered, and if it be odd, it changes all the signs.

As a transformation of the direction of the axes of co-ordinates without changing the origin does not introduce any new dimensions of the variables, it follows that if the co-ordinates be placed as above, and that the axis of x be a counter-diameter, all right lines through the origin are also counter-diameters. It appears therefore, that if a curve admits of any counter-diameter, it admits of an infinite number, and that they all intersect in the same point.

From the property of counter-diameters, it appears that all right lines through their point of intersection, and terminated in the curve, are bisected at that point, and it is hence called the *centre* of the curve.

In order that a curve should admit of counter-diameters and a centre, it is necessary that the dimensions of the variables which enter the even rows of the analytic triangle in a descending order should be capable of being removed by the transformation of co-ordinates. As the existence of a centre and counter-diameters has been proved to be independent of the direction of the co-ordinates, this transformation can only be effected by a change of origin. If then a transformation of origin, which will make the necessary terms disappear, gives finite and determinate values for the co-ordinates of the new origin, that point will be the centre; otherwise the curve admits of no centre.

(608.) The distinction by which algebraic lines of any proposed order are subdivided into classes is the number of their infinite branches. We shall not enter here into the detail of this subject, as the specific properties of the different orders of lines beyond the second offer no particular interest to the student. The general methods given in the preceding part are sufficient to determine the figure and properties of any particular curve which may present itself to our inquiries. It may not be uninteresting, however, simply to detail the subdivision of lines of the third order.

Newton has divided the lines of the third order into four principal classes, included under equations of the forms;

$$\text{A}x^3 + \text{B}xy^2 + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0 \quad (1),$$
$$\text{A}x^3 + \text{B}xy + \text{C}x^2 + \text{E}x + \text{F} = 0 \quad (2),$$
$$\text{A}x^3 + \text{B}y^2 + \text{C}x^2 + \text{E}x + \text{F} = 0 \quad (3),$$
$$\text{A}x^3 + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0 \quad (4);$$

under the first are included 65 different species, to which 8 more have since been added.

These 65 are again subdivided into the eleven following classes:

1. 2. 3. 4. *Redundant hyperbolas.* Six hyperbolic branches and three asymptotes, characterised by $\text{A} > 0$.

5. 6. *Defective hyperbolas.* Two hyperbolic branches and one asymptote, characterised by $\text{A} < 0$.

7. 8. *Parabolic hyperbolas.* Two parabolic branches, two hyperbolic branches, and one asymptote, characterised by $\text{A} = 0$.

9. *Hyperbolisms of an hyperbola.* Six hyperbolic branches and three asymptotes, characterised by $\text{A} = 0$, $\text{C} = 0$, $\text{E} > 0$.

10. *Hyperbolisms of an ellipse.* Two hyperbolic branches

and one asymptote, characterised by A $= 0$, C $= 0$, E < 0.

11. *Hyperbolisms of a parabola.* Four hyperbolic branches and two asymptotes, characterised by A $= 0$, C $= 0$, E $= 0$.

The second equation represents a curve called the *Trident.* It consists of two parabolic branches, two hyperbolic branches, and one asymptote.

The third represents curves having two parabolic branches.

The fourth represents the cubical parabola.

This classification, numerous as are its parts, does not contain all the species.

SECTION XXII.

Geometrical problems, illustrative of the application of the preceding parts of Algebraic Geometry.

PROP. CCXCIX.

(609.) *Given the base of a triangle, and the ratio of the rectangle under the sides to the difference of their squares, to find the locus of the vertex.*

The base and a perpendicular through its middle point being taken as axes of co-ordinates, and the given ratio being $m : 1$, and half the base being expressed by a, the condition in the proposition may be expressed thus:

$$\sqrt{y^2 + (x + a)^2} \cdot \sqrt{y^2 + (x - a)^2} = 4max,$$

which reduced to a rational form, becomes

$$(y^2 + x^2 + a^2)^2 - 4a^2x^2 (1 + 4m^2) = 0,$$

which is resolved into the factors

$$y^2 + x^2 - 2a\sqrt{1+4m^2}\,.\,x + a^2 = 0,$$
$$y^2 + x^2 + 2a\sqrt{1+4m^2}\,.\,x + a^2 = 0,$$

which are the equations of two circles, whose centres are on the axis of x, and determined by

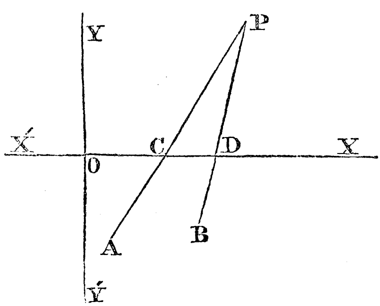
$$x = \pm\, a\sqrt{1+4m^2},$$

and whose radii are equal, each being $2ma$.

If $m = 1$, the two circles thus determined cut the base and produced base in extreme and mean ratio.

PROP. CCC.

(610.) *Two right lines, each of which passes through a given point, intersect in such a manner as to intercept between them a given magnitude of a right line given in position to find the curve traced by their intersection,*



Let AP and BP be the right lines passing always through the given points A and B, and intercepting CD, a part of the right line XX' given in position, always equal to the given magnitude m.

Assuming the line X'X as axis of x, and a perpendicular YY' intersecting it at any point O as axis of y, let the points A and B be $y'x'$, and $y''x''$, and let the equations of AP and BP be

$$(y - y') - a'(x - x') = 0,$$
$$(y - y'') - a''(x - x'') = 0.$$

By supposing $y = 0$ in each of these, we find

$$\text{OC} = x' - \frac{y'}{a'},$$

$$\text{OD} = x'' - \frac{y''}{a''};$$

and therefore by subtraction,

$$\text{CD} = x'' - x' + \frac{y'}{a'} - \frac{y''}{a''} = m.$$

By means of this equation and the first two, the quantities a' and a'' being eliminated, and the result arranged by the dimensions of the variables, we find

$$(x'' - x' - m)y^2 + (y' - y'')yx + (my' + my'' + y''x' - y'x'')y - my'y'' = 0;$$

which being an equation of the second degree, in which $\text{C} = 0$ and $\text{E} = 0$, shows that the locus is an hyperbola, and that the right line XX' is an asymptote.

The position of the centre and axes may be found with facility by the general formulæ already given.

PROP. CCCI.

(611.) *Given the base and the locus of the vertex of a triangle, to investigate the locus of the points where a square inscribed on the given base meets the sides, and also the locus of its centre.*

The base and a perpendicular through one of its extremities being taken as axes of co-ordinates, let the co-ordinates of the vertex be $y'x'$, and those of the point where the angle of the square meets the side passing through the origin be yx. By (66), we have the conditions

$$y = \frac{ay'}{a+y'},$$

$$x = \frac{yx'}{y'},$$

where a is the base of the triangle.

The values of $y'x'$ resulting from these equations are

$$y' = \frac{ay}{a-y},$$

$$x' = \frac{ax}{a-y}.$$

If yx be the co-ordinates of the centre of the inscribed square, these formulas become

$$y' = \frac{2ay}{a-2y},$$

$$x' = \frac{a(x-y)}{a-2y}.$$

The equations of the loci of these points respectively will therefore be found by substituting the values of $y'x'$ in the equation of the locus of the vertex.

From the form of the values of $y'x'$, it follows, that if the locus of the vertex be an algebraic curve of any proposed order, the loci of these points will be a curve of the same order. But it is not necessarily a curve of the same species.

Thus for example, if the locus of the vertex be the circle represented by the equation

$$y'^2 + x'^2 - a^2 = 0,$$

the equation of the locus of the point where the angle of the square meets the side of the triangle, is

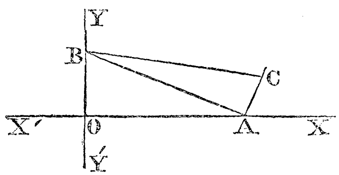
$$x^2 + 2ay - a^2 = 0,$$

which is the equation of a parabola, whose axis is the axis of y.

The student will easily observe various other particular applications of the general formulæ determined above.

PROP. CCCII.

(612.) *A given right-angled triangle* (BAC) *is so moved, that the vertex* (A) *of the right angle, and one extremity* (B) *of the hypotenuse, describe right lines perpendicular to each other, and given in position, to find the nature of the curve described by the other extremity* (C) *of the hypotenuse.*



Let the right lines XX′ and YY′ described by the points A and B be assumed as axes of co-ordinates, and let the co-ordinates of the point C be yx, and AB $= b$, AC $= a$; then by the conditions of the question,

$$\text{AO} = \frac{by}{a},$$

$$\therefore y^2 + (x - \frac{by}{a})^2 = a^2,$$

which being arranged by the dimensions of the variables, becomes

$$(a^2 + b^2)\, y^2 - 2abxy + a^2x^2 - a^4 = 0.$$

Since $\text{B}^2 - 4\text{AC} = -4a^4 < 0$, the curve must be an ellipse, whose centre is at the origin of co-ordinates.

PROP. CCCIII.

(613.) *To determine the curve in which the sine of the angle at which the radius vector is inclined to the tangent, varies inversely as the square of the radius vector.*

Let zt be the angle under the radius vector and tangent, and m^2 being assumed as constant, the condition of the proposition is expressed thus:

$$\sin. zt = \frac{m^2}{z^2};$$

but by the general formula,

$$\sin. zt = \frac{z d\omega}{(z^2 d\omega^2 + dz^2)^{\frac{1}{2}}}.$$

By eliminating sin. zt, the result will be

$$z^6 d\omega^2 - m^4 z^2 d\omega^2 - m^4 dz^2 = 0,$$

$$\therefore d\omega = \frac{m^2 dz}{z(z^4 - m^4)^{\frac{1}{2}}}.$$

To integrate this, let $z^2 = \frac{1}{y}$, and therefore

$$dz = -\frac{dy}{2y^{\frac{3}{2}}}.$$

By which substitutions we have

$$d\omega = -\frac{m^2 dy}{2(1 - m^4 y^2)^{\frac{1}{2}}}.$$

The integral of which is

$$\omega = \tfrac{1}{2} \cos.^{-1} m^2 y;$$

and therefore,

$$\cos. 2\omega = \frac{m^2}{z^2},$$

$$\therefore z^2(\cos.^2 \omega - \sin.^2 \omega) = m^2,$$

which, related to rectangular co-ordinates, is

$$y^2 - x^2 = - m^2.$$

The locus sought is therefore the equilateral hyperbola.

PROP. CCCIV.

(614) *To find the locus of a point from which several right lines being drawn to several given points, the sum of their* 2m*th powers will be given,* m *being supposed a positive integer.*

Let the co-ordinates of the several given points be $y'x'$, $y''x''$, ... $y^{(n)}x^{(n)}$, and those of the point whose locus is sought yx. By the conditions of the question,

$\{(y-y')^2+(x-x')^2\}^m+\{(y-y'')^2+(x-x'')^2\}^m$, &c. $-\text{v}=0$,

v being the given magnitude. This equation, after the terms are severally expanded and arranged according to their dimensions, must be of the form

$$\text{A}y^{2m}+\text{B}y^{2m-1}x+\text{C}y^{2m-2}x^2\ \ldots\ldots -\text{v}=0,$$

which being an equation of the $2m$th degree, shows that the locus is a line of that order.

The case in which $m=1$ was given in (262).

PROP. CCCV.

(615.) *To find the locus of a point, the difference of the* 2m*th powers of whose distances from two given points is given,* m *being supposed a positive integer.*

The co-ordinates of the given points being as before, the condition, expressed algebraically, is

$$\{(y-y')^2+(x-x')^2\}^m-\{(y-y'')^2+(x-x'')^2\}^m-\text{v}=0,$$

where v is the given difference after expanding the terms and expunging those which destroy one another, the resulting equation between yx is one of the $2(m-1)$th degree, therefore the sought line is one of the $2(m-1)$th degree.

If $m=1$, the equation is that of a right line, and becomes

$$2(y''-y')y+2(x''-x')x+y'^2+x'^2-(y''^2+x''^2)-\text{v}=0,$$

which is the equation of a right line perpendicular to that which joins the two points, and which divides it into segments, the difference of the squares of which is equal to the given difference. This will more readily appear by supposing the axis of x to pass through the two given points, and one of them to be the origin, which renders $y' = y'' = x'' = 0$, by which the equation becomes

$$x'^2 - 2xx' - \text{V} = 0,$$

which involves the condition,

$$(x' - x)^2 - x^2 = \text{V},$$

which shows that the right line divides the line joining the two points into segments, the difference of whose squares is equal to V.

PROP. CCCVI.

(616.) *To find the locus of a point from which the sum of the* m*th powers of right lines drawn at given angles to several given right lines shall be given,* m *being supposed a positive integer.*

Let the equations of the given right lines be

$$\text{A}y + \text{B}x + \text{C} = 0,$$
$$\text{A}'y + \text{B}'x + \text{C}' = 0,$$
$$\text{A}''y + \text{B}''x + \text{C}'' = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and the given angles be ϕ, ϕ', ϕ'', &c. the lengths of the several lines will be

$$-\frac{\text{A}y + \text{B}x + \text{C}}{\sin.\ \phi\sqrt{\text{A}^2 + \text{B}^2}},$$
$$-\frac{\text{A}'y + \text{B}'x + \text{C}'}{\sin.\ \phi'\sqrt{\text{A}'^2 + \text{B}'^2}},$$

$$-\frac{A''y+B''x+C''}{\sin.\varphi''\sqrt{A''^2+B''^2}}.$$

. . . .

. . . .

The sum of the mth powers being taken and equated with a constant quantity, will give an equation of the mth degree between y and x, which is that of the locus sought.

The case in which $m = 1$ was investigated in (63).

The case in which $m = 2$ was given in art. (269).

PROP. CCCVII.

(617.) *To find the equation of a curve of a given species passing through any proposed number of given points.*

The following demonstrations are taken with some inconsiderable change from Lagrange, *Cahiers de l'Ecole Normale.*

Let the co-ordinates of the given points be $y'x'$, $y''x''$, $y'''x'''$, &c. These being successively substituted for y and x in the equation of the curve, will give as many equations as there are given points, which will eliminate as many constants as there are points, which will determine the equation of the required curve.

Although the circle is, after the right line, the line most easily described, it is not so by its equation related to rectangular co-ordinates. In this respect the class of curves which may be considered as the simplest, are those of which the values of y are integral and rational functions of x, and which are therefore included in the general equation

$$y = A + Bx + Cx^2 + Dx^3. \quad . \quad . \quad .$$

This class of curves are called parabolic, because the equation of the parabola is a particular case of this equation, scil. the case in which the first three terms only occur. We have already pointed out a striking application of this class of curves in illustrating the theory of equations in Sec. XX., and they are also useful in the investigation of curves in general; for a curve of this kind can always be made to pass through any number of points of the proposed curve, since it is only necessary to take as many indeterminate co-efficients, A, B, C, D, as there are points through which it is required to pass, and to determine these co-efficients by the values of the co-ordinates of the given points. Hence it is clear that whatever be the nature of the proposed curve, the parabolic curve thus determined will differ from it the less the greater number of points they have in common.

Let the co-ordinates of the points through which the curve is required to pass be $y'x'$, $y''x''$, $y'''x'''$,......and we have the following equations:

$$y' = A + Bx' + Cx'^2 + Dx'^3,$$
$$y'' = A + Bx'' + Cx''^2 + Dx''^3,$$
$$y''' = A + Bx''' + Cx'''^2 + Dx'''^3,$$
$$y'''' = A + Bx'''' + Cx''''^2 + Dx''''^3,$$

. . . .

from which equations the values of A, B, C, D, are determined. By subtracting each equation from the preceding, we find

$$y' - y'' = B(x' - x'') + C(x'^2 - x''^2) + D(x'^3 - x''^3),$$
$$y'' - y''' = B(x'' - x''') + C(x''^2 - x'''^2) + D(x''^3 - x'''^3),$$
$$y''' - y'''' = B(x''' - x'''') + C(x'''^2 - x''''^2) + D(x'''^3 - x''''^3).$$

. . . .

By dividing these equations by $x' - x''$, $x'' - x'''$, &c.

$$\frac{y'-y''}{x'-x''} = B + C(x' + x'') + D(x'^2 + x'x'' + x''^2),$$

$$\frac{y''-y'''}{x''-x'''} = B + C(x'' + x''') + D(x''^2 + x''x''' + x'''^2),$$

$$\frac{y'''-y''''}{x'''-x''''} = B + C(x''' + x'''') + D(x'''^2 + x'''x'''' + x''''^2).$$

. . . .

Subtracting each of these equations from the preceding, and dividing the results respectively by $x' - x''''$, $x'' - x'''''$......

$$\frac{a - a'}{x' - x'''} = \text{c} + \text{d}(x' + x'' + x''''),$$

$$\frac{a' - a''}{x'' - x''''} = \text{c} + \text{d}(x'' + x''' + x'''''),$$

. . . .

. . . .

In which

$$a = \frac{y' - y''}{x' - x''},$$

$$a' = \frac{y'' - y'''}{x'' - x'''}.$$

. . . . .

. . . . .

By continuing the process, the quantities A, B, C, may be successively eliminated, and the value of the last found, and thence the values of the others. As an example of this, let the curve represented by the equation

$$y = \text{a} + \text{b}x + \text{c}x^2$$

be required to pass through three points $y'x'$, $y''x''$, $y'''x'''$,

$$\frac{y' - y''}{x' - x''} = \text{b} + \text{c}(x' + x''),$$

$$\frac{y'' - y'''}{x'' - x'''} = \text{b} + \text{c}(x'' + x'''),$$

$$\frac{a - a'}{x' - x'''} = \text{c}.$$

Substituting the last of these in the first,

$$\frac{y' - y''}{x' - x''} - \frac{(a - a')\,(x' + x'')}{x' - x'''} = \text{b}.$$

And substituting the values of B and C in the equation,

$$y' = \text{a} + \text{b}x' + \text{c}x'^2,$$

the value of A is found,

$$\text{A} = y' - ax' - \frac{a - a'}{x' - x'''}x'x''.$$

Hence the equation of the curve sought is

$$y = y' + a(x - x') + \frac{a - a'}{x' - x'''}(x - x')\,(x - x'').$$

A general formula for the equation sought in these cases may be found with somewhat greater simplicity. Since y must successively become y', y'', y'''...when x is x', x'', x''',... the value of y found by eliminating the indeterminate constants; the expression for y must be of the form

$$y = \text{A}'y' + \text{B}'y'' + \text{C}'y''' + \text{D}'y''''. \ . \ . \ .$$

Where A', B', C', are such functions of x, as when $x = x'$ will become $\text{A}' = 1$, $\text{B}' = 0$, $\text{C}' = 0$. . . . . when $x = x''$, $\text{A}' = 0$, $\text{B}' = 1$, $\text{C}' = 0$, $\text{D}' = 0$. . . . . when $x = x'''$, $\text{A}' = 0$, $\text{B}' = 0$, $\text{C}' = 1$, $\text{D}' = 0$, ... and so on. Whence it is obvious that the values of A', B', C', &c. are

$$\text{A}' = \frac{(x - x'')\,(x - x''')\,(x - x'''')\ \ldots\ldots}{(x' - x'')\,(x' - x''')\,(x' - x'''')\ \ldots\ldots}$$

$$\text{B}' = \frac{(x - x')\,(x - x''')\,(x - x'''')\ \ldots\ldots}{(x'' - x')\,(x'' - x''')\,(x'' - x'''')\ \ldots\ldots}$$

$$\text{C}' = \frac{(x - x')\,(x - x'')\,(x - x'''')\ \ldots\ldots}{(x''' - x')\,(x''' - x'')\,(x''' - x'''')\ \ldots\ldots}.$$

. . . .

. . . .

The number of factors in the numerators and denominators of each of these is one less than the number of given points.

This last expression for y, although under a different form from that found by the first method, yet ultimately is the same, as may be proved by arranging the result given by the former process according to the quantities y', y'', y''', . . . and substituting for a, a', &c. their values. But the latter method is preferable, as well on account of the simplicity of the analysis by which we are conducted to it, as on account of the elegance and symmetry of its form, and its commodiousness for calculation.

It follows also from this, that in any series composed of several terms, as many intermediate terms may be supplied as may be required. This is useful in supplying the links in systems of observations or experiments, or in tables calculated by formulas or by constructions. In this consists the *method of interpolation*.

PROP. CCCVIII.

(618.) *To investigate the figure and area of a curve represented by the equation* $a^2y^2 - y^2x^2 - 16x^4 = 0$.

By differentiating the equation, the result is

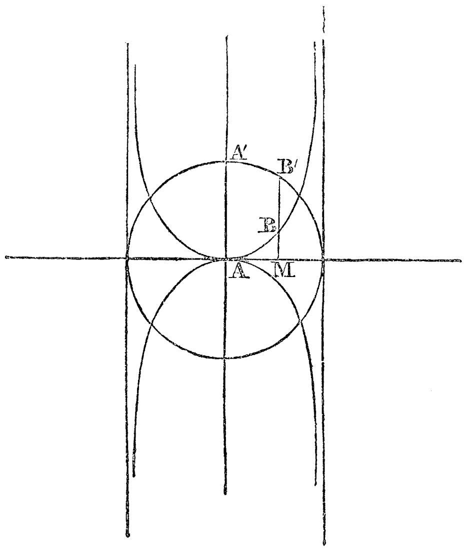
$$\frac{dy}{dx} = \frac{4x(2a^2 - x^2)}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Hence the axis of x touches the curve at the origin.

If $x = a$, $\frac{dy}{dx}$ is infinite, and for this value of x, y is also infinite. Hence the lines, whose equations are $x = \pm a$ are asymptotes, and between them the curve is entirely included. To find the area, let the equation be solved for y, and the result multiplied by dx, gives

$$ydx = \frac{4x^2dx}{\sqrt{a^2 - x^2}}.$$

To integrate this, let a circle be described with the origin as centre, and a radius equal to a. Let the arc A'B', whose sign is x, be ϕ; and let the area sought be A,



$$\text{A} = \int y dx = 4\int . \frac{\sin.^2 \phi d \sin. \phi}{\cos. \phi} = -4\int \sin. \phi d \cos. \phi.$$

This integral taken between the limits $+a$ and $-a$, proves the area included between the curve and its asymptotes to be equal to the area of the circle.

If $y' = \text{MB}'$, $\therefore y' = \sqrt{a^2 - x^2}$; hence by the equation of the curve we have

$$yy' = 4x^2.$$

This curve therefore is the locus of a point B assumed on the ordinate to the diameter of a circle such, that MB shall be a third proportional to B′M, and twice AM.

PROP. CCCIX.

(619.) *To investigate the figure and quadrature of the curve represented by the equation* $x^4 - a^2x^2 + a^2y^2 = 0$.

By solving the equation for y, we find

$$y = \frac{x\sqrt{a^2 - x^2}}{a}.$$

Hence the curve is included within the limits $x = \pm a$, and passes through the origin; by differentiating, we find

$$\frac{dy}{dx} = \frac{a^2 - 2x^2}{a\sqrt{a^2 - x^2}};$$

which shows that parallels to the axis of y at the distances $\pm a$ are tangents; that the origin is a multiple point at which the two tangents are inclined to the axis of x at angles of 45°.

The figure of this curve is therefore similar to that of the lemniscata.

To find the area

$$\text{A} = \int y dx = \int \frac{x dx \sqrt{a^2 - x^2}}{a} = -\frac{(a^2 - x^2)^{\frac{3}{2}}}{3a} + \text{C};$$

which taken between the limits $x = + a$ and $x = - a$, gives the whole area

$$\text{A} = \tfrac{4}{3}a^2.$$

Hence the area of this curve is to that of the lemniscata with the same axis, in the ratio of 4 : 3.

PROP. CCCX.

(620.) *The ordinate to the axis of a cycloid being produced until it becomes equal to the cycloidal arc intercepted between it and the vertex; to find the locus of its extremity.*

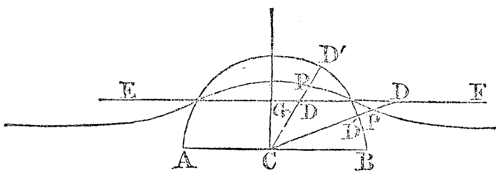
If A = the axis of the cycloid, it is evident, from the rectification of this curve, that the equation of the locus sought is

$$y^2 = 4\text{A}x.$$

It is therefore a parabola whose axis and vertex coincide with those of the cycloid, and whose focus is at the middle point of the base.

PROP. CCCXI.

(621.) *A semicircle being described upon a given right line* AB *as diameter, let an indefinite right line* EF *be drawn parallel to that diameter, and intersecting the circle; to find the locus of a point* P *in a right line drawn from the centre of the circle intersecting the circle, and the parallel at* D′ *and* D, *so that* P *will divide the intercept* DD′ *in a given ratio* m : n.



Let CP $= z$, and the angle PCB $= \omega$, CG $= b$, CB $= r$; hence

$$z = \frac{b}{\sin.\,\omega} + \left\{ r - \frac{b}{\sin.\,\omega} \right\} \frac{m}{m+n}.$$

This equation is obviously deduced from the conditions of the question, and, after reduction, becomes

$$z = \frac{b}{\sin.\,\omega} \cdot \frac{n}{m+n} + r \cdot \frac{m}{m+n}.$$

This is the equation of a *conchoid*, whose *modulus* is

$$\frac{rm}{m+n};$$

and the equation of whose *rule* is

$$y = \frac{bn}{m+n}.$$

The centre of the given circle being the pole of the *conchoid.*

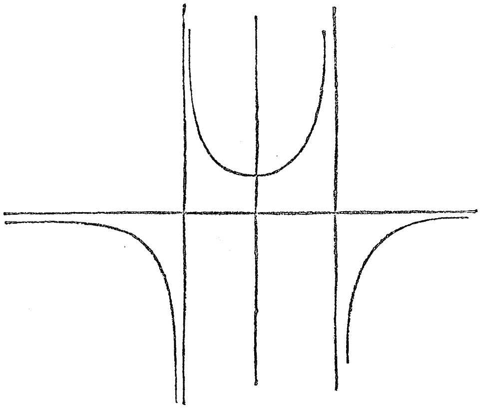
PROP. CCCXII.

(622.) *To investigate the figure and quadrature of the curve represented by the equation* $a^2y - x^2y - a^3 = 0$.

By solving the equation for y,

$$y = \frac{a^3}{a^2 - x^2}.$$

Hence it appears, that if $x = \pm a$, y is infinite, and therefore the parallels to the axis of y represented by the equation $x = \pm a$ are asymptotes. Also y is positive for all values of x between $+a$ and $-a$. The minimum positive value of y is $= a$, and corresponds to $x = 0$, therefore a parallel to the axis of x intersecting the axis of y at a distance from the origin equal to a touches the curve at that point, and the part of the curve included between the parallel asymptotes is extended indefinitely above this tangent. For all values of x beyond the parallel asymptotes y is negative, and diminishes without limit as x increases without limit. Hence the axis of x is an asymptote. This curve is represented thus:



It is included in the ninth class of Newton's enumeration of lines of the third order, and comes under the generic name of *redundant hyperbola*, as having a greater number of hyperbolic branches than the hyperbola of the second degree.

The particular species of *redundant hyperbola*, investigated in this proposition, is called an *hyperbolism of the hyperbola.*

To effect the quadrature of this curve, let the area be A,

$$A = \int y dx = a^3 \int . \frac{dx}{a^2 - x^2},$$

which, by integration, gives

$$A = \tfrac{1}{2} a^2 . l . \frac{a+x}{a-x}.$$

The area being supposed to commence from the axis of y, no constant is introduced.

PROP. CCCXIII.

(623.) *To find the equation of the curve, whose subtangent varies as the rectangle under the co-ordinates.*

This condition, expressed analytically, is

$$\frac{xdy}{dx} . a = yx,$$

which gives

$$ady - ydx = 0;$$

which integrated, is

$$y = b^x,$$

b being the base, whose modulus is a. The curve sought is therefore *the logarithmic.*

PROP. CCCXIV.

(624.) *To find the equation of a curve whose area always equals twice the rectangle under its co-ordinates.*

Let the co-ordinates of any point be yx. The condition stated in the proposition is

$$\int y dx = 2yx;$$

which, by differentiation, gives

$$2xdy + ydx = 0.$$

This equation, multiplied by y, and integrated, gives

$$2xy^2 = a,$$

a being an arbitrary constant, which is therefore the equation of a curve possessing the proposed property.

PROP. CCCXV.

(625.) *To find the equation of a spiral in which the area is proportional to the logarithm of the radius vector.*

This condition, expressed analytically, is

$$\int z^2 d\omega = alz;$$

by differentiation,

$$z^2 d\omega = a\frac{dz}{z},$$

which, integrated, is

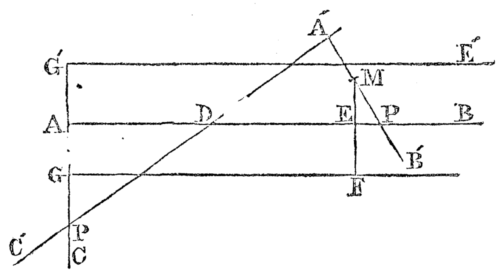
$$2z^2\omega = -a,$$

which is the equation sought.

PROP. CCCXVI.

(626.) *A right angle* CAB *is given in position, and another right angle* C′A′B′ *is so moved, that the points of intersection,* P, P′, *of the sides of the angles respectively, shall be always at a given distance* AP = A′P′ *from their vertices; to find the curve described by the middle point* M *of the intercept* A′P′ *between the side of the angle given in position and the vertex of the other angle.*

Through the middle point G of PA let GF be drawn per-



pendicular to it, and let this and PA be assumed as axes of co-ordinates. Hence if the angle $A'DP' = \omega$, and AG or $MP' = a$, we find

$$AD = 2a \cot. \omega,$$

$$P'D = \frac{2a}{\sin. \omega},$$

$$P'E = a \sin. \omega,$$

$$ME = a \cos. \omega.$$

Hence we find

$$x = 2a \cot. \omega + \frac{2a}{\sin. \omega} - a \sin. \omega,$$

$$y^2 = a(1 + \cos. \omega).$$

By eliminating ω by these equations, the result, after reduction, is

$$x^2(2a - y) - y^3 = 0,$$

which is the equation of the *cissoid of Diocles.*

If $AG' = AG$, and $G'E'$ be drawn parallel to AB, GG' is the diameter of the generating circle, and the line $G'E'$ is the asymptote.

PROP. CCCXVII.

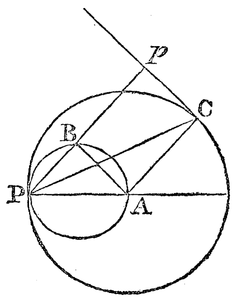
(627.) *To find the locus of the intersection of a tangent to a given circle, and a line perpendicular to it passing through a given point in the circle.*

Let the diameter passing through the given point be assumed as the axis from which the values of the angle ω are measured, the perpendicular being represented by z, the conditions of the question give the equation

$$z = 2r \cos.^2 \tfrac{1}{2}\omega = r(1 - \cos.\ \omega),$$

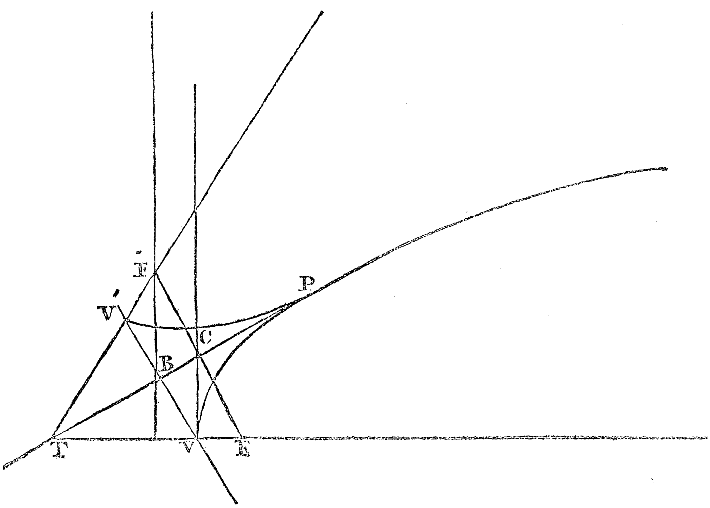
which is the equation of the cardioide.

It is otherwise evident from geometrical construction, that the cardioide is the locus: for let P be the given point on the circle, p the intersection of the perpendicular and tangent. Draw Pp and CA, and on the radius PA as diameter let a circle be described intersecting Pp in B, and draw BA. BpCA is evidently a rectangle, and therefore Bp = AC. Hence Bp is constant, which is a property of the *cardioide*.



PROP. CCCXVIII.

(628.) *Two equal parabolas being placed in the same plane, and so as to touch at their vertices, let one of them be supposed to roll upon the other; to find the loci of its focus and vertex.*



By the conditions of the question, it appears that if from the vertex V of the fixed parabola a perpendicular VB be drawn to a tangent through any point P, and produced until V′B = VB, the point V′ is the vertex of the moveable parabola; and if TV′ be produced until V′F′ = VF, F′ is its focus; and FF′ being perpendicular to PT, and bisected by it; since by art. (253), the locus of C is the vertical tangent to the fixed parabola, the locus of F′ is its directrix. The equation of the tangent PT being

$$2yy' - p(x + x') = 0.$$

The value of VB is x' cos. BVT. Let the co-ordinates of the point V′ be yx,

$$\frac{y}{x} = \frac{2y'}{p},$$

$$y'^2 = px',$$

$$y^2 + x^2 = 4x'^2 \cdot \frac{p}{p+4x'}.$$

Eliminating $y'x'$ by these equations, the result is

$$x^3 + y^2x - \tfrac{1}{2}py^2 = 0.$$

The equation of the locus of the vertex, which is therefore a *cissoid*, the diameter of whose generating circle is $\frac{1}{2}p$.

PROP. CCCXIX.

(629.) *The ordinate* P′M *to the diameter of a circle being produced until the rectangle under* PM *and the absciss* AM *is equal to the rectangle under the ordinate* P′M *and the diameter, to find the equation, figure, and properties of the curve, which is the locus of the extremity of the produced ordinate.*

The origin of co-ordinates being at A, let AM $= x$, P′M $= y'$, PM $= y$, and AB $= a$. By the conditions of the question,

$$ay' = yx;$$

but by the equation of the circle,

$$y' = \sqrt{ax - x^2}.$$

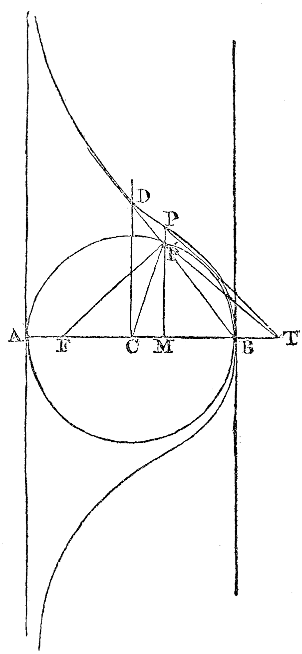
Hence the equation of the locus, after clearing it of the radical, is

$$y^2x + a^2x - a^3 = 0;$$

which solved for y, is

$$y = \frac{a\sqrt{a-x}}{\sqrt{x}}.$$

Hence, when $x = a$, $y = 0$, therefore the curve intersects the axis of x at B. All positive values of x greater than a, and all negative values of x whatever, give impossible values of y; but all values of x between a and 0 give real values of y: hence the curve is entirely included between the parallels through A and B. Since, for every value of x, there are two equal and opposite values of y, the parts of the curve on each side of the axis of x are symmetrical.



By differentiating the equation of the curve, we find

$$\frac{dy}{dx} = -\frac{a^3}{2yx^2} = -\frac{a^2}{2x\sqrt{ax - x^2}}.$$

Hence the subtangent S is

$$S = -\frac{y'^2}{\frac{1}{2}a}.$$

Hence follows a geometrical construction for drawing a tangent. Let MF $= \frac{1}{2}a$, and draw FP′ and P′T perpendicular to it. Then TP is the tangent.

Also $\frac{dy}{dx} = \infty$ when $x = 0$, and since at the same time $y = \infty$, the axis of y is an asymptote.

Let the equation be differentiated a second time, and the result is

$$\frac{d^2y}{dx^2} = \frac{a^2(3a-4x)}{4x(ax-x^2)^{\frac{3}{2}}}.$$

Hence the points whose co-ordinates are $x = \frac{3}{4}a$, $y = \frac{1}{\sqrt{3}}a$ are points of inflection; therefore if the radius CB be bisected, the ordinate passing through the point of bisection meets the curve at the points of inflection.

Let A be the area of the segment PMB, supposed to begin from B,

$$\text{A} = -\int y dx = -a\int \frac{\sqrt{ax-x^2}}{x} dx.$$

Let the angle P′CB be ϕ;

$$\tfrac{1}{2}a \sin. \phi = \sqrt{ax - x^2} \;.\; \tfrac{1}{2}a(1 + \cos. \phi) = x : \text{hence}$$

$$\text{A} = \tfrac{1}{2}a^2\int \frac{\sin.^2 \phi}{1 + \cos. \phi} \,.\, d\phi;$$

$$\because \text{A} = \tfrac{1}{2}a^2\int(1 - \cos. \phi)d\phi,$$

which, by integration, gives

$$\text{A} = \tfrac{1}{2}a^2(\phi - \sin. \phi).$$

No constant is added, since A and ϕ are simultaneously evanescent.

The quantity $\frac{1}{2}a^2\phi$ is equal to four times the sector P′CB, and $\frac{1}{2}a^2 \sin. \phi$ is four times the triangle P′CB, and therefore the area is four times the difference between these. Hence the area PMB is equal to four times the segment P′B.

The entire area included between the curve and its asymptote is therefore equal to twice the area of the circle.

If a perpendicular CD to AB meet the curve at D, the line joining D and B will be a tangent.

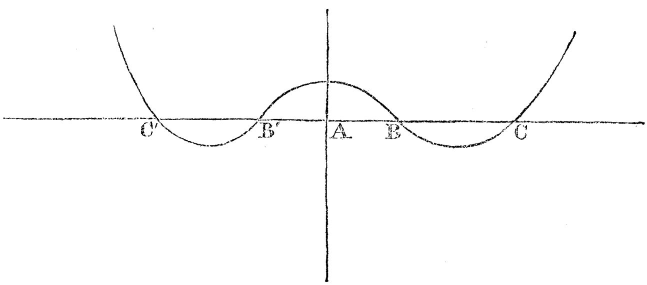
PROP. CCCXX.

(630.) *To investigate the figure of the curve, whose equation is* $x^4 - a^2x^2 - b^2x^2 + a^2b^2 - c^3y = 0$.

The proposed equation may be expressed thus:

$$y = \frac{(x^2-a^2)(x^2-b^2)}{c^3}.$$

Hence the curve meets the axis of x at the points $x = +a$, $x = -a$, $x = +b$, $x = -b$; and if $a > b$, for all values of $x > a$, y is positive, and continually increases for all greater values. For the values of x between a and b, y is negative, and positive for those between b and 0. Hence it is easily seen that the figure of the curve is



where AB $= +b$, AB$' = -b$, AC $= +a$, AC$' = -a$.

The equation of the tangent by differentiating is found to be

$$y - y' = \frac{2x'\{2x'^2 - (a^4 + b^2)\}}{c^3}(x - x');$$

therefore the points at which the tangents are parallel to the axis of x are $x = 0$, $x = \pm \frac{\sqrt{a^2+b^2}}{2}$.

If a and b both $= 0$, the four points A, A$'$, B, B$'$, unite in one, and become a point of undulation: the equation of the curve becomes in this case $y = \frac{x^4}{c^3}$, which is one of the numerous family of parabolic curves represented by the general equation $y = ax^n$.

PROP. CCCXXI.

(631.) *To find the locus of the intersection of the tangent to an ellipse, with a perpendicular to it passing through the centre.*

The equation of the ellipse being

$$a^2y'^2 + b^2x'^2 = a^2b^2,$$

and that of the tangent

$$a^2y'y + b^2x'x = a^2b^2,$$

the equation of the perpendicular to the tangent from the centre is

$$b^2x'y - a^2y'x = 0.$$

If y' and x' be eliminated by these equations, the result will be

$$(y^2 + x^2)^2 - a^2x^2 - b^2y^2 = 0,$$

which is therefore the equation of the locus sought.

To investigate the figure of this curve, let $y = 0$, and the corresponding values of x are

$$x = 0,$$
$$x = + a,$$
$$x = - a.$$

In like manner if $x = 0$, the corresponding values of y are

$$y = 0,$$
$$y = + b,$$
$$y = - b.$$

Hence it appears that the four vertices and the centre are points of the proposed locus.

If the equation be solved for y^2, we find

$$y^2 = \tfrac{1}{2}b^2 - x^2 \pm \sqrt{\tfrac{1}{4}b^4 + c^2x^2},$$

where $c^2 = a^2 - b^2$.

Hence it appears that for each value of x there are four values of y, but of these four two are impossible. For since

$$(\tfrac{1}{2}b^2 - x^2)^2 < (\tfrac{1}{4}b^4 + c^2x^2),$$

therefore

$$\tfrac{1}{2}b^2 - x^2 < \sqrt{\tfrac{1}{4}b^4 + c^2x^2}.$$

Hence

$$y = \pm\sqrt{\tfrac{1}{2}b^2 - x^2 - \sqrt{\tfrac{1}{4}b^4 + c^2x^2}}$$

are both impossible values of y, except when $x = 0$, which gives $y = 0$. Therefore the two real values are

$$y = \pm\sqrt{\tfrac{1}{2}b^2 - x^2 + \sqrt{\tfrac{1}{4}b^4 + c^2x^2}}.$$

The parallels to the conjugate axis of the ellipse therefore meet the curve in but two points, and are all bisected by the transverse axis, which is therefore an axis of the locus.

By differentiating the real values of y, we find

$$\frac{dy}{dx} = \frac{x(c^2 - 2\sqrt{\tfrac{1}{4}b^4 + c^2x^2})}{2y\sqrt{\tfrac{1}{4}b^4 + c^2x^2}} = -\frac{x}{y}\cdot\frac{a^2 - 2(y^2 + x^2)}{b^2 - 2(y^2 + x^2)}.$$

Hence if $x = 0$, $y = \pm b$, $\dfrac{dy}{dx} = 0$.

Also if $y = 0$ and $x = \pm a$, $\dfrac{dy}{dx} = \frac{1}{0}$. Hence the tangents through the four vertices of the ellipse are also tangents to the locus at these points.

Since the numerator of the value of $\dfrac{dy}{dx}$ consists of two factors, $x = 0$ is not the only condition on which it may vanish. If

$$c^2 - 2\sqrt{\tfrac{1}{4}b^4 + c^2x^2} = 0,$$

$$\text{or } x = \frac{a\sqrt{c^2 - b^2}}{2c},$$

we shal. also have $\frac{dy}{dx} = 0$. This value of x is impossible if $c < b$, but real if otherwise. Hence if $c > b$, there are six points at which the tangent to the locus is parallel to the transverse axis; which points are determined by

$$x = 0,$$

$$x = + \frac{a\sqrt{c^2 - b^2}}{2c},$$

$$x = - \frac{a\sqrt{c^2 - b^2}}{2c},$$

and the corresponding values of y. If $c = b$, three of these points unite in one, and form a point of *undulation.*

If $c < b$, there are only two points where the tangent is parallel to the transverse axis, which are determined by $x = 0, y = \pm b$.

To find whether the tangent through the vertex of the conjugate axis intersects the locus, let b be substituted for y in the value of x, and we find

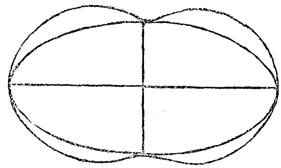
$$x = \pm \sqrt{c^2 - b^2}.$$

Hence if $c > b$, the tangent intersects the curve at two points determined by these values of x.

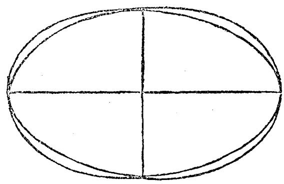
If c be not less than b, the tangent does not intersect the curve.

Of the two factors in the denominator of $\frac{dy}{dx}$, one cannot $= 0$, therefore the value can be infinite only when $y = 0$ and $x = \pm a$; hence the only points at which the tangent through the vertex is perpendicular to the transverse axis are the vertices.

In the case where $c > b$, the figure of the locus is therefore represented thus.



If c be not greater than b, the figure of the locus is represented thus.



It appears also that the centre is a *conjugate point*.

When $c > b$, the curve has four points of inflection.

To determine the polar equation, let z sin. ω, and z cos. ω be substituted for y and x, and the result is

$$z^2 = a^2 \cos.^2 \omega + b^2 \sin.^2 \omega.$$

This equation bears an obvious analogy to the polar equation of the ellipse itself, related to the centre as pole, which may be expressed

$$\frac{1}{z^2} = \frac{\cos.^2 \omega}{a^2} + \frac{\sin.^2 \omega}{b^2}.$$

To find the area (A$'$) of the locus: by the general formula

$$\text{A}' = \tfrac{1}{2}\int z^2 d\omega + \text{C}'.$$

Hence in this case

$$\text{A}' = \tfrac{1}{2}a^2\int \cos.^2 \omega\, d\omega + \tfrac{1}{2}b^2\int \sin.^2 \omega\, d\omega + \text{C}';$$

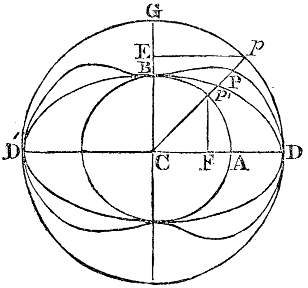
but since

$$d \sin. \omega = \cos. \omega\, d\omega,$$
$$d \cos. \omega = - \sin. \omega\, d\omega,$$

we have

$$\text{A}' = \tfrac{1}{2}a^2\int \cos. \omega d \sin. \omega - \tfrac{1}{2}b^2\int \sin. \omega d \cos. \omega + \text{C}'.$$

Let circles be described on the axes of the ellipse as diameters: if CP $= z$ and PCD $= \omega$, let p, p', be the points



where the radius vector meets the two circles, and let p'F be parallel to CB, and pE to CD. Then $-a^2 \cos. \omega d \sin. \omega$ is the differential of the area GEp, and $- b^2 \sin. \omega d \cos. \omega$ is the differential of the area AFp'. Hence,

$$\text{CPD} = \tfrac{1}{2}(-\text{GE}p + \text{AF}p') + \text{c}'.$$

To determine c$'$, we should observe that when CPD $= 0$, that is, when CP coincides with CD, the point p will coincide with D, and p' with A. Hence

$$\text{c}' - \tfrac{1}{2}\text{GCD} = 0,$$

$$\therefore \text{c}' = \tfrac{1}{2}\text{GCD}.$$

Hence we find

$$\text{CPD} = \tfrac{1}{2}(\text{CE}p\text{D} + \text{AF}p').$$

Hence we find the entire area (A) of the curve

$$\text{A} = \tfrac{1}{2}\left\{ a^2\pi + b^2\pi \right\}.$$

The area of the entire curve is therefore an arithmetical mean between the areas of the two circles, and is equal to half the area of the circle described with the line joining the extremities of the axes as radius.

It appears from this that the curve BPD bisects the space ABGD included between the circles.

The transverse axis of the ellipse being supposed fixed, if the conjugate axis be continually diminished, the ellipse will *ultimately* coincide with the transverse axis. The corresponding limit of the locus will be found by supposing $b = 0$ in its equation, which gives

$$(y^2 + x^2)^2 - a^2x^2 = 0.$$

This equation is resolvable into two factors,

$$y^2 + x^2 - ax = 0,$$

$$y^2 + x^2 + ax = 0,$$

which are the equations of two circles described on CD and CD′ as diameters, which are therefore the limit of the curve in this case.

If the ellipse becomes a circle, scil. if $b = a$, the equation of the locus is

$$(y^2 + x^2)^2 - a^2x^2 - a^2y^2 = 0,$$

which is resolved into the factors

$$y^2 + x^2 - a^2 = 0,$$
$$y^2 + x^2 = 0.$$

The first gives the circle on the transverse axis, and the other the centre.

PROP. CCCXXII.

(632.) *To determine the locus of the intersection of the tangent to an hyperbola, and a perpendicular to it through the centre.*

The equation of the locus found in the last proposition becomes in this case

$$(y^2 + x^2)^2 - a^2x^2 + b^2y^2 = 0.$$

If in this equation $x = 0$, it is necessary that $y = 0$ also, therefore the conjugate axis can meet the locus only at the centre. But $y = 0$ gives, as before,

$$x = 0,$$
$$x = + a,$$
$$x = - a;$$

hence the locus meets the tranverse axis at the vertices.

It appears, as before, that two of the four values of y are impossible, and that therefore the perpendiculars to the transverse axis can each meet the curve in but one point, and that the transverse axis is an axis of the curve.

It appears, as before, that the tangents to the hyperbola, passing through the vertices of the curve, are also tangents to the locus at those points; for the differential co-efficient becomes, in this case,

$$\frac{dy}{dx} = \frac{x}{y} \cdot \frac{a^2 - 2(y^2 + x^2)}{b^2 + 2(y^2 + x^2)},$$

which becomes infinite when $y = 0$ and $x^2 = a^2$.

For all values of $x^2 > a^2$, y is impossible; therefore the locus is included between the vertical tangents.

To find the value of $\frac{dy}{dx}$ when $y = 0$ and $x = 0$, we are to consider that the value of $\frac{dy}{dx}$ consists of two factors, $\frac{x}{y}$ and $\frac{a^2 - 2(y^2 + x^2)}{b^2 + 2(y^2 + x^2)}$. When $x = 0$ and $y = 0$, the former assumes the form $\frac{0}{0}$, and the latter becomes $\frac{a^2}{b^2}$. To find the true value of the first factor, or what is more readily done, of its square, let the numerator and denominator be both differentiated, and we find

$$\frac{d(x^2)}{d(y^2)} = \frac{2(\frac{1}{4}b^4 + c^2x^2)^{\frac{1}{2}}}{c^2 - 2(\frac{1}{4}b^4 + c^2x^2)^{\frac{1}{2}}},$$

which, when $x = 0$, becomes $\frac{b^2}{a^2}$; therefore, when $x = 0$ and $y = 0$,

$$\frac{x}{y} = \pm \frac{b}{a};$$

and in this case, therefore,

$$\frac{dy}{dx} = \pm \frac{a}{b}.$$

Hence the centre is a multiple point. The tangents to the curve at this point are perpendicular to the asymptotes of the hyperbola. The equations of the tangents to this point are

$$by - ax = 0,$$
$$by + ax = 0.$$

To determine whether these tangents meet the curve again, let b^2y^2 be substituted for a^2x^2 in the equation of the curve, and it becomes

$$y^2 + x^2 = 0,$$

which gives $y = 0$ and $x = 0$. The tangents therefore do not meet the curve again.

It appears from this investigation, that the figure of the curve is like that of the lemniscata, which are a species of it.

Its polar equation is

$$z^2 = a^2 \cos.^2 \omega - b^2 \sin.^2 \omega,$$

which, when $a^2 = b^2$, becomes

$$z^2 = a^2 \cos. 2\omega,$$

the equation of the lemniscata.

PROP. CCCXXIII.

(633.) *Given the base and rectangle under the sides of a triangle, to determine the locus of the vertex.*

If the base and a perpendicular through its middle point be assumed as axes of co-ordinates, the condition expressed in the proposition is

$$\sqrt{y^2 + (x + a)^2} \cdot \sqrt{y^2 + (x - a)^2} = a^2 + b^2,$$

where a is half the base, and $a^2 + b^2 =$ the rectangle. This equation, when reduced to a rational form, becomes

$$(y^2 + x^2)^2 + 2a^2(y^2 - x^2) = b^2(b^2 + 2a^2).$$

This equation, solved for y^2 and x^2, gives

$$y^2 = -(a^2 + x^2) \pm \sqrt{(a^2 + b^2)^2 + 4a^2x^2},$$
$$x^2 = a^2 - y^2 \pm \sqrt{(a^2 + b^2)^2 - 4a^2y^2}.$$

Hence, for each value of x, two of the four values of y are impossible, and the other two are real and equal with opposite signs. The axis of x is therefore an axis of the curve. When $x = 0$, $y = \pm b$: the axis of y meets the curve, therefore, at two points determined by $y = +b$ and $y = -b$.

When $y = 0$, $x = \pm\sqrt{2a^2 + b^2}$, or $x = \pm b\sqrt{-1}$. The latter values are impossible, and the former determine the two points where the curve meets the axis of x.

All values of $x^2 > 2a^2 + b^2$ render y impossible; therefore the curve is included between the parallels to the axis of y through the points determined by $x = \pm\sqrt{2a^2 + b^2}$.

If the equation of the curve be differentiated, the result is

$$\frac{dy}{dx} = \frac{x}{y} \cdot \frac{a^2 - (y^2 + x^2)}{a^2 + (y^2 + x^2)}.$$

If $x = 0$, and $\therefore y = \pm b$, $\frac{dy}{dx} = 0$; therefore the parallels to the axis of x, through the points determined by these values of y and x, are tangents.

If $y = 0$, and $\therefore x = \pm\sqrt{2a^2 + b^2}$, the values of $\frac{dy}{dx}$ are infinite; therefore the parallels to the axis of y, through the points determined by these values of y and x, are tangents.

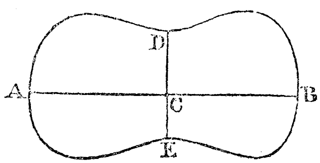
To determine whether the tangents through the points determined by $x = 0$, $y = \pm b$, meet the curve again, let b be substituted for y in the value of x^2, and we find

$$x = 0,$$
$$x = \sqrt{2(a^2 - b^2)}.$$

The latter values are real, $= 0$, or imaginary, according as $a^2 > b^2$, $a^2 = b^2$, or $a^2 < b^2$.

From these circumstances, it appears that when $a^2 > b^2$ is represented thus:

If C be the origin, and A and B the points where it meets the axis of x, and D, E, those where it meets the axis of y, there are points of inflection on each side of the points D and E.



In this case $\frac{dy}{dx} = 0$, on the condition,

$$a^2 = y^2 + x^2.$$

The points (besides D and E), therefore, where the tangent is parallel to AB, are determined by the intersection of a circle, whose centre is C, and radius a with the curve. This circle will not meet the curve if $a^2 < b^2$, and will touch it at D if $a^2 = b^2$.

If a^2 be not greater than b^2, the figure of the curve is similar to that of the ellipse.

PROP. CCCXXIV.

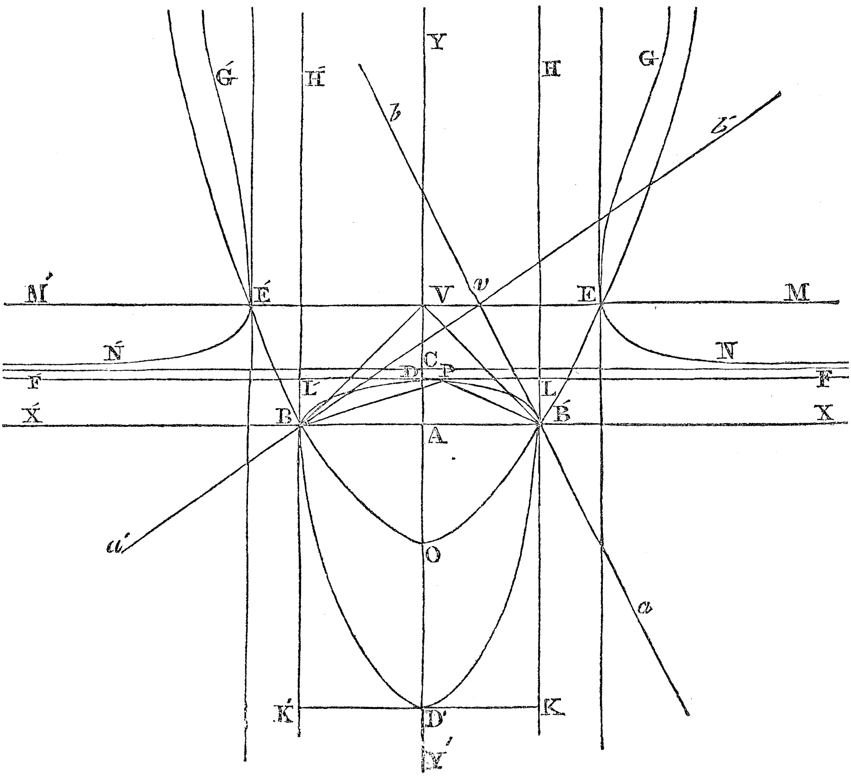
(634.) *Given the base and area of a triangle, to find the locus of the centre of the inscribed circle.*

Let the base BB′ and a perpendicular YY′ through its middle point be the axes of co-ordinates.

Since the area of the triangle is given, the locus of the vertex is a right line parallel to the base, at a distance AV from the base such, that the rectangle under AV and AB shall be equal to twice the given area. Let v be the vertex of the triangle in any position. If BP and B′P bisect the angles vBA and vB′A, P is the centre of the circle which touches the three sides. Let the co-ordinates of v be $y'x'$, and those of P, yx, therefore

$$\tan.\, v\text{BA} = \frac{y'}{a - x'},$$

$$\tan.\, v\text{B}'\text{A} = \frac{y'}{a + x'},$$



$$\tan.\, \text{PBA} = \frac{y}{a - x},$$

$$\tan.\, \text{PB}'\text{A} = \frac{y}{a + x}.$$

But the angles vBA and vB$'$A are respectively equal to twice the angles PBA and PB$'$A; therefore, by trigonometry,

$$\tan. vBA = \frac{2 \tan. PBA}{1 - \tan.^2 PBA},$$

$$\tan. vB'A = \frac{2 \tan. PB'A}{1 - \tan^2 PB'A}.$$

By substituting in these the values of the tangents already found, we find

$$\frac{y'}{a - x'} = \frac{2y(a-x)}{(a-x)^2 - y^2},$$

$$\frac{y'}{a + x'} = \frac{2y(a+x)}{(a+x)^2 - y^2}.$$

Eliminating x' by these equations, the result, arranged by the dimensions of the variables, is

$$2x^2y - y'y^2 - y'x^2 - 2a^2y + a^2y' = 0,$$

which being an equation of the third degree, shows that the locus is a line of that order.

To examine the figure of the locus, let its equation be solved for each of the variables, and the results are

$$y = \frac{-(a^2 - x^2) \pm \sqrt{(a^2 - x^2)(b^2 - x^2)}}{y'},$$

$$x = \pm \frac{\sqrt{y'y^2 + 2a^2y - a^2y'}}{\sqrt{2y - y'}}.$$

In which $b^2 = a^2 + y'^2$, which is the square of the line BV; $\therefore$ BV $= b$.

To determine the points where the curve meets the axis of y, let $x = 0$, which gives

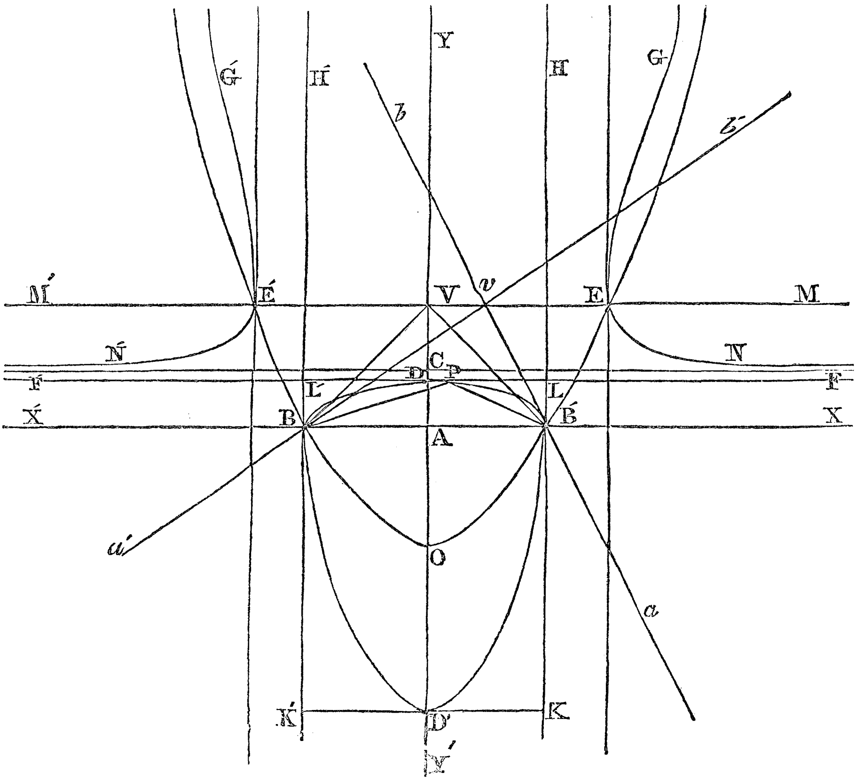
$$y'y^2 + 2a^2y - a^2y' = 0;$$

$$\therefore y = -\frac{a(a \mp b)}{y'}.$$

Let AD $= \frac{a(b-a)}{y'}$, AD $= -\frac{a(a+b)}{y'}$, and the points D and D$'$ are those at which the locus meets the axis of y. These points may be obviously found geometrically by bisecting the angle VBA and its external supplement; the bisectors will meet YY′ at D, D′.

If AD = r, and AD′ = $-r'$, the value of x may be expressed thus,

$$x = \sqrt{\frac{y(y-r)(y+r)}{2y-y'}}.$$



By the value of y, it appears that for all values of x^2 between b^2 and a^2, that is, $< b^2$ and $> a^2$, the values of y are imaginary.

Hence if VE = VE′ = VB, and parallels to YY′ be drawn through the four points E, B, E′, and B′, the curve is excluded from between the parallels through E and B, and also from between those through E′ and B′. But since for all other values of x the values of y are real, a part of the curve is included between the parallels BH and B′H′; and the remaining parts of it extend infinitely in opposite directions beyond the parallels through E and E′.

Since $y = y'$ gives $x = \pm b$, the curve passes through the points E and E′.

Since $y = 0$ gives $x = \pm a$, the curve passes through the points B and B′.

Since for all negative values of y greater than AD′, the value of x^2 is negative, and therefore that of x is imaginary, the curve cannot pass below a parallel to the axis of x through D′.

From the value of x, it appears that it is impossible for all values of y between r and $\frac{1}{2}y'$; and hence it follows, that the curve is excluded from between the parallels to the axis of x through the points C and D, CA being half of VA, and DA being equal to r.

From these circumstances, it appears that the part of the curve included between the parallels BH and B′H′ is inclosed in the rectangle KL′, whose sides are parallel to the axes of co-ordinates, and that the curve meets this rectangle at the points B, B′, D, and D′.

Since for each value of y there are two equal and opposite values of x, the axis of y is an axis of the curve.

If the equation be differentiated, we find

$$\frac{dy}{dx} = \frac{x(2y-y')^2}{yy'(y-y')}.$$

This vanishes on either of the conditions,

$$x = 0,$$

$$y = \tfrac{1}{2}y'.$$

The first condition indicates that the parallels to XX′ through D, D′, are tangents to the curve. The second would also show the parallel through C to be a tangent; but the corresponding value of x being infinite, proves it to be an asymptote.

The value of $\frac{dy}{dx}$ becomes infinite on any one of three conditions,

$$y = 0,$$
$$y = y',$$
$$y = \infty.$$

The first shows that the parallels BH and B′H′ touch the curve at B, B′.

The second shows that the parallels to YY′ through E, E′, touch the curve at these points.

The third shows that the tangent to a point in the branch EG or E′G′ approaches without limit to parallelism with YY′.

From all these circumstances, it appears that the figure of the curve is as it has been represented.

By the value of y, it appears that the parabola represented by the equation

$$x^2 - yy' - a^2 = 0$$

is a diameter of the curve bisecting a system of chords parallel to the axis of y. If the equation of this diameter be put under the form

$$x^2 - y'(y + \frac{a^2}{y'}) = 0,$$

it is plain that the axis of the parabola is the line YY′, and that if AO $= -\frac{a^2}{y'}$, O is the vertex. Also, since by its equation $y = 0$, gives $x = \pm a$, and $y = y'$ gives

$$x = \pm \sqrt{a^2 + y'^2} = \pm b,$$

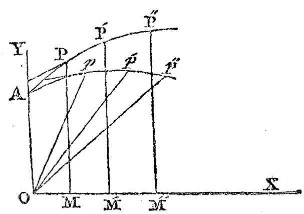
it must pass through the points B, B′, and E, E′. The point O is evidently the point of bisection of DD′.

It is not difficult to explain the genesis of the different parts of the curve by the motion of the centres of the circles touching the three sides of the triangle. While the vertex v moves in the direction VM, the centre of the circle touching the three sides, vB, vB$'$, and BB$'$, moves through DPB. At the same time the centre of the circle which touches vb' and BX, the productions of the sides B$'v$ and BB$'$, and also the side Bv, describes the asymptotic branch EN. Also the centre of the circle touching vb, B$'$X$'$, and B$'v$, is describing the infinite branch E$'$G$'$, and finally, the centre of the circle touching Ba, B$'a'$, and B$'$B, is describing the part B$'$D$'$.

In like manner, while the vertex of the triangle is moving in the direction VM$'$, the centres of these four circles describe the several parts, DB$'$, E$'$N$'$, EG, and B$'$D$'$.

PROP. CCCXXV.

(635.) *Two given curves* APP$'$P$''$ *and* A$pp'p''$, *one related to rectangular, and the other to polar co-ordinates, and represented by equations of the forms* $\text{F}(yx) = 0$, $z = \text{F}(\omega)$, *are so related, that if the curve* APP$'$P$''$ *be wrapped upon* A$pp'p''$, *the ordinates* PM, P$'$M$'$, P$''$M$''$, . . . *preserving their inclinations to the curve, shall be equal to and coincident with the radii vectores* Op, Op', Op'' . . . . . . *the points* p, p', p'', . . . *being the positions of the points* P, P$'$, P$''$, ... *when the curve* APP$'$P$''$ *is wrapped upon* A$pp'p''$; *to determine the conditions by which the equation of either of these curves may be found from that of the other.*



By the conditions expressed, the lines PM and Op are equally inclined to the tangents at the points P and p; and the same being true for all corresponding points, we have the condition

$$\frac{dx}{dy} = \frac{zd\omega}{dz},$$

the angle ω being measured from the axis of y: but since

$$y = z \text{ and } dy = dz, \because$$

$$zd\omega = dx.$$

Hence by differentiating the equation $\text{F}(yx) = 0$, and changing y into z, and dx into $zd\omega$, and eliminating x, we shall find by integrating the result, the equation $z = \text{F}(\omega)$. Also, if this latter equation be given, a similar process will discover the equation $\text{F}(yx) = 0$.

It is evident that either curve may be supposed to produce the other; APP′P″ by being bent into such a form that the points M, M′, M″, shall all unite in the origin O, and the ordinates become radii vectors, will produce A$pp'p''$; or A$pp'p''$ by being bent into such a form, that the radii vectors will become parallel, and their extremities lie in a straight line, OMM′M″ passing through O will produce APP′P″.

By these means every curve related to rectangular coordinates produces a corresponding spiral and vv.

Since $zd\omega = dx$, and $z = y$, we have $z^2d\omega = ydx$.

Hence it follows, that the area included between two ordinates of the one curve is double the area included between the two radii vectores of the other, which are equal to those ordinates. Also, since

$$dy^2 + dx^2 = dz^2 + z^2d\omega^2.$$

The arc of the one intercepted between two ordinates is equal to the arc of the other intercepted between the corresponding radii vectores. We shall apply these general principles to some examples.

1°. Let the curve APP′P″ be represented by the equation

$$y^m = ax.$$

By differentiating, we find

$$my^{m-1}dy = adx;$$

and by making the necessary substitutions,

$$mz^{m-2}dz = ad\omega,$$

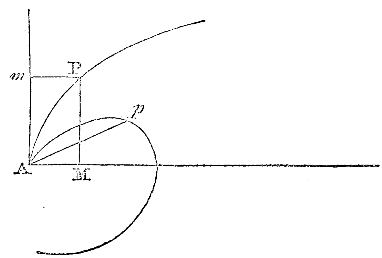
which being integrated, gives

$$mz^{m-1} = (m-1)a\omega.$$

When $m = 2$, the curve is the common parabola, and the corresponding spiral is that of Archimedes, represented by the equation

$$z = \tfrac{1}{2}a\omega.$$

Hence if AP be any arc of the parabola, and PM = Ap, the arc Ap will be equal to AP, and the area APM will be double that of the segment Ap, and therefore the segment Ap will be one-third of the rectangle Mm, and equal to the area AmP.



It appears therefore that the rectification of the spiral of Archimedes depends on that of the parabola.

2°. Let the curve APP′ be a line of the second degree related to its axis and vertical tangent as axes of co-ordinates, and represented by the equation

$$y^2 + \frac{p}{2\text{A}}x^2 - px = 0\,;$$

by differentiating

$$2ydy + \frac{p}{\text{A}}xdx - pdx = 0,$$

which, after the necessary substitutions, becomes

$$d\omega = \frac{dz}{\sqrt{\frac{p}{2\text{A}}\left(\frac{p\text{A}}{2} - z^2\right)}}.$$

When A is infinite, the integration of this gives

$$z = \tfrac{1}{2}p\omega,$$

a result which coincides with that already found. Otherwise the integration, after substituting $\frac{2B^2}{A}$ for p, gives

$$z = B \sin. \frac{B}{A}\omega.$$

If $B = A$, this equation becomes

$$z = B \sin. \omega.$$

Hence if the curve APP' be a circle whose diameter is $2B$, the curve App' is a circle whose diameter is B.

3°. Let the curve App' be the logarithmic spiral represented by the equation

$$z = a^{\omega}.$$

By differentiating, we find

$$\tan. \theta . dz = zd\omega.$$

By the proper substitutions, this becomes

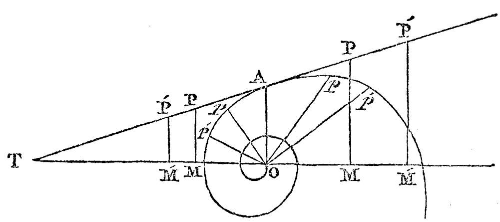
$$\tan. \theta dy = dx;$$

which being integrated and a constant introduced,

$$(y - y') - \cot. \theta . x = 0,$$

which is the equation of a right line.

Hence if App' be a logarithmic spiral, APP' will be a right line touching it at A, and if PM, $P'M'$, be inflected parallel to OA, and equal to op, op', the arc pp' will be equal to PP', and the area pop' will be half the area $PMM'P'$. Also AT is equal to the arc of the spiral continued from A to the pole, and the triangle AOT is double the corresponding area. These results agree with those found in art. (437), (438).



4°. Let the curve $PP'P''$ be the equilateral hyperbola related to its asymptotes, and represented by the equation

$$yx = m^2.$$

By differentiating, we find

$$ydx + xdy = 0,$$

which, after the necessary substitutions, becomes

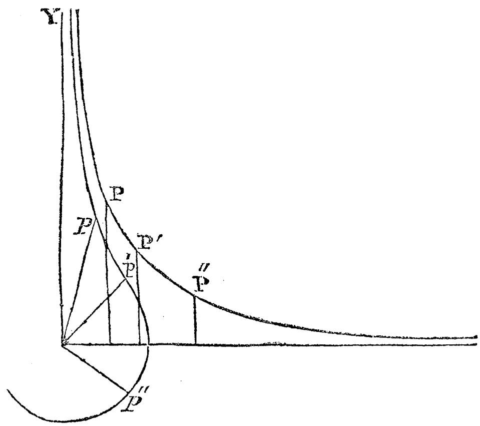
$$z^3 d\omega + m^2 dz = 0;$$

which being integrated, gives

$$z^2 \omega = \tfrac{1}{2} m^2,$$

which is the equation of the *lituus*.

Hence if PM = Ap, and P′M′ = Ap', the arc PP′ = pp', and the area PMM′P′ is equal to twice the area pAp'; also the area of the entire lituus continued to the pole is equal to half the space included between the hyperbola and its asymptotes, and the entire length of the lituus is equal to that of the hyperbola.



5°. The preceding is only a particular case of a more general class of curves included under the equation

$$y^m x = a^{m+1},$$

which, differentiated, gives

$$y^m dx + my^{m-1} xdy = 0,$$

$$\therefore ydx + mxdy = 0;$$

which, after the necessary substitutions, becomes

$$(m + 1) z^{m+1} \omega = ma^{m+1}.$$

SECTION XXIII.

Propositions illustrative of the application of the preceding part of Algebraic Geometry to various parts of Physical Science.

PROP. CCCXXVI.

(636.) *A right line of a given length being drawn perpendicular to an horizontal plane, to find the nature of the curve traced out by the extremity of its shadow.*

The meridian line being assumed as the fixed axis, and the angle which the shadow r makes with it being ω, and L the length of the perpendicular. Let z be the zenith distance of the sun, ϖ its polar distance, and λ the latitude. By the conditions of the question, we have the equation

$$r = \text{L} \tan. z;$$

and since the angle ω is the sun's azimuth, we have by spherical trigonometry,

$$\cos. \omega = \frac{\cos. \varpi - \cos. z \sin. \lambda}{\sin. z \cos. \lambda}.$$

By these equations, eliminating z, the result is

$$r . \cos. \lambda . \cos. \omega + \text{L} \sin. \lambda = \sqrt{r^2 + \text{L}^2} . \cos. \varpi.$$

If the meridian line be taken as the axis of x, and the intersection of the planes of the horizon and prime vertical as the axis of y, and this equation be reduced to one between rectangular co-ordinates yx, the result is

$$\cos.^2 \varpi . y^2 + (\cos.^2 \varpi - \cos.^2 \lambda)x^2 - \sin. 2\lambda . \text{L}x$$
$$+ (\cos.^2 \varpi - \sin.^2 \lambda)\text{L}^2 = 0.$$

The locus is therefore a line of the second degree.

It is an *ellipse*, *hyperbola*, or *parabola*, according as

$\cos.\varpi > \cos.\lambda$, $\cos.\varpi < \cos.\lambda$, or $\cos.\varpi = \cos.\lambda$.

In other words, when the sun's polar distance is less than the latitude, it is an *ellipse*; when it is greater, an *hyperbola*; and when it is equal to the latitude, it is a *parabola*.

At the pole $\cos.\lambda = 0$; the locus is therefore a circle represented by the equation

$$y^2 + x^2 = L^2 \tan.^2 \varpi.$$

At places within the polar circle the locus has at different times of the year all its varieties, inasmuch as the sun's polar distance is, at different times of the year, greater, equal to, and less than the latitude.

At the polar circle the locus is a parabola at the solstice, and an hyperbola at all other times.

At all latitudes less than $66\frac{1}{2}^\circ$ the locus is always an hyperbola, since the sun's polar distance is never less than $66\frac{1}{2}^\circ$.

At the equator the locus is the intersection of the planes of the prime vertical and horizon at the equinoxes; for in that case $\varpi = 90^\circ$, and the equation of the locus becomes

$$x = 0.$$

PROP. CCCXXVII.

(637.) *To find the curve described by the vertex of the earth's conical shadow.*

Let the semidiameters of the sun and earth be r, r', and let z, z', be the distances of the vertex of the shadow and the centre of the earth from the sun, we have the equations

$$\frac{z}{z'} = \frac{r}{r-r'},$$

$$z' = \frac{p}{2(1+e\cos.\omega)};$$

the latter being the polar equation of the earth's orbit, eliminating z', the result is

$$z = \frac{pr}{2(r-r')\,(1+e\cos.\omega)}.$$

Hence the vertex of the shadow describes an ellipse similar to that of the earth, and whose parameter is

$$\frac{pr}{r-r'}.$$

PROP. CCCXXVIII.

(638.) *If a body revolves in any proposed curve, to find the curve of a fixed star's aberration as seen from this body.*

As the aberration is in direction always parallel to the tangent to the orbit in which the body is supposed to move, and in quantity reciprocally proportional to the perpendicular from the centre of force upon the tangent, the nature of the curve of aberration may be investigated by finding the locus of the extremity of a line drawn from the centre of force parallel to the tangent, and such that the rectangle under it, and the perpendicular on the tangent, shall be constant.

These conditions may with great facility be reduced to equations. Let $\text{F}(y'x') = 0$ be the equation of the curve, the origin of rectangular co-ordinates being at the centre of force, and yx being the co-ordinates of any point of the sought locus, the condition of the radius vector being always parallel to the tangent, gives the equation

$$\frac{dy'}{dx'} = \frac{y}{x}, \text{ or}$$

$$xdy' - ydx' = 0.$$

Let x'' be the distance from the origin at which the tangent meets the axis of x. By the equation of the tangent

$$x'' = \frac{x'dy' - y'dx'}{dy'}.$$

Let p = the perpendicular on the tangent

$$p = x'' \cdot \frac{dy'}{\sqrt{dy'^2 + dx'^2}},$$

and therefore we find

$$p = \frac{x'dy' - y'dx'}{\sqrt{dy'^2 + dx'^2}};$$

and since by the first condition,

$$\frac{dy'}{dx'} = \frac{y}{x};$$

therefore

$$p = \frac{x'y - y'x}{\sqrt{y^2 + x^2}}.$$

And by the condition, that the rectangle under the perpendicular and radius vector is constant,

$$p\sqrt{y^2 + x^2} = m^2.$$

The locus sought will therefore be found by eliminating $y'x'$ from the equations

$$x'y - y'x = m^2 \quad (1),$$
$$ydx' - xdy' = 0 \quad (2),$$
$$F(y'x') = 0 \quad (3).$$

PROP. CCCXXIX.

(639.) *The orbit being a line of the second degree with the centre of force at the focus, to find the curve of aberration.*

The polar equation of a line of the second degree, the focus being the pole, is

$$r = \frac{p}{2(1 - e\cos.\omega)},$$

which reduced to rectangular co-ordinates, and arranged by the dimensions of the variables, is

$$y'^2 + (1 - e^2)x'^2 - pex' - \tfrac{1}{4}p^2 = 0.$$

By differentiating this, we have

$$\frac{dy'}{dx'} = \frac{pe - 2(1-e^2)x'}{2y'}.$$

Hence equation (2) of the last proposition becomes

$$pex - 2(1 - e^2)xx' - 2yy' = 0.$$

By means of this, the equation of the orbit, and the equation (1) of the last proposition, eliminating y' and x', the result, arranged according to the dimensions of the variables, is

$$y^4 + (1 - e^2)x^4 + (2 - e^2)y^2x^2 + 4\frac{em^2}{p}y^3$$
$$+ 4\frac{em^2}{p}(1 - e^2)yx^2 - \frac{4m^4(1-e^2)}{p^2}y^2 - \frac{4m^4(1-e^2)^2}{p^2}x^2 = 0;$$

which is obviously resolvable into the factors

$$y^2 + (1 - e^2)x^2 = 0,$$
$$y^2 + x^2 + \frac{4em^2}{p}y - \frac{4m^4(1-e^2)}{p^2} = 0.$$

The former gives $y = 0$, $x = 0$ scil., the origin of the co-ordinates. The latter is the equation of the locus sought, which is therefore a circle, the co-ordinates of whose centre are $x = 0$, $y = -\frac{2em^2}{p}$, and whose radius is $\frac{2m^2}{p}$. From which it also appears that the origin of co-ordinates is within the circumference, on it, or outside it, according as $e < 1$, $= 1$, or > 1.

Hence it follows that the curve of aberration is always a circle when the orbit is a line of the second degree, with the centre of force in the focus; but that the true place of the star is within the circle if it be an ellipse, on it if it be a parabola, and outside it if it be an hyperbola. It is obvious, that if the orbit be a circle, the true place of the star is in the centre of the circle.

PROP. CCCXXX.

(640.) *The orbit being an ellipse or hyperbola, with the centre of force at its centre, to find the curve of aberration.*

The three equations in proposition (638), become in this case

$$\text{A}^2 y'^2 + \text{B}^2 x'^2 = \text{A}^2\text{B}^2,$$
$$\text{A}^2 y'y + \text{B}^2 x'x = 0,$$
$$x'y - y'x = m^2.$$

By eliminating y' and x' from these, we find

$$\text{A}^2 y^2 + \text{B}^2 x^2 = m^4.$$

The equation of the curve of aberration, which is therefore a curve similar to the orbit, its semiaxes being

$$\frac{m^2}{\text{A}} \text{ and } \frac{m^2}{\text{B}}.$$

PROP. CCCXXXI.

(641.) *The orbit being a parabola, the force acting along the diameters, to find the curve of aberration.*

The three equations in prop. (638), become in this case

$$y'^2 - px' = 0,$$
$$2yy' - px = 0,$$
$$x'y - y'x = m^2.$$

By eliminating y', x', by these equations, we find

$$x^2 + \frac{4m^2}{p}y = 0.$$

The locus is therefore a parabola, whose axis is perpendicular to that of the orbit.

PROP. CCCXXXII.

(642.) *The orbit of the planet being supposed a circle, with the sun in the circumference, to find the curve of aberration.*

The equations in prop. (638) become in this case

$$x'y - y'x = m^2,$$
$$yy' + xx' = rx,$$
$$y'^2 + x'^2 - 2rx' = 0.$$

Finding values for $y'x'$ from the first, and substituting them in the last, the result, divided by $y^2 + x^2$, is

$$r^2x^2 + 2rm^2y - m^4 = 0,$$

the equation of the curve of aberration, which is therefore a parabola, whose axis is the axis of y, and the co-ordinate of whose vertex is $y = \frac{m^2}{2r}$.

PROP. CCCXXXIII.

(643.) *The orbit of a comet being supposed parabolic, to find the place of perihelion from two distances from the sun and the included angle.*

Let the equation of the orbit related to the axis and focus be

$$z = \frac{p}{2(1 - \cos. \omega)}.$$

Let the two distances given be z', z'', and the corresponding anomalies ω', ω'', hence

$$z' = \frac{p}{2(1 - \cos. \omega')},$$

$$z'' = \frac{p}{2(1 - \cos. \omega'')};$$

by dividing, we find

$$\frac{z'}{z''} = \frac{1 - \cos. \omega''}{1 - \cos. \omega'}.$$

The given angle under z' and z'' being ϕ, we have

$$\omega' + \omega'' = \phi.$$

By eliminating one of the anomalies by these equations, we shall find a value for the other. This elimination may be effected thus,

$$1 - \cos.\ \omega' = 2 \sin.^2 \tfrac{1}{2}\omega',$$
$$1 - \cos.\ \omega'' = 2 \sin.^2 \tfrac{1}{2}\omega''.$$

Hence

$$\frac{\sqrt{z'}}{\sqrt{z''}} = \frac{\sin.\ \frac{1}{2}\omega''}{\sin.\ \frac{1}{2}\omega'},$$

$$\therefore \frac{\sqrt{z'} + \sqrt{z''}}{\sqrt{z'} - \sqrt{z''}} = \frac{\sin.\ \frac{1}{2}\omega' + \sin.\ \frac{1}{2}\omega''}{\sin.\ \frac{1}{2}\omega' - \sin.\ \frac{1}{2}\omega''},$$

$$\therefore \frac{\sqrt{z'} + \sqrt{z''}}{\sqrt{z'} - \sqrt{z''}} = \frac{\cot.\ \frac{1}{4}\phi}{\tan.\ \frac{1}{4}(\omega' - \omega'')}.$$

Hence by finding the value of $\omega' - \omega''$, the value of each anomaly is known.

PROP. CCCXXXIV.

(644.) *The parabolic orbits of several comets having a common tangent, to find the locus of the perihelia.*

Let P be the perpendicular distance of the common tangent from the sun, z the perihelion distance, and z' the distance of the point of contact, and let the angle under P and z be ω. By the polar equation of the parabola

$$z' = \frac{2z}{1 - \cos.\ 2\omega};$$

and since by the properties of this curve P is a mean proportional between z and z',

$$\text{P}^2 = zz',$$

z' being eliminated by these equations, the result is

$$z^2 = \text{P}^2 \frac{1 - \cos.\ 2\omega}{2};$$

but by trigonometry,

$$\sin.^2 \omega = \frac{1-\cos. 2\omega}{2}.$$

Hence the equation related to rectangular co-ordinates is

$$y^2 + x^2 - \text{P}x = 0.$$

The locus is therefore a circle described on the perpendicular P as diameter.

PROP. CCCXXXV.

(645.) *The parabolic orbits of several comets intersecting at the same point, to find the locus of the perihelia.*

Let z' be the distance of the intersection of their orbits from the sun, and z the perihelion distance, and ω the angle under them. By the polar equation of the parabola,

$$z' = \frac{2z}{1-\cos. \omega}.$$

Hence the equation of the locus sought is

$$z = z' \sin.^2 \tfrac{1}{2}\omega,$$

which is the equation of a *cardioide,* the diameter of whose generating circle is the distance z'.

PROP. CCCXXXVI.

(646.) *Projectiles being thrown from a given point with the same velocity in different directions, to find the loci of the vertices and foci of the parabolæ described by them.*

Let the height due to the velocity be H, and the angle of elevation ε, the equation of the path of the projectile in free space is

$$y = x \tan. \varepsilon - \frac{x^2}{4\text{H} \cos.^2 \varepsilon}.$$

The axis of y being vertical, and x horizontal, let the co-ordinates of the vertex be $y'x'$, and those of the focus $y''x''$.

To find x', let $y = 0$, and x' will be half the resulting value of x: hence

$$x' = 2\text{H} \sin. \varepsilon \cos. \varepsilon = \text{H} \sin. 2\varepsilon.$$

To find y', let x' be substituted in the equation of the curve, and we find

$$y' = \text{H} \sin.^2 \varepsilon.$$

The values of $y''x''$ are found by observing that $x'' = x'$, and the line drawn from the origin to the focus is equal to H and the angle under it, and the axis of y is, by the properties of the parabola, bisected by the direction of projection. Hence we find

$$x'' = \text{H} \sin. 2\varepsilon \qquad y'' = -\text{H} \cos. 2\varepsilon.$$

To find the locus of the vertices, let ε be eliminated by means of the values of $y'x'$, and the resulting equation is

$$4y'^2 + x'^2 - 4\text{H}y' = 0,$$

which is the equation of an ellipse, whose conjugate axis is vertical, having its extremity at the point of projection, and transverse axis horizontal. To find the magnitude of these axes, let $x' = 0$, $\because y' = \text{H}$; and if $y' = \frac{\text{H}}{2}$, the corresponding value of x' is H. Hence the conjugate axis is equal to H, and the transverse axis to 2H.

To find the locus of the foci, let ε be eliminated by means of the values of $y''x''$, which gives

$$y''^2 + x''^2 = \text{H}^2.$$

The locus sought is therefore a circle whose centre is at the point of projection, and whose radius is the height due to the velocity.

PROP. CCCXXXVII.

(647.) *Several projectiles being thrown in the same direction with different velocities, to find the loci of their vertices and foci.*

To find the locus of their vertices, let H be eliminated by means of the values of $y'x'$ found in the last proposition. The resulting equation is

$$y' = \tfrac{1}{2}\tan.\ \varepsilon\ .\ x'.$$

The locus is therefore a right line through the origin.

To find the locus of the foci, let H be eliminated by means of the values of $y''x''$; the result of which is

$$y'' = \cot.\ 2\varepsilon\ .\ x''.$$

Hence the locus sought is a straight line through the point of projection, and inclined to the vertical line at an angle which is bisected by the direction of projection.

PROP. CCCXXXVIII.

(648.) *Given the velocity and direction of a projectile, to find the point where it will meet a given plane, and also the time of flight.*

The projectile must meet the given plane in some point of the intersection of the given plane with the vertical plane in which the projectile moves. Let the equation of this line be

$$y = \tan.\ \varpi\ .\ x + b;$$

where ϖ is the angle at which the plane is inclined to the horizon, and b the distance of the point where it meets the axis of y from the origin. By this equation, and the equation

$$y = \tan.\ \varepsilon\ .\ x - \frac{x^2}{4\text{H}\cos.^2\varepsilon},$$

we find the co-ordinates of the point sought are

$$x = 2\text{H}\frac{\cos.\varepsilon}{\cos.\varpi}\left\{\sin.(\varepsilon-\varpi) \pm \left\{\sin.^2(\varepsilon-\varpi) - \frac{b}{\text{H}}\cos.^2\varpi\right\}^{\frac{1}{2}}\right\}$$

$$y = 2\text{H}\frac{\sin.\varpi\cos.\varepsilon}{\cos.^2\varpi}\left\{\sin.(\varepsilon-\varpi) \pm \left\{\sin.^2(\varepsilon-\varpi) - \frac{b}{\text{H}}\cos.^2\varpi\right\}^{\frac{1}{2}}\right\} + b.$$

To find (T) the time of flight. Let $m = 16\frac{1}{12}$ inches. Since the vertical through the projectile moves uniformly with a velocity expressed by $2\cos.\varepsilon\sqrt{\text{H}m}$, we find

$$\text{T} = \frac{x}{2\cos.\varepsilon\sqrt{\text{H}m}}.$$

Hence

$$\text{T} = \left\{\frac{\text{H}}{m\cos.^2\varpi}\right\}^{\frac{1}{2}}.\left\{\sin.(\varepsilon-\varpi) \pm \left\{\sin.^2(\varepsilon-\varpi) - \frac{b}{\text{H}}\cos.^2\varpi\right\}^{\frac{1}{2}}\right\}$$

being the time of flight expressed in seconds.

PROP. CCCXXXIX.

(649.) *Given the velocity of projection, to find the angle of projection, at which the distance of the point where the projectile meets a given plane shall be a maximum.*

This problem is in effect to investigate the value of ε, which renders the value of x in the last proposition a maximum. For this purpose, let it be differentiated, and its differential equated with cypher; the result of which is

$$\cos.\varepsilon\cos.(\varepsilon-\varpi) - \sin.\varepsilon\sin.(\varepsilon-\varpi) - \text{R}\sin.\varepsilon + \frac{\cos.\varepsilon\sin.(\varepsilon-\varpi)\cos.(\varepsilon-\varpi)}{\text{R}} = 0;$$

$$\text{where R} = \left\{\sin.^2(\varepsilon-\varpi) - \frac{b}{\text{H}}\cos.^2\varpi\right\}^{\frac{1}{2}}.$$

This equation gives the sought value of ε; but to extricate it, requires some trigonometrical artifice. Observing that

$$\cos.\varepsilon\cos.(\varepsilon-\varpi) - \sin.\varepsilon\sin.(\varepsilon-\varpi) = \cos.(2\varepsilon-\varpi),$$

multiplying by R, and substituting for R^2 its value, the equation becomes

$$\cos.(2\varepsilon - \varpi)\left\{\text{R} + \sin.(\varepsilon - \varpi)\right\} + \frac{b}{\text{H}}\sin.\varepsilon\cos.^2\varpi = 0,$$

$$\therefore \text{R} = -\sin.(\varepsilon - \varpi) - \frac{b}{\text{H}} \cdot \frac{\sin.\varepsilon\cos.^2\varpi}{\cos.(2\varepsilon - \varpi)}.$$

By equating this with the other value of R, and squaring and expunging the terms which mutually destroy each other, we have

$$\cos.^2(2\varepsilon - \varpi) + 2\sin.(\varepsilon - \varpi)\cos.(2\varepsilon - \varpi)\sin.\varepsilon$$
$$+ \frac{b}{\text{H}}\sin.^2\varepsilon\cos.^2\varpi = 0.$$

But by trigonometry,

$$2\sin.(\varepsilon - \varpi)\sin.\varepsilon = \cos.\varpi - \cos.(2\varepsilon - \varpi).$$

By this substitution, the expression being cleared of the terms which destroy each other, and divided by cos. ϖ, becomes

$$\cos.(2 \quad - \varpi) + \frac{b}{\text{H}}\sin.^2\varepsilon\cos.\beta = 0.$$

By trigonometry,

$$\cos.(2\varepsilon - \varpi) = \cos.2\varepsilon\cos.\varpi + \sin.2\varepsilon\sin.\varpi,$$
$$\cos.2\varepsilon = \cos^2\varepsilon - \sin.^2\varepsilon,$$
$$\sin.2\varepsilon = 2\sin.\varepsilon\cos.\varepsilon.$$

Making these substitutions, and dividing by cos. ϖ cos.^{2} ε, and arranging for solution, we find

$$\tan.^2\varepsilon + 2 \cdot \frac{\text{H}}{b - \text{H}} \cdot \tan.\varpi\tan.\varepsilon + \frac{\text{H}}{b - \text{H}} = 0;$$

which solved, gives

$$\tan.\varepsilon = \frac{\text{H}}{\text{H} - b} \cdot \left\{\tan.\varpi \pm \left\{\sec.^2\varpi - \frac{b}{\text{H}}\right\}^{\frac{1}{2}}\right\}$$

which gives the required value of ε.

The cause of the two values of ε is obvious; for the angle of elevation being the same, the projectile may be thrown either *up* or *down* the plane, and one of the values of the elevation gives the maximum range *up* the plane, and the other *down* it.

If the projectile be thrown from one point to another on the same plane, $b = 0$, and the formula for the range becomes

$$x = 4\text{H} \cdot \frac{\cos. \varepsilon \sin. (\varepsilon - \varpi)}{\cos. \varpi}.$$

In this case also the maximum range is given by the elevation resulting from

$$\tan. \varepsilon = \tan. \varpi \pm \sec. \varpi;$$

but by trigonometry,

$$\tan. \varpi \pm \sec. \varpi = \frac{\sin. \varpi \pm 1}{\cos. \varpi} = \tan. \tfrac{1}{2}(\varpi \pm 90^0).$$

Hence the direction which produces the greatest range is that which bisects the angle under the plane, and the vertical through the point of projection.

It is observable, that if ε and ε' be two angles of elevation so related that

$$\varepsilon + \varepsilon' = 90^0 + \varpi,$$

we shall always have

$$\cos. \varepsilon \sin. (\varepsilon - \varpi) = \cos. \varepsilon' \sin. (\varepsilon' - \varpi).$$

Hence two such elevations always give equal ranges with the same velocity of projection; and it appears from the way these angles are related to the value of ε, giving the maximum range, that any two directions of projection, equally inclined to that which gives the maximum range, will, with the same velocity of projection, give equal ranges.

It also appears from what has been said, that the elevation which produces the greatest range on the horizon is 45^0, and that complemental elevations give equal ranges.

PROP. CCCXL.

(650.) *To find the locus of the empty foci of the orbits of several planets having a common point of intersection, and at that point having the same velocity.*

Let the distance of the point of intersection from the sun be D, and the distance of the same point from the empty focus D'. By the properties of the ellipse,

$$D + D' = 2A,$$

where A represents the semitransverse axis.

Since the velocity at the point of intersection is the same in all the orbits, the osculating circles at those points must have a common chord passing through the sun. Let this be C. By the properties of the ellipse,

$$C = \frac{2DD'}{A}.$$

Eliminating A, we find

$$D' = \frac{CD}{4D - C}.$$

Hence the value of D' is constant, and therefore the locus of the empty focus is a circle with the common point as centre, and D' as radius.

PROP. CCCXLI.

(651.) *To investigate the figure of the earth from the horizontal parallax of the moon accurately observed at the same time in different latitudes.*

Let ϕ be the latitude, and r the corresponding semidiameter of the earth, and let the equation

$$r^2 = \frac{a^2 b^2}{a^2 \sin.^2 \phi + b^2 \cos.^2 \phi}$$

be assumed as that of a meridian, a representing the equatorial, and b the polar semidiameters. Also let r' be the semidiameter corresponding to another latitude ϕ', ∵

$$r'^2 = \frac{a^2 b^2}{a^2 \sin.^2 \phi' + b^2 \cos.^2 \phi'}.$$

Dividing one of these equations by the other, and supposing $a = mb$, we have

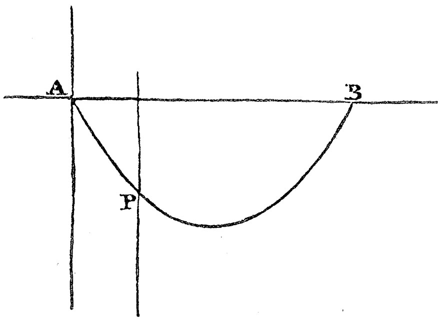
$$\frac{r^2}{r'^2} = \frac{m^2 \sin.^2 \phi' + \cos.^2 \phi'}{m^2 \sin.^2 \phi + \cos.^2 \phi}.$$

Hence, observations of two horizontal parallaxes at given latitudes will give the value of $\frac{r}{r'}$; from whence that of m, or the relation of the equatorial and polar diameters, is easily found.

Therefore, assuming the earth to be an oblate spheroid, we can find its axes. And the truth of the assumption may be proved by every pair of observed parallaxes giving the same value of m.

PROP. CCCXLII.

(652.) *A perfectly flexible and inelastic chain of uniform density and thickness being suspended from two fixed points, to find the curve into which it will form itself by the effect of its own weight.*



Let the fixed points A and B be the points of suspension, and the horizontal and vertical lines through A being taken axes of co-ordinates, let yx be the co-ordinates of any point P, and let the arc AP be s. The part AP of the chain may be considered as a rigid body retained in a state of equilibrium by three forces, scil. the weight of the chain s acting in the vertical direction, the tension a at the point A acting in the direction of a tangent at that point, and the tension t at the point P acting in the direction of the tangent at the point P.

Let the angles under the tangents at the points A and P, and a horizontal line, be respectively α and ϖ.

By the principles of Statics, since the forces all act in the same plane, their component parts in the direction of each axis of co-ordinates must be in equilibrio. Hence we have the equations

$$a \cos. \alpha - t \cos. \varpi = 0,$$
$$a \sin. \alpha - t \sin. \varpi - s = 0.$$

Eliminating t by these equations, the resulting equation solved for tan. ϖ, gives

$$\tan. \varpi = \frac{a \sin. \alpha - s}{a \cos. \alpha};$$

but in general $\tan. \varpi = \frac{dy}{dx}$, $\because$

$$\frac{dy}{dx} = \frac{a \sin. \alpha - s}{a \cos. \alpha};$$

which is equivalent to the differential equation of the catenary found in (546). By a comparison with this, it appears that the curve sought is a catenary, whose axis is vertical.

PROP. CCCXLIII.

(653.) *A flexible and elastic chain is attached to two fixed points, to find the curve into which it will form itself by its own gravity.*

The chain being supposed of uniform density and thickness, let the ratio of any assumed length of it to the length of the same, when extended by the tension t, be $1 : 1 + et$. Let the tension at the lowest point be equal to the weight of a length c of the unextended chain. Let s' be any length of the unextended, and s of the extended chain. The axes of co-ordinates being vertical and horizontal, we have by the conditions of the question

$$ds = ds'(1 + et).$$

The forces which keep s at rest are 1°, the tension t, 2°, the tension c, 3°, the weight s' of the chain. These forces are therefore as the sides of a triangle, which are parallel to their directions, that is, as dx, dy, ds; hence

$$\frac{t}{c} = \frac{ds}{dx}, \text{ and } \frac{s'}{c} = \frac{dy}{dx}.$$

By differentiating the latter, and eliminating ds', we find

$$\frac{d^2y}{dx^2} = \frac{ds}{cdx + c^2eds}.$$

Now if $\frac{dy}{dx} = p$, $\therefore ds = \sqrt{1+p^2} \,.\, dx$, $\therefore$

$$\frac{dp}{dx} = \frac{\sqrt{1+p^2}}{c+c^2e\sqrt{1+p^2}},$$

$$\therefore dx = \frac{cdp}{\sqrt{1+p^2}} + c^2edp,$$

$$\text{and } dy = \frac{cpdp}{\sqrt{1+p^2}} + c^2epdp.$$

By integration we find

$$x = cl\left\{p + \sqrt{1+p^2}\right\} + c^2ep,$$

$$y = c\sqrt{1+p^2} + \tfrac{1}{2}c^2ep^2.$$

The integral being assumed, so that when $p = 0$ we have $x = 0$, $y = c$.

In these equations the variable p is the tangent of the angle, which the curve at the point yx makes with the horizon.

By the elimination of p, we should find the equation of the curve expressed between the variables yx.

To express the equation between s and p, we should observe that

$$ds = \sqrt{1+p^2}\, dx.$$

In which the value of dx already found being substituted, and the result integrated, gives

$$s = cp + \tfrac{1}{2}c^2e\left\{p\sqrt{1+p^2} + l(p + \sqrt{1+p^2})\right\}$$

PROP. CCCXLIV.

(654.) *A given orbit is described by a body round a given point as centre of force, and from any point in it the body is projected with the velocity in the orbit in a direction immediately opposed to the action of the force; to find the locus of the point at which it shall cease to recede from the orbit.*

Let D be the distance of any point in the orbit from the centre of force, and H the distance at which the body shall cease to recede from the curve. Since in moving through H − D the body loses the velocity with which it is projected, it would acquire the same velocity in moving through the same space in the opposite direction. Hence, if V be the velocity in the curve,

$$V^2 = -4m\int F\,dD,$$

where $2m$ is the velocity communicated by the unit of force in the unit of time. Let the law of the force be such that

$$F = D^{n-1},$$

the force being unity at the distance unity. Hence, by integrating, we find

$$V^2 = \frac{4m}{n}(H^n - D^n).$$

If C be the chord of the osculating circle passing through the centre of force, since V is equal to the velocity in the curve, we have

$$V^2 = mFC = mD^{n-1}C,$$

which combined with the former, gives

$$H^n = D^n + nD^{n-1} \cdot \tfrac{1}{4}C.$$

The values of C and D, known by the equation of the given orbit, give the equation sought. We shall give a few examples of the application of this formula.

1. Let the orbit be an ellipse, the centre of force being the focus. In this case $n = -1$, and $C = \frac{2D(2A - D)}{A}$, where A expresses the semitransverse axis. Hence

$$H = 2A.$$

The locus is therefore in this case a circle, whose radius is equal to the transverse axis of the orbit.

2. If the curve be an ellipse, the centre being the centre of force. In this case $n = 2$, $C = \frac{2B'^2}{D}$, where B′ is the semidiameter conjugate to D. Hence

$$H^2 = D^2 + B'^2 = A^2 + B^2,$$

A and B being the semiaxes. The locus is therefore a circle, whose radius is equal to the line joining the extremities of the axes.

3. If the curve be a circle, the centre being the centre of force, the formula becomes

$$H^n = (1 + \frac{n}{2})D^n.$$

PROP. CCCXLV.

(655.) *A given orbit is described round a given point as centre of force, and a body being placed at any point in the orbit, is moved by the action, and in the direction of the force, until it acquires the velocity it would have in the orbit; to find the locus of the point at which this velocity shall be acquired.*

Let H be the distance at which the velocity shall be acquired, and D the distance of the point in the orbit from the centre of force. As before,

$$\text{V}^2 = \frac{4m}{n}(\text{D}^n - \text{H}^n),$$

$$\text{V}^2 = m\text{D}^{n-1} \cdot \text{C},$$

$$\because \text{H}^n = \text{D}^n - n\text{D}^{n-1} \cdot \tfrac{1}{4}\text{C}.$$

The values of D and C resulting from the equation of the given orbit being substituted, give the locus sought.

To apply this formula to some examples:

1. If the curve be a circle, with the force at the centre,

$$\text{H}^n = (1 - \frac{n}{2})\text{D}^n.$$

2. If the curve be a circle, with the force on the circumference, let its equation be

$$z = 2r \cos. \omega.$$

Since in this case $z = \text{D}$, and $n = -4$, and $\text{C} = \text{D} = z$, $\because$

$$\text{H} = 2^{\frac{3}{4}} \cdot r \cos. \omega,$$

which shows that the locus is a circle touching the given one internally at the centre of force, and whose radius is $\dfrac{r}{2^{\frac{1}{4}}}$.

3. If the curve be an ellipse or hyperbola, the force being at the focus. Let the equation be

$$z = \frac{p}{2(1 + e \cos. \omega)}.$$

In this case $z = \text{D}$, $n = -1$, $\text{C} = \dfrac{2z(2a - z)}{a}$, a being the semitransverse axis, $\because$

$$\text{H} = \frac{2az}{4a - z}.$$

Substituting for z, this becomes

$$\text{H} = \frac{p'}{2(1+e' \cos. \omega)},$$

where

$$p' = \frac{4ab^2}{4a^2-b^2},$$

$$e' = \frac{4a\sqrt{a^2-b^2}}{4a^2-b^2},$$

b being the semiconjugate axis of the given orbit. The locus is therefore an ellipse or hyperbola, whose parameter and eccentricity are p' and e', and whose transverse axis coincides with that of the given ellipse.

4. If the given orbit be a parabola, the force being at the focus. In this case $\text{C} = 4\text{D}$, $\therefore$

$$\text{H} = \tfrac{1}{2}\text{D},$$

$$\therefore \text{H} = \frac{p}{4(1-\cos. \omega)}.$$

The locus is therefore a parabola, whose axis is coincident with that of the given orbit, and which has the same focus.

5. If the orbit be an ellipse, with the force at the centre.

In this case $n = 2$, $\text{C} = \dfrac{2b'^2}{a'}$, $\text{D} = a'$, $\therefore$

$$\text{H}^2 = a'^2 - b'^2 = 2a'^2 - (a^2 + b^2).$$

And since by (173),

$$a'^2 = \frac{b^2}{1-e^2 \cos.^2 \omega};$$

the equation of the locus sought is

$$\text{H}^2 = \frac{2b^2}{1-e^2 \cos.^2 \omega} - (a^2 + b^2),$$

which is the equation of a curve of the fourth degree, similar in figure to the lemniscata.

For the hyperbola the equation becomes

$$\text{H}^2 = \frac{-2b^2}{1-e^2 \cos.^2 \omega} - (a^2 - b^2),$$

which, when $a^2 = b^2$, becomes

$$\text{H}^2 = \frac{2b^2}{2 \cos.^2 \omega - 1},$$

which is the equation of an equilateral hyperbola.

6. If the given orbit be the logarithmic spiral represented by

$$z = a^{\omega}.$$

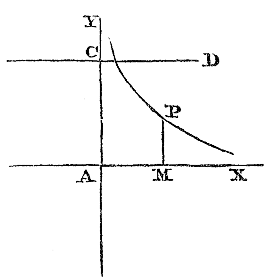
In this case $\text{D} = z$, $\text{C} = 2z$, and $n = -2$, ∴

$$\text{H} = \frac{1}{\sqrt{2}} z = \frac{1}{\sqrt{2}} a^{\omega}.$$

Hence the locus sought is also a logarithmic spiral.

PROP. CCCXLVI.

(656.) *A material point is moved by its own weight on a curve, the plane of which is vertical; to determine the perpendicular pressure on the curve.*



Let AY the axis of Y be vertical, and AX that of x horizontal; and let the co-ordinates of any point P be yx. Let the velocity V at P be that which would be acquired in falling freely from the horizontal line CD; and it follows from the principles of Mechanics, that the velocity at every point will be that which would be acquired in falling from the same line. Hence, if $\text{AC} = y'$,

$$\text{V}^2 = 4m(y' - y),$$

m being the space described freely in the unit of time.

The pressure exerted by the moving point at P in the direction of the normal is the effect of two causes; 1°, the weight g of the moving point resolved in the direction of the normal, and 2°, the force arising from the angular swing of the point round the centre of the osculating circle. The former is

$$g \cdot \sin. \theta = g \cdot \frac{dx}{(dy^2+dx^2)^{\frac{1}{2}}},$$

where θ is the angle at which the tangent at the point P is inclined to the horizon. The latter is

$$g \cdot \frac{v^2}{2mr} = g \cdot \frac{2(y'-y)}{r},$$

where r is the radius of the osculating circle, the value of which being substituted for it, gives

$$g \cdot \frac{v^2}{2mr} = g \cdot \frac{2(y'-y)d^2y \cdot dx}{(dy^2+dx^2)^{\frac{3}{2}}},$$

These two forces being united, give the whole pressure,

$$p = g \cdot \frac{dx}{(dy^2+dx^2)^{\frac{3}{2}}} \left\{ dy^2 + dx^2 + 2(y' - y)d^2y \right\}$$

It is evident that when the curve is convex towards the horizon, these two forces act in conjunction, and when concave, in opposition. The formula thus determined, however, accommodates itself to these cases by the sign of d^2y.

To determine the point at which the pressure is equal to the weight, let $p = g$, which gives

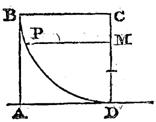
$$(dy^2 + dx^2)^{\frac{3}{2}} - (dy^2 + dx^2)dx - 2(y' - y)d^2y \cdot dx = 0,$$

$$\text{or } 2(y' - y)\frac{d^2y}{dx^2} + (1 + \frac{dy^2}{dx^2}) - (1 + \frac{dy^2}{dx^2})^{\frac{3}{2}} = 0.$$

If part of the pressure arising from the weight resolved in the direction of the normal act in opposition to the part arising from the centrifugal force, which will be always the case when the curve is concave towards the horizon; when these become equal, the pressure on the curve will vanish, and the body will fly off in the path of a projectile. To determine this point, let $p = 0$, ∴

$$2(y' - y) \cdot \frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} + 1 = 0.$$

These general formulæ serve for the determination of the pressure when the equation of the curve is given, or, when the law of the pressure is given, the species of the curve may be found by integration. We shall proceed to give some examples.



1. Let the curve be a circle, the point commencing its motion from the extremity (B) of the horizontal diameter. The equation of the circle being

$$y^2 + x^2 = r^2,$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y},$$

$$\frac{d^2y}{dx^2} = -\frac{r^2}{y^3}.$$

By these substitutions, and observing that $y' = 0$, we find

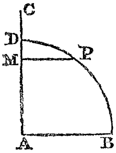
$$p = -g \cdot \frac{3y}{r}.$$

In this case, the part of the weight of the moving point resolved in the direction of the normal is $g \cdot \frac{y}{r}$, and therefore the part arising from the centrifugal force is twice this. It follows also that the pressure at the lowest point D is equal to three times the weight. This result evidently gives the tension of the string in a common pendulum; for the perpendicular pressure being exerted in the direction of the radius, is the force which would draw the radius were it the string of a pendulum.

The point P at which the pressure is equal to the weight, is found by taking CM $= \frac{1}{3}r$, and drawing MP.

2. Let the curve be a circle, the point commencing its

motion with a given velocity from the extremity D of the vertical diameter.



Let the velocity at D be that which would be acquired in falling through CD, and let AC $= y'$. In this case, making similar substitutions, we find

$$p = g \cdot \frac{3y - 2y'}{r}.$$

To find the point at which it will fly off, we have

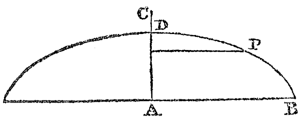
$$y = \tfrac{2}{3}y'.$$

Hence if CM $= \frac{1}{3}$AC, P will be the point.

If $y' = r$, that is, if C coincide with D, we have

$$p = g\,\frac{3y - 2r}{r}.$$

3. Let the curve be the cycloid, having its axis vertical, and the point commence to move from the vertex with a given velocity.



Let the velocity at D be that which would be acquired in falling through CD, and let AC $= y'$. By differentiating the equation of the cycloid twice, we have

$$\frac{dy^2}{dx^2} = \frac{2r - y}{y},$$

$$\frac{d^2y}{dx^2} = -\frac{r}{y^2},$$

which being substituted in the general formula, gives

$$p = g \cdot \frac{2y - y'}{\sqrt{2ry}}.$$

To determine the point where it will fly off, we have

$$y = \tfrac{1}{2}y'.$$

Hence if CA be bisected at M, the parallel MP determines the point.

If $y' = 2r$, and therefore the initial velocity be nothing, the point where it will fly off is determined by a parallel to the base through the middle point of the axis.

If the convexity of the cycloid is turned towards the horizon, y becomes negative, and the formula for the pressure becomes

$$p = -g \cdot \frac{2y + y'}{\sqrt{2ry}}.$$

When $y' = 0$, this becomes

$$p = -g \cdot \frac{2y}{\sqrt{2ry}} = -2g \cos. \theta.$$

The negative sign points out that the pressure is exerted against the concavity, and not the convexity. This result shows that the pressure is twice that which would be exerted if the body were quiescent on the curve. At the lowest point the pressure therefore is equal to double the weight.

These results apply to the tension of the string in the cycloidal pendulum ; and it may in general be observed, that the formula for the pressure always applies to the tension of a string to which the moving point may be supposed to be attached, and which is wound off from the evolute of the proposed curve (342).

A change of sign in the formula for the pressure indicates a change in the direction of the pressure, the positive sign indicating a pressure on the convexity, and a negative on the concavity. The sign changes at the point where the two forces already mentioned are equal and opposite, and at that point the moving point will fly off, unless it be supposed to change its position to the concave side of the curve in which the formula continues to represent the pressure. Thus, in the second example, if the point at P, where it would fly off, be supposed to be changed to the concave side of the curve, the formula continues to represent the pressure, and we thus find the pressure at B

$$p = -g \cdot \frac{2y'}{r}.$$

In this case, if $y' = r$, the pressure at B is equal to twice the weight.

In like manner, in the third example we find the pressure at B infinite. This is accounted for by the cusp at B through which it would require an infinite force to carry the moving point.

PROP. CCCXLVII.

(657.) *To determine a curve such, that a material point constrained to move in it by the force of gravity will descend with an uniform vertical velocity.*

Let the uniform vertical velocity be a, and the point being supposed to begin its motion from the axis of x, the velocity at any point yx of the curve will be $2\sqrt{gy}$, which being resolved in the vertical direction, gives

$$\frac{2\sqrt{gy}\,dy}{\sqrt{dy^2+dx^2}} = a,$$

$$\therefore 4gy\,dy^2 = a^2dy^2 + a^2dx^2,$$

$$\therefore dx = \frac{\sqrt{4gy-a^2}\,.\,dy}{a};$$

which being integrated, gives

$$x = \frac{(4gy-a^2)^{\frac{3}{2}}}{6ga};$$

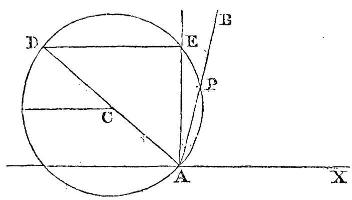
which by a transformation of origin may be reduced to the form

$$y^{\frac{3}{2}} = px:$$

the curve sought is therefore the semicubical parabola.

PROP. CCCXLVIII.

(658.) *A right line* AB, *fixed at* A *to an horizontal line* AX, *is moveable round* A *in a vertical plane passing through* AX, *to determine the locus of the point* P, *so assumed, that the time of descent of a body through* PA *shall be the same at all elevations, the friction being supposed proportional to the pressure.*



Let the space which a body will descend through freely in a vertical direction in the time of falling through PA be s, and let AE, perpendicular to AX, be equal to s.

By the principles of Mechanics, if T be the time of falling through s,

$$T^2 = \frac{s}{m},$$

where $m = 16\frac{1}{12}$ feet, and T being expressed in seconds.

If unity express the weight of the body, and $\omega = $ PAX, the pressure on AP is expressed by $\cos. \omega$, and the part of the weight acting in the direction PA by the $\sin. \omega$. Let the friction be $a \cos. \omega$, a being constant. Hence if F be the whole force in the direction PA, we have

$$F = \sin. \omega - a \cos. \omega.$$

This expression will be somewhat simpler by assuming an angle θ such, that $\tan. \theta = a$, which substituted in the above equation, gives

$$F = \sin. \omega - \tan. \theta \cos. \omega,$$

$$\text{or } F = \frac{\sin. (\omega - \theta)}{\cos. \theta}.$$

Now if PA $= z$, we have

$$z = m\text{FT}^2.$$

Substituting for F and T their values found above, we have

$$z = \frac{s}{\cos.\ \theta} \cdot \sin.\ (\omega - \theta),$$

which is the equation of the locus of P.

If $\omega' = \frac{\pi}{2} + \theta - \omega$, $\because \cos.\ \omega' = \sin.\ (\omega - \theta)$, by which the equation becomes

$$z = \frac{s}{\cos.\ \theta} \cos.\ \omega',$$

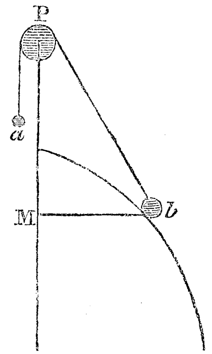
which is the equation of a circle whose centre is placed on the line from which the values of ω' are measured, and whose diameter is equal to $\frac{s}{\cos.\ \theta}$.

Hence the locus may be thus constructed. Let AD be drawn, making the angle EAD $= \theta$, and through E draw AE perpendicular to ED. Hence AD $= \frac{s}{\cos.\ \theta}$, and DAP $= \omega'$. The circle described on the diameter DA is therefore the locus.

The segment APE is the only part of the circle which will fulfil the conditions of the proposition. For in the other segment the tangent of θ would be negative, which would be equivalent to supposing the friction to act, not in opposition to, but in conjunction with the force down the line. Strictly speaking, therefore, the locus sought is the segment of a circle described upon AE containing an obtuse angle, whose tangent expresses the ratio of the friction to the pressure.

PROP. CCCXLIX.

(659.) *Two weights* (a *and* b) *are connected by a string which passes over a fixed pulley* (P); *one* (a) *hangs vertically; the other* (b) *is supported upon a curve, the plane of which is vertical; to determine the point on the curve at which the weights will be in equilibrio.*



Let PM and Mb be x and y, and P$b = r$. If X be the result of the forces acting on b, resolved in the vertical, and Y their result, resolved in the horizontal directions, by the principles of Statics, the condition of equilibrium is

$$Xdx + Ydy = 0;$$

but these forces are a acting in the direction bP, and b acting vertically; hence we find

$$Y = -a\frac{y}{r},$$

$$X = b - a\frac{x}{r}.$$

Hence the equation of equilibrium becomes

$$bdx - a\frac{xdx + ydy}{r} = 0;$$

but since

$$r^2 = y^2 + x^2,$$

$$\therefore rdr = ydy + xdx,$$

therefore the equation becomes

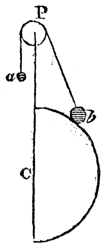
$$bdx - adr = 0;$$

or if the curve be represented by a polar equation,

$$dx = \cos.\ \omega dr - r \sin.\ \omega\, d\omega,$$

$$\therefore (b \cos.\ \omega - a)dr - br \sin.\ \omega\, d\omega = 0.$$

Either of these equations combined with that of the curve are sufficient to determine the point.



Ex. 1. If the curve be a circle whose centre is in the vertical line passing through the pulley, and which is therefore represented by the equation

$$(x - x')^2 + y^2 = R^2,$$

$$\text{or } r^2 - 2x'x = R^2 - x'^2,$$

$$\therefore rdr - x'dx = 0,$$

which, combined with the general formula, gives

$$r = \frac{ax'}{b}.$$

Hence if CP $= x'$, C being the centre, CP : Pb :: b : a, which determines the position of b.

If $a : b :: \sqrt{x'^2 - R^2} : x'$, the equilibrium will take place when Pb is a tangent.

If $a = b$, $\therefore x' = r$, a circle described with PC as radius, and P as centre, gives the point.

Ex. 2. Let the curve be an hyperbola, whose tranverse axis coincides with the vertical passing through the pulley, and so placed that the pulley is at the centre.

Let the equation of the curve be

$$A^2y^2 - B^2x^2 = -A^2B^2,$$

$$\therefore r^2 = e^4x^2 - B^2,$$

where $e = \frac{\sqrt{A^2 + B^2}}{A}$. By differentiating, we find

$$rdr = e^2xdx.$$

By this equation and

$$bdx - adr = 0,$$

the differentials being eliminated, we find

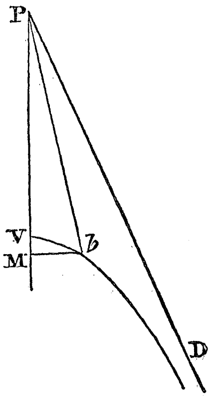
$$e^2x = \frac{b}{a} r.$$

If $x = r \cos. \omega$, we find

$$\cos. \omega = \frac{b}{a} \times \frac{A^2}{A^2 + B^2}.$$

Let PD be the asymptote to the curve, and let VPD $= \theta$, and VP$b = \omega$, therefore

$$\cos. \omega = \frac{b}{a} \cos.^2 \theta,$$



which determines the sought point.

If $\omega = 0$, the sought point is at an infinite distance, or what amounts to the same thing, the weight rests in equilibrio on the asymptote.

As ω cannot be greater than θ.

Hence, if $\frac{a}{b} > \cos. \theta$, the equilibrium is impo ible.

PROP. CCCL.

(660.) *To find the centre of gravity of the arc or area of any plane curve, its equation, related to rectangular co-ordinates, being given.*

By the principles of Mechanics, if any number of particles of matter p', p'', p''', be placed in the same plane, the perpendicular distance of their centre of gravity from any line in that plane is equal to the sum of the products of the particles into their respective distances from that line divided by the sum of the particles. Hence, if the sum of the products be represented by s(py), with respect to the axis of x, and by s(px), with respect to the axis of y, and the sum of the particles by s(p), the co-ordinates YX of the centre of gravity are

$$\left.\begin{aligned} Y &= \frac{s(py)}{s(p)} \\ X &= \frac{s(px)}{s(p)} \end{aligned}\right\} (1).$$

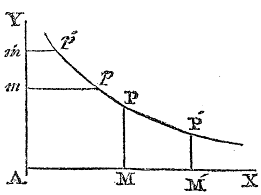
The numerators of these expressions are called the *statical moments* of the particles with respect to the respective axes.

If therefore it be required to find the centre of gravity of an arc (a), we have $p = da$, and s$(p) = a$, ∵

$$\left.\begin{aligned} Y &= \frac{\int y da}{a} \\ X &= \frac{\int x da}{a} \end{aligned}\right\} (2).$$

If it be required to find the centre of gravity of an area (a'), we have $dydx = p$, and $\int dydx = a'$: hence we find

$$\left.\begin{aligned} Y &= \frac{\iint ydxdy}{\iint dydx} \\ X &= \frac{\iint xdydx}{\iint dxdy} \end{aligned}\right\} (3).$$



If the area be intercepted between two values of y, scil. PM $= y$, and P'M' $= y'$; these formulæ, integrated for y, become

$$\left.\begin{aligned} Y &= \frac{\int y^2 dx}{2\int ydx} \\ X &= \frac{\int xydx}{\int ydx} \end{aligned}\right\} (4),$$

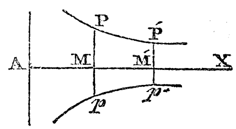
the integrals being taken between the limits y and y'.

If the area be intercepted between $pm = x$, and $p'm = x'$, the formulæ integrated for x become

$$\left.\begin{aligned} Y &= \frac{\int yxdy}{\int xdy} \\ X &= \frac{\int x^2dy}{2\int xdy} \end{aligned}\right\} (5).$$

In both the systems of values (4) and (5), one of the variables must be eliminated by means of the equation of the curve and its differential, and the results integrated between the required limits give the co-ordinates of the centre of gravity.

If the axis of x be an axis of the curve, and the centre of gravity of the area intercepted between two ordinates $\text{P}p$, and $\text{P}'p'$ must be upon the axis AX, and in this case the formulæ (4) become



$$\left.\begin{aligned} Y &= 0, \\ X &= \frac{\int xydx}{\int ydx} \end{aligned}\right\} (6).$$

And if the axis of y be an axis, and the centre of gravity of a similar area be sought, we have

$$\left.\begin{aligned} X &= 0, \\ Y &= \frac{\int xydy}{\int xdy} \end{aligned}\right\} (7).$$

It is plain that these systems of formulæ are independent of the angle of ordination.

We shall proceed to apply these formulæ to some examples.

1°. *To find the centre of gravity of a given straight line.*

The given line itself being axis of x, the formulæ (2) become

$$\text{Y} = 0,$$

$$\text{X} = \frac{\int x dx}{x} = \frac{x}{2};$$

the centre of gravity is therefore at the point of bisection.

2°. *To find the centre of gravity of a circular arc.*

Let the axis of x be the radius bisecting the arc, and the origin being at the centre, the equation is

$$y^2 + x^2 = r^2,$$

$$\therefore ydy = -xdx,$$

$$\therefore dy^2 + dx^2 = \frac{r^2}{x^2} \cdot dy^2;$$

but $(dy^2 + dx^2) = da^2$, and the formulæ (2) become

$$\text{Y} = 0,$$

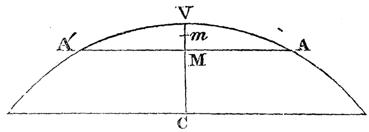
$$\text{X} = \frac{2ry}{a}.$$

Let 2ω be the angle subtended by the arc at the centre, $\therefore y = r \sin. \omega$, $a = 2r\omega$; hence

$$\text{X} = r\frac{\sin. \omega}{\omega}.$$

Hence in general, " The centre of gravity of a circular arc is that point of the bisecting radius, whose distance from the centre is to the radius as the sine of half the arc is to half the arc, or as the chord of the arc is to the arc itself." And it follows from this, that " The centre of gravity of a semicircle is at that point of the bisecting radius whose distance from the centre is a third proportional to the circumference and the diameter."

3°. *To find the centre of gravity of the arc of a cycloid, which is bisected at the vertex.*



The origin of co-ordinates being at the vertex V, let AA′ be the arc, V being its middle point. The centre of gravity must lie upon the axis VC = $2r$, and, by the properties of this curve, if the origin be at the vertex,

$$\text{VA} = 2\sqrt{2ry},$$
$$\therefore \text{AA}' = 4\sqrt{2ry},$$
$$\therefore da = 2\sqrt{2r} \cdot \frac{dy}{\sqrt{y}}.$$

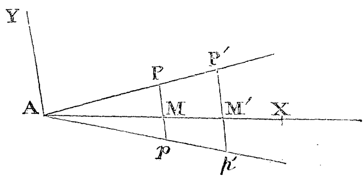
By this substitution, the formulæ (2) become

$$\text{Y} = \tfrac{1}{3}y,$$
$$\text{X} = 0.$$

Hence, if Vm = $\frac{1}{3}$VM, the point m is the centre of gravity.

The centre of gravity of the entire cycloid is at the point of trisection of the axis next the vertex.

4°. *To find the centre of gravity of the area intercepted by two parallels intersecting the sides of a given angle.*



Let AX be assumed bisecting the parallels, and AY parallel to them.

The equations of the sides of the angle are

$$y = \pm\, ax,$$

which substituted for y in (6), gives

$$\text{X} = \frac{a\!\int\! x^2 dx}{a\!\int\! x dx} = \tfrac{2}{3}\left(\frac{x^3 - x'^3}{x^2 - x'^2}\right) = \tfrac{2}{3} \cdot \frac{x^2 + xx' + x'^2}{x + x'};$$

$$\therefore \text{X} = \tfrac{2}{3}\left(x + \frac{x'^2}{x + x'}\right),$$

the integration being effected between the limits $x' =$ AM and $x =$ AM′.

If $x' = 0$, the formula gives the centre of gravity of the triangle AP′p', for which

$$\text{X} = \tfrac{2}{3}x.$$

It follows also that "The centre of gravity of a triangle is the point where the bisectors of the sides drawn from the opposite angles intersect."

5°. *To find the centre of gravity of the area intercepted between two parallel chords of a circle.*

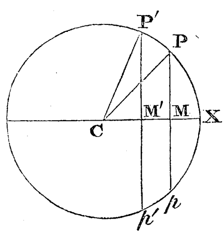
Axes of co-ordinates being assumed, with the origin at the centre parallel and perpendicular to the given chords, the equation of the circle is

$$y^2 + x^2 = r^2,$$
$$\therefore xdx = -ydy.$$

This substitution being made in the formulæ (6), gives

$$\text{X} = \frac{-\int y^2 dy}{\int ydx};$$

but $-\int y^2 dx = \frac{y'^3 - y^3}{3}$, the integral being assumed between



the limits $y =$ PM, and $y' =$ P′M, and $\int ydx =$ PM′ $= \frac{1}{2}$Pp', therefore if P$p' = a$,

$$\text{X} = \frac{2(y'^3 - y^3)}{3a}.$$

Or if the angle PCX $= \omega$, and P′CX $= \omega'$, we find

$$y'^3 - y^3 = r^3(\sin.^3 \omega' - \sin.^3 \omega),$$
$$a = r^2\{(\omega' - \omega) - \sin.(\omega' - \omega)\cos.(\omega' + \omega)\},$$

which substitutions give

$$\text{X} = \frac{2r}{3} \cdot \frac{\sin.^3 \omega' - \sin.^3 \omega}{(\omega' - \omega) - \sin.(\omega' - \omega)\cos.(\omega' + \omega)}.$$

If $\omega = 0$, the formula gives the centre of gravity of the segment $\text{P}'\text{X}p'$, which is determined by

$$\text{X} = \tfrac{2}{3}r \cdot \frac{\sin.^3 \omega'}{\omega' - \sin.\omega' \cos.\omega'}.$$

In this case, if $\omega' = \frac{\pi}{2}$, the result becomes

$$\text{X} = \tfrac{4}{3} \cdot \frac{r}{\pi},$$

which determines the centre of gravity of a semicircle. Hence "the centre of gravity of the area of a semicircle is that point in the bisecting radius, whose distance from the centre is a third proportional to three times the circumference and the diagonal of the circumscribed square."

6°. *To determine the centre of gravity of the area intercepted between two parallel chords of a parabola.*

Let the diameter to which those chords are ordinates be the axis of x, and the tangent through its vertex the axis of y, the equation will be

$$y^2 = px,$$

$$\therefore \frac{2ydy}{p} = dx.$$

This substitution in the formulæ (6) gives

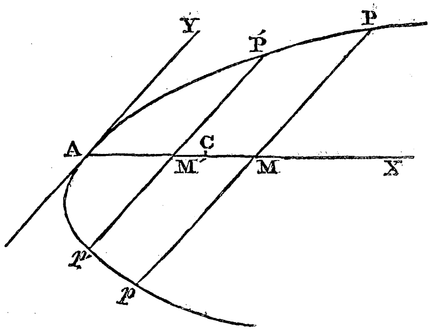
$$\text{X} = \frac{\int y^4 dy}{p\int y^2 dy},$$

which being integrated between the limits $y = \text{PM}$, and $y' = \text{P}'\text{M}'$, gives

$$\text{X} = \tfrac{3}{5} \cdot \frac{y^5 - y'^5}{p(y^3 - y'^3)}.$$

If $y = 0$, the result is

$$\text{X} = \tfrac{3}{5} \cdot \frac{y^2}{p} = \tfrac{3}{5}x.$$



Hence the centre of gravity of the parabolic segment PAp is determined by assuming AC equal to three-fifths of AM.

7°. *To determine the centre of gravity of the area intercepted between two parallel chords of an ellipse or hyperbola.*

Let the diameter bisecting those chords and its conjugate be assumed as axes of co-ordinates, and the equation of the curve is

$$a^2y^2 + b^2x^2 = a^2b^2,$$

$$\therefore xdx = -\frac{a^2}{b^2}ydy,$$

which being substituted in (6), gives

$$\text{X} = -\frac{2a^2}{b^2}\cdot\frac{\int y^2dy}{\text{A}}\cdot \sin. yx;$$

where A expresses the area intercepted between the parallel chords, and yx the angle of ordination. This being integrated between the usual limits, gives

$$\text{X} = \frac{2a^2}{3b^2}\cdot\frac{y'^3 - y^3}{\text{A}}\cdot \sin. yx.$$

If $y = 0$, this becomes

$$\text{X} = \frac{2a^2y'^3}{3b^2\text{A}}\cdot \sin. yx,$$

which determines the centre of gravity of an elliptic or hyperbolic segment.

If the curve be an ellipse, and $y' = b$, the formula becomes

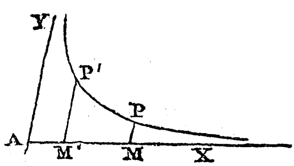
$$\text{X} = \frac{2a^2b}{3\text{A}} \cdot \sin. yx;$$

but in this case, the area being that of the semiellipse, becomes $\text{A} = \frac{1}{2}a'b'\pi$, a' and b' being the semiaxes; and since $ab \sin. yx = a'b'$, the formula becomes

$$\text{X} = \frac{4a}{3\pi}.$$

Hence the centre of gravity is independent of the diameter $2b$, and therefore all semiellipses, bisected by the same semidiameter, have the same centre of gravity, and which is determined in the same manner as that of a semicircle.

8°. *To find the centre of gravity of the area intercepted between two parallels to one asymptote of an hyperbola terminated in the other asymptote.*



The equation of the hyperbola, related to its asymptotes, is

$$yx = m^2,$$
$$\therefore ydx = -xdy,$$

$$\therefore xydx = -x^2dy = -\frac{m^4dy}{y^2}.$$

Also, $y^2dx = -yxdy = -m^2dy$. Making the substitutions in the formulæ (4),

$$\text{Y} = -m^2 \cdot \frac{\int dy}{2\int ydx},$$

$$\text{X} = -m^4 \cdot \frac{\int y^{-2}dy}{\int ydx};$$

but $\int y dx = -\int x dy = -m^2 \int \frac{dy}{y} = -m^2 ly + \text{C}$. The formulæ being integrated between the limits $y = \text{PM}$, and $y' = \text{PM}$, give

$$\text{Y} = \tfrac{1}{2} \cdot \frac{y'-y}{ly'-ly},$$

$$\text{X} = \frac{m^2}{y'y} \cdot \frac{y'-y}{ly'-ly}.$$

9°. *To determine the centre of gravity of the area intercepted between parallels to the axis of* Y *of a parabolic curve represented by the equation* $y^m = a^{m-1}x$.

By differentiation,

$$\frac{my^{m-1}}{a^{m-1}} dy = dx.$$

By this substitution, the formulæ (4) become

$$\text{Y} = \frac{m+1}{2(m+2)} \cdot \frac{y'^{m+2} - y^{m+2}}{y'^{m+1} - y^{m+1}},$$

$$\text{X} = \frac{m+1}{2m+1} \cdot \frac{y'^{2m+1} - y^{2m+1}}{a^{m-1}(y'^{m+1} - y^{m+1})}.$$

If $y = 0$, the values become

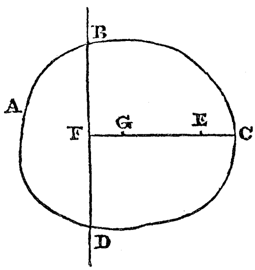
$$\text{Y} = \frac{m+1}{2(m+2)} \cdot y',$$

$$\text{X} = \frac{m+1}{2m+1} \cdot x'.$$

The examples 4° and 6° are cases of this.

PROP. CCCLI.

(661.) *To investigate the centres of gyration and percussion or oscillation of an arc or area of a plane curve.*



Let ABCD be a body moveable on an axis BD, and G its centre of gravity, and let CF be drawn through the centre of gravity perpendicular to the axis BD. In revolving on the axis, let the body be supposed to strike a fixed obstacle at any distance, FE $= d$ from the axis of rotation, with any force f; there is a certain point in the line FE, at which, if the whole mass of the body were concentrated, the line FE would strike the obstacle at E with the same force; this point is called the *centre of gyration.*

If the angular velocity at the moment of impact be given, the force (f) with which the body will strike the obstacle at E, will be a function of the distance (d). But since the moving force of the entire mass is independent of the point of impact, and the same at whatever distance the obstacle may be applied, the whole of it cannot be in all cases expended on the obstacle. What is not expended upon it must, from the nature of inertia, be exerted upon the fixed axis, which therefore sustains an equivalent shock. The entire moving force of the mass is then equal to the sum of the forces of the impact on the obstacle at E, and the shock upon the axis of rotation. There is a certain point in the line EF, at which, if the obstacle were placed, it would receive an impact equal to the whole moving force of the revolving mass, and at which therefore the axis would suffer no force. This point is called the *centre of percussion.*

The determination of the centre of percussion involves that of *spontaneous rotation*. As a fixed obstacle at the centre of percussion would destroy all the moving force of the body, without producing any effect on the axis of rotation, it follows, that a force applied at the same point upon the body supposed quiescent, would produce a rotatory motion round BD. Relatively therefore to any point not coinciding with the centre of gravity, there is a corresponding centre of *spontaneous rotation*, which may be determined by considering the former as the centre of percussion, and thence determining the axis of rotation.

The centre of percussion possesses also a still more remarkable property. If the axis of motion be horizontal, and the body vibrate as a pendulum, the time of its vibration will be the same as that of a single particle suspended at the centre of percussion. From this property, the centre of percussion is more generally called the *centre of oscillation*. We shall now proceed to determine these points.

Let p be any particle of the body, and z its distance from the axis of motion, and let ω be the angular velocity on the axis of motion at the moment of impact; the velocity of the particle p will therefore be $z\omega$, and its moving force $pz\omega$. The quantity of this which acts at the point E is $\frac{pz^2\omega}{d}$. As the whole impact upon the obstacle at E is composed of the forces of all the particles in the body, the value of f may be thus expressed:

$$f = \frac{\omega \text{s}(pz^2)}{d},$$

where $\text{s}(pz^2)$ signifies the sum of the products found by multiplying every particle of the body by the square of its distance from the axis of rotation.

The quantity $\text{s}(pz^2)$, which is called the *moment of inertia* of the body with respect to the axis on which it is supposed to revolve, is of considerable importance in all theorems relating to the motion of a body upon a fixed axis. The determination of its value depends upon the figure and mass of the body, and the line on which it is supposed to revolve. We shall presently give the methods of determining it where the particles of the body form any line or plane surface.

Let G be the distance of the centre of gyration from the axis of rotation. By the conditions of its definition, we find

$$f = \frac{\omega \text{S}(p\text{G}^2)}{d} = \frac{\omega \text{MG}^2}{d},$$

where M expresses the mass of the body. Hence,

$$\text{G}^2 = \frac{\text{S}(pz^2)}{\text{M}};$$

that is, "the square of the distance of the centre of gyration from the axis of motion is found by dividing the moment of inertia by the mass of the body."

It appears, therefore, that the same body may have different centres of gyration corresponding to different axes of motion. Of these, that which corresponds to an axis passing through the centre of gravity is called the principal centre of gyration. A remarkable relation subsists between the position of this and any other centre of gyration. Let z' express the distance of any particle p from the axis of motion passing through the centre of gravity, and D the distance between this and any other axis parallel to it. It is obvious that

$$z^2 = z'^2 + \text{D}^2 + 2\text{D}z',$$

$$\therefore pz^2 = pz'^2 + p\text{D}^2 + 2\text{D}pz',$$

$$\therefore \text{S}(pz^2) = \text{S}(pz'^2) + \text{S}(p\text{D}^2) + 2\text{D}\text{S}(pz').$$

If G$'$ be the distance of the principal centre of gyration from the centre of gravity,

$$\text{G}'^2 = \frac{\text{S}(pz'^2)}{\text{M}}.$$

Hence, since $\text{G}^2\text{M} = \text{S}(pz^2)$, and $\text{G}'^2\text{M} = \text{S}(pz'^2)$, and as $\text{S}(pz')$ is the statical moment of the body with respect to a line through the centre of gravity, and therefore $\text{S}(pz') = 0$, we find

$$G^2M = G'^2M + D^2M,$$
$$\therefore G^2 = G'^2 + D^2.$$

Hence the centre of gyration, with respect to a y axis, can be found if the centre of gyration, with respect to an axis parallel to it through the centre of gravity, be given.

To determine the centre of percussion, it will be necessary to estimate the shock sustained by the axis of rotation by the impact of the body on the point E. Let this be f'. Now, if the entire moving force of the mass be F, the condition

$$F = f + f'$$

must be fulfilled. As the moving force of the whole mass is composed of the moving forces of its parts, the value of F may be found by considering that the moving force of any particle p is $pz\omega$, and therefore

$$F = \omega S(pz),$$
$$\therefore \omega S(pz) = \frac{\omega S(pz^2)}{d} + f',$$
$$\therefore f' = \omega \left\{ S(pz) - \frac{S(pz^2)}{d} \right\}.$$

This equation expresses the shock sustained by the axis of rotation. To determine the centre of percussion, d must be assumed of such a value as to fulfil the condition

$$f' = 0.$$

Let this value be L; $\therefore$

$$S(pz) - \frac{S(pz^2)}{L} = 0,$$
$$\therefore L = \frac{S(pz^2)}{S(pz)};$$

that is, "the distance of the centre of percussion from the axis of rotation is found by dividing the moment of inertia by the statical moment."

It is not difficult to prove that the centre of percussion is also the centre of oscillation.

Let the axis of motion be conceived to be placed in an horizontal position, and the force of gravity acting on the body will cause it to have a motion of vibration. The body being supposed to have descended from any position, let it be supposed to have acquired an angular velocity ω, when the angular distance of its centre of gravity from a vertical passing through the axis of motion is ϕ, and let the angular distance of any particle p from the vertical be ψ. The part of the force of gravity (g) which acts in accelerating the particle p is g sin. ψ, and therefore the force impressed on the particle is gp sin. ψ. In the equation

$$fd = \varpi s(pz^2)$$

if $gs(pz \sin. \psi)$ be substituted for fd, and $\frac{d\omega}{dt}$ for ω, the result is

$$gs(pz \sin. \psi) = \text{MD} \sin. \phi,$$

$$\therefore g\text{MD} \sin. \phi = \frac{d\omega}{dt} \, . \, s(pz^2).$$

Also, since $\omega = -\frac{d\phi}{dt}$, $\therefore d\omega = -\frac{d^2\phi}{dt}$. Hence,

$$g\text{MD} \sin. \phi = -\frac{d^2\phi}{dt^2} s(pz^2).$$

Multiplying both sides of this equation by $2d\phi$, and integrating, the result is

$$2g\text{MD} \cos. \phi + \text{A} = \frac{d\phi^2}{dt^2} s(pz^2);$$

$$\text{or, } 2g\text{MD} \cos. \phi + \text{A} = \omega^2 \, . \, s(pz^2).$$

To determine the constant A, let ϕ' be the value of ϕ when $\omega = 0$; $\therefore$

$$2g\text{MD} \cos. \phi' + \text{A} = 0.$$

Subtracting this equation from the former, and solving for ω^2

$$\omega^2 = \frac{2g\text{MD} (\cos. \phi - \cos. \phi')}{s(pz^2)}.$$

This equation determines the angular velocity acquired by the body in falling from a quiescent state through the angle

$\varphi' - \varphi$. To determine the distance of a single particle p from the axis, which, in moving through the same angle would acquire the same velocity, let d be its distance.

To apply the general formula just found to this case, we must suppose

$$\text{M} = p,\ \text{D} = d, \text{ and } \text{S}(pz^2) = pd^2. \quad \text{Hence,}$$

$$\omega^2 = \frac{2gpd\,(\cos.\ \varphi - \cos.\ \varphi')}{pd^2}.$$

Dividing the former equation by this, we find

$$d = \frac{\text{S}(pz^2)}{\text{MD}} = \frac{\text{S}(pz^2)}{\text{S}(pz)},$$

$$\therefore d = \text{L}.$$

Hence, a point placed at the distance L from the centre of motion would vibrate in the same time as the body itself. The calculation of the time of vibration of any pendulous body is thus reduced to the case of a simple pendulum.

Since $\text{S}(mz^2) = \text{MG}^2$, and also

$$\text{G}^2 = \text{G}'^2 + \text{D}^2,$$

$$\therefore \text{L} = \frac{\text{G}'^2}{\text{D}} + \text{D}.$$

By this formula, the centre of oscillation may be immediately found from the principal centre of gyration.

From the formula

$$\text{L} = \frac{\text{S}(mz^2)}{\text{S}(mz)} = \frac{\text{G}^2}{\text{D}}$$

it follows, that "the distances of the centres of gravity, gyration, and oscillation, from the centre of motion, are in continued proportion."

Also, since

$$\text{L} - \text{D} = \frac{\text{G}'^2}{\text{D}},$$

it follows, that "the distance between the centres of gravity and oscillation is a third proportional to the distances between the centres of gravity and motion, and between the principal centre of gyration and the centre of gravity."

If the axis of suspension be changed to the centre of oscillation, let L' be the distance of the new centre of oscillation, and D' the distance of the centre of gravity from the new axis of motion. Hence,

$$L' = \frac{G'^2}{D'} + D';$$

but $D' = L - D = \frac{G'^2}{D}$. Therefore,

$$L' = D + \frac{G'^2}{D},$$

$$\text{or, } L' = L;$$

that is, the former point of suspension becomes the new centre of oscillation. The centres of oscillation and rotation are therefore convertible.

In applying these principles to the arcs and areas of plane curves, we shall confine ourselves to the case where the axis of rotation is in the plane of the curve.

I. *To determine the moment of inertia of the arc of a plane curve revolving round an axis in its own plane.*

Let the axis of motion be taken as axis of y, and a perpendicular to it as axis of x, and let s equal the arc. In this case, $p = ds$, and $z = x$; therefore,

$$S(pz^2) = \int x^2 ds.$$

Ex. 1. If the revolving line be a right line represented by the equation

$$y - \tan.\ \phi\ .\ x + b = 0.$$

By differentiating, we find

$$dy - \tan.\ \phi\ .\ dx = 0,$$

$$\because ds = \sqrt{dy^2 + dx^2} = \sec.\ \phi\ .\ dx,$$

$$\therefore S(pz^2) = \sec.\ \phi \int x^2 dx.$$

Integrating this between the limits x and x',

$$\text{s}(pz^2) = \text{sec. } \phi \,.\, \frac{x^3 - x'^3}{3},$$

x and x' being the values of x for the extremities of the line.

If the length of the line be M, we find

$$\text{M} = \text{sec. } \phi(x - x'),$$

$$\therefore \text{s}(pz^2) = \tfrac{1}{3}\text{M}(x^2 + x'x + x'^2).$$

Hence we find the centre of gyration

$$\text{G}^2 = \tfrac{1}{3}(x^2 + x'x + x'^2).$$

And since the centre of gravity is the middle point of the line

$$\text{D} = \tfrac{1}{2}(x + x'),$$

$$\therefore \text{L} = \frac{2(x^2 + x'x + x'^2)}{3(x + x')},$$

which determines the centre of oscillation.

If the line be parallel to the axis of y, $x = x'$, $\therefore$

$$\text{G} = x,$$

$$\text{L} = x.$$

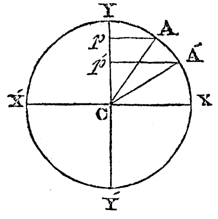
The centres of oscillation, gyration, and gravity, therefore, coincide in this case.

If the axis of y pass through the middle point of the line $x = -x'$, hence,

$$\text{G}'^2 = \tfrac{1}{3}x^2.$$

Ex. 2. Let the revolving line be the arc of a circle represented by the equation

$$y^2 + x^2 = r^2,$$



$$\therefore ds = \frac{rdy}{x},$$

$$\therefore \text{s}(pz^2) = r\int xdy.$$

Hence, if AA′ be the arc, and YY′ the axis of rotation, $\int xdy$ is the area pAA′p'. To express this, let ACX $= \phi$, and A′CX $= \phi'$, $\therefore$

$$p\text{AA}'p' = \text{CA}p + \text{CAA}' - \text{CA}'p',$$

$\therefore \int x dy = \frac{1}{2}r^2 \{\phi - \phi' + \sin.\phi \cos.\phi - \sin.\phi' \cos.\phi'\}$

$\therefore s(pz^2) = \frac{1}{2}r^3 \{\phi - \phi' + \frac{1}{2}\sin. 2\phi - \frac{1}{2}\sin. 2\phi'\}.$

Let the length of the arc AA$'$ be M, and

$$M = r((\phi - \phi').$$

To find the centre of gyration, we have, therefore,

$$G^2 = \frac{1}{2}r^2 \left\{1 + \frac{\sin. 2\phi - \sin. 2\phi'}{2\phi - 2\phi'}\right\}.$$

The distance of the centre of gravity from the centre of the circle being

$$r \cdot \frac{\sin. \frac{1}{2}(\phi - \phi')}{\frac{1}{2}(\phi - \phi')}.$$

Its distance from the axis of rotation is

$$D = r\frac{2\sin. \frac{1}{2}(\phi - \phi') \cos. \frac{1}{2}(\phi + \phi')}{\phi - \phi'},$$

$$\therefore D = r\frac{\sin. \phi - \sin. \phi'}{\phi - \phi'}.$$

Hence, for the centre of oscillation, we have

$$L = \frac{1}{2}r\left\{\frac{\phi - \phi'}{\sin. \phi - \sin. \phi'} + \frac{\sin. 2\phi - \sin. 2\phi'}{2\sin. \phi - 2\sin. \phi'}\right\}.$$

If the point X bisects the arc AA$'$, we have $\phi = -\phi'$, and the several formulæ become

$$s(pz^2) = \frac{1}{2}r^3\{2\phi + \sin. 2\phi\},$$

$$G^2 = \frac{1}{2}r^2\left\{1 + \frac{\sin. 2\phi}{2\phi}\right\},$$

$$D = r\frac{\sin. \phi}{\phi},$$

$$L = \frac{1}{2}r\left\{\frac{\phi}{\sin. \phi} + \frac{\sin. 2\phi}{2\sin. \phi}\right\}.$$

For the semicircle YXY$'$, $\phi = \frac{\pi}{2}$, and $\phi' = -\frac{\pi}{2}$, and the formulæ become

$$s(pz^2) = \frac{1}{2}r^3\pi,$$

$$G^2 = \frac{1}{2}r^2,$$

$$\text{D} = \frac{2r}{\pi},$$

$$\text{L} = \frac{r\pi}{4}.$$

For the whole circle these become

$$\text{s}(pz^2) = r^3\pi,$$
$$\text{G}^2 = \tfrac{1}{2}r^2,$$
$$\text{D} = 0,$$
$$\text{L} = \infty.$$

Ex. 3. To determine the moment of inertia of a cycloidal arc terminated at the vertex, and revolving on its base.

The base being assumed as axis of x, and the axis being expressed by a,

$$s = 2\sqrt{a(a-y)},$$

$$\therefore ds = -\sqrt{a} \cdot \frac{dy}{\sqrt{a-y}},$$

$$\therefore \text{s}(pz^2) = -\sqrt{a}\int \frac{y^2 dy}{\sqrt{a-y}}.$$

To integrate this, let $a - y = y'$, $\therefore dy = -dy'$,

$$\therefore \text{s}(pz^2) = \sqrt{a}\frac{(a-y')^2 dy'}{\sqrt{y'}},$$

$$\therefore \text{s}(pz^2) = \sqrt{a}\left\{a^2 \cdot \int \frac{dy'}{\sqrt{y'}} + \int y'^{\frac{3}{2}} dy' - 2a\int y'^{\frac{1}{2}} dy'\right\}$$

$$\therefore \text{s}(pz^2) = \sqrt{a}\left\{2a^2 y'^{\frac{1}{2}} + \tfrac{2}{5}y'^{\frac{5}{2}} - \tfrac{4}{3}ay'^{\frac{3}{2}}\right\}.$$

No constant is supplied, as the arc is supposed to terminate at the vertex; and if the vertex be taken as the middle point of the arc, the expression becomes

$$\text{s}(pz^2) = 4\sqrt{ay'}\left\{a^2 + \tfrac{1}{5}y'^2 - \tfrac{2}{3}ay'\right\}.$$

In this formula it should be observed, that y' is measured upon the axis from the vertex.

Since $s = 4\sqrt{ay'}$, we find

$$\text{G}^2 = a^2 + \tfrac{1}{5}y'^2 - \tfrac{2}{3}ay';$$

and since D $= a - \frac{1}{3}y'$,

$$L = \frac{a^2 + \frac{1}{5}y'^2 - \frac{2}{3}ay'}{a - \frac{1}{3}y'}.$$

For the whole cycloid $y' = a$, and the formulæ become

$$s(pz^2) = \tfrac{32}{15}a^3,$$
$$G^2 = \tfrac{8}{15}a^2,$$
$$L = \tfrac{4}{5}a.$$

Hence it appears that the time of vibration of the entire cycloid is equal to the time of vibration of a simple pendulum, the length of which is equal to four-fifths of the axis.

II. *To determine the moment of inertia of the area of a plane curve revolving on an axis in its own plane.*

As before, let the axis of motion be assumed as axis of y, and a perpendicular to it as axis of x, and let A be the area. Since $p = dA = dydx$, we find

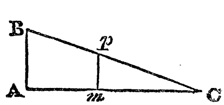
$$s(pz^2) = \int\int x^2 dydx,$$

which being integrated for y, gives

$$s(pz^2) = \int x^2 ydx,$$

either of the co-ordinates x or y being eliminated by means of the equation of the curve, and the result integrated for the other, the integral so determined, taken between proper limits, will be the moment of inertia sought.

Ex. 1. To determine the moment of inertia of the area of a right angled triangle revolving round one of the sides of the right angle.



Let BA $= y'$, and AC $= x'$, the equation of BC is

$$x'y + y'x - y'x' = 0,$$

$$\therefore y = \frac{y'(x' - x)}{x'},$$

$$\therefore \text{s}(pz^2) = \frac{y'}{x'}\int x^2(x' - x)dx,$$

which, by integration, gives

$$\text{s}(pz^2) = \frac{y'x^3}{x'}\left(\frac{x'}{3} - \frac{x}{4}\right).$$

No constant is added, the area being supposed $= 0$ when $x = 0$. The integral, as it stands, expresses the moment of inertia of the trapezium BpmA, in which A$m = x$. To determine that of the triangle, let $x = x'$, $\therefore$

$$\text{s}(pz^2) = \frac{y'x'^3}{12}.$$

And since the area of the triangle is $\frac{1}{2}y'x'$, we find

$$\text{G}^2 = \tfrac{1}{6}x'^2;$$

but $\text{D} = \frac{1}{3}x'$: hence

$$\text{L} = \tfrac{1}{2}x'.$$

Ex. 2. If the triangle be supposed to move on an axis passing through C, perpendicular to CA, to determine the moment of inertia.

In this case, if the co-ordinates of the point B be $y'x'$, the equation of CB is

$$x'y - y'x = 0,$$

$$\therefore \text{s}(pz^2) = \frac{y'}{x'}\int x^3dx,$$

$$\therefore \text{s}(pz^2) = \frac{y'}{x'} \cdot \frac{x^4}{4},$$

which, extended to the entire area, gives

$$\text{s}(pz^2) = \frac{y'x'^3}{4}.$$

Since the area $= \frac{1}{2}y'x'$, and $\text{D} = \frac{2}{3}x'$, we find

$$\text{G}^2 = \tfrac{1}{2}x'^2,$$

$$\text{L} = \tfrac{3}{4}x'.$$

Ex. 3. If the curve be a parabola of any order represented by the equation

$$y = px^m.$$

By substituting this value in the general formula, and integrating

$$\text{s}(pz^2) = \frac{p}{m+3} \cdot x^{m+3}.$$

Since

$$\int y dx = \frac{p}{m+1} x^{m+1},$$

$$\therefore \text{G}^2 = \frac{m+1}{m+3} x^2;$$

but $\text{D} = \frac{m+1}{m+2} x$: hence

$$\text{L} = \frac{m+2}{m+3} x.$$

In the common parabola $m = \frac{1}{2}$, $\therefore$

$$\text{G}^2 = \tfrac{3}{7} x^2,$$

$$\text{L} = \tfrac{5}{7} x.$$

Ex. 4. Let the curve be a circle represented by the equation

$$y^2 + x^2 - r^2 = 0,$$

$$\therefore \text{s}(pz^2) = \int x^2 \sqrt{r^2 - x^2}\, dx.$$

If $x = r \cos. \varphi$, and $y = r \sin. \varphi$, this becomes

$$\text{s}(pz^2) = r^4 \int \sin.^2 \varphi \cos.^2 \varphi d\varphi,$$

which being integrated, gives

$$\text{s}(pz^2) = \tfrac{1}{4} r^4 \left\{ \sin.^3 \varphi \cos. \varphi - \tfrac{1}{2} \sin. \varphi \cos. \varphi + \tfrac{1}{2} \varphi \right\};$$

no constant being added, this expresses the moment of inertia of the area included by the sine, versed sine, and arc. To express that of the segment contained by the arc, the chord of which is $2y$, it becomes

$$\text{s}(pz^2) = \tfrac{1}{2} r^4 \left\{ \sin.^3 \varphi \cos. \varphi - \tfrac{1}{2} \sin. \varphi \cos. \varphi + \tfrac{1}{2} \varphi \right\}.$$

If A be the area of the segment

$$\text{A} = r^2(\varphi - \sin. \varphi \cos. \varphi).$$

Hence we find

$$G^2 = \tfrac{1}{2}r^2\left\{\tfrac{1}{2} + \frac{\sin.^3 \varphi \cos. \varphi}{\varphi - \sin. \varphi \cos. \varphi}\right\}.$$

Extending this to the semicircle, let

$$\varphi = \frac{\pi}{2}, \because$$

$$G^2 = \tfrac{1}{4}r^2;$$

but in this case,

$$D = \frac{4r}{3\pi},$$

$$\because L = \frac{3r\pi}{16}.$$

PROP. CCCLII.

(662.) *A vessel sails between two light-houses; to find the track she must describe so as to receive an equal quantity of light from each.*

The intensity of the lights being expressed by m, m', and the distances z, z', the quantities of light will be $\frac{m}{z^2}$, $\frac{m'}{z'^2}$; hence we have

$$mz'^2 = m'z^2.$$

This equation being expressed with relation to rectangular axes, one of which is the line joining the light-houses, becomes

$$m\left\{y^2 + (x - x')^2\right\} = m'\left\{y^2 + (x - x'')^2)\right\},$$

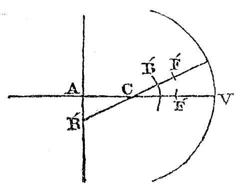
x', x'', being the distances of the light-houses from the origin of co-ordinates. This equation, when disposed according to the dimensions of the variables, is

$$y^2 + x^2 - \frac{2(mx' - m'x'')}{m - m'}x + \frac{mx'^2 - m'x''^2}{m - m'} = 0.$$

When m and m' are not equal, this is the equation of a circle, the centre of which is on the axis of x at the distance $\frac{mx' - m'x''}{m - m'}$ from the origin. But if $m = m'$, it is a right perpendicular to and bisecting the right line joining the light-houses.

PROP. CCCLIII.

(663.) *To find the image of a straight line in a spherical reflector.*



Let C be the centre of the reflector, and CV that radius which is perpendicular to the straight line AB, and F the principal focus. Let CA $= a$, CF $= b$, ACB $= \omega$, and B' being the image of any point, let CB' $= z$.

By the principles of optics,

$$\text{BF} \cdot \text{B'F} = \text{CF}^2,$$

which gives the equation

$$\left(\frac{a}{\cos.\omega} + b\right)(b - z) = b^2,$$

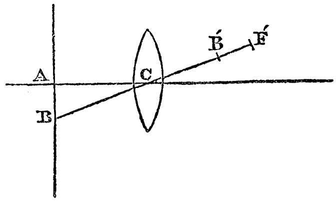
which solved for z, is

$$z = \frac{b}{1 + \frac{b}{a}\cos.\omega}.$$

Hence the image is a line of the second degree, whose species depends on the distance of the straight line from the centre of the reflector; if this distance be less than half the radius, the image is an ellipse; if equal, a parabola; and if greater, an hyperbola.

PROP. CCCLIV.

(664.) *To find the image of a straight line made by a lens.*



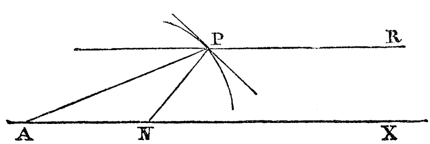
Let C be the centre of the lens, and, as in the last proposition, let CA $= a$, CF $= b$, ACB $= \omega$, CB$' = z$. By the principles of optics, as before, we find

$$z = \frac{b}{1 + \frac{b}{a} \cos. \omega}.$$

Hence the same conclusions follow as in the last proposition.

PROP. CCLV.

(665.) *To find a refracting curve such, that parallel homogeneous rays incident on it shall be all refracted to the same point.*



Let the incident rays PR be parallel to AX, and let PN be the normal to the point P. By the principles of optics, the sine of the angle of incidence bears an invariable ratio to the sine of the angle of refraction, the medium being supposed of uniform density. Now, the angle PNX is the angle of incidence, and the angle APN is the angle of refraction, therefore

$$\frac{\sin. \text{APN}}{\sin. \text{PNX}} = m,$$

m being constant. By (327) we find

$$\tan. \text{APN} = -\frac{dz}{zd\omega}$$

where $z = \text{AP}$, and $\omega = \text{PAX}$. Hence

$$\sin. \text{PNX} = \sin. \omega \cos. \text{APN} + \cos. \omega \sin. \text{APN}.$$

By these substitutions in the first equation, the result is

$$\frac{dz}{zd\omega} = m\left(\frac{dz}{zd\omega}\cos. \omega - \sin. \omega\right),$$

$$\text{or } dz = m(\cos. \omega dz + zd\cos. \omega),$$

which, by integration, gives

$$z = mz\cos. \omega + mc,$$

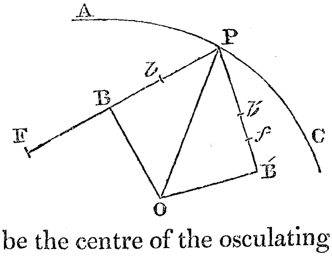
$$\text{or } z = \frac{mc}{1 - m\cos. \omega}.$$

This is the equation of a line of the second degree, whose eccentricity is m, and whose axis coincides with AX, and whose focus is A. The curve will be an ellipse or hyperbola, according as $m < 1$, or $m > 1$, that is, according as the rays are supposed to pass into a denser or rarer medium.

The value of C remains indeterminate, which shows that any curve having the eccentricity equal to m, whatever its axes may be, will fulfil the proposed conditions.

PROP. CCCLVI.

(666.) *To determine the caustic by reflection of a given curve.*



Let APC be the curve, F the focus of incident rays, FP being a ray incident at P; if PO be normal to the curve, its reflection PB′ will make the angle OPB′ = OPB. Now, by the principles of optics, if O be the centre of the osculating circle, and the lines OB, OB′, be perpendicular to PB, PB′, and the intercepts PB, PB′, be bisected at b, b', the corresponding point f of the caustic is found by taking $b'f$ a third proportional to Fb and bP. Since OP is the radius of the osculating circle, PB must be half the chord which coincides with the incident ray; let this chord be C, and let FP $= z'$, and P$f = v$. By the conditions already expressed,

$$v = \frac{z'\text{C}}{4z'-\text{C}}.$$

From this equation, that of the curve and the known value of C, the locus of f may, without difficulty, be determined.

If the incident rays be parallel, z' is infinite, therefore, in this case, the formula becomes

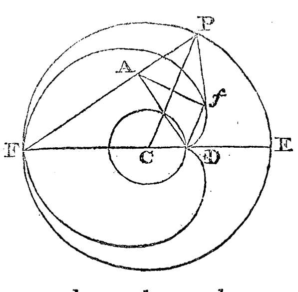
$$v = \tfrac{1}{4}\text{C}.$$

PROP. CCCLVII.

(667.) *To find the caustic when the reflecting curve is a circle, the focus of incident rays being on its circumference.*

In this case C $= z' =$ PF, therefore

$$v = \frac{z'^2}{3z'} = \tfrac{1}{3}z'.$$



Since fP $= \tfrac{1}{3}$FP, a parallel to CP through f will intercept CD equal to a third of the radius, therefore D is a fixed point. Let fA be perpendicular to CP, and join DA. It is obvious that DfA is a right angle, and since DE and PA bear the same ratio to DF and AF, the angle DAF is right. Let the angle PFE be ω', and fDE $= \omega$, and CP $= r$, and D$f = z$,

$$z = \text{DA} \sin.\ \omega',$$

$$\text{DA} = \tfrac{4}{3}r \sin.\ \omega',$$

$$\therefore z = \tfrac{4}{3}r \sin.^2 \omega';$$

and since $\omega' = \tfrac{1}{2}\omega$, the equation of the caustic is

$$z = \tfrac{4}{3}r \sin.^2 \tfrac{1}{2}\omega,$$

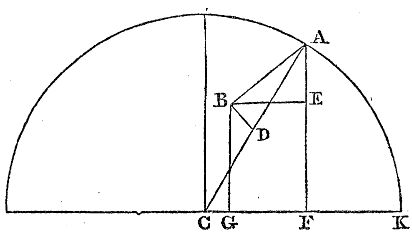
$$\text{or } z = \tfrac{2}{3}r(1 - \cos.\ \omega).$$

which is the equation of a cardioide, whose base is a circle concentrical with the given circle, and the radius of which is one-third of the radius of the given circle.

If therefore CD $= \frac{1}{3}$CF, and a circle be described with C as centre, and CD as radius, the cardioide, whose base is this circle, is the *caustic*.

PROP. CCCLVIII.

(668.) *To find the caustic by reflection, when the reflecting curve is a circle, and parallel rays are incident in the plane of the circle.*



Let CK be a radius of the circle drawn perpendicular to the incident rays; and let FA be a ray incident at A and let CAF = CAB, AB will be the reflected ray; and if CA be bisected at D, and DB be drawn perpendicular to BA, the point B will be the focus of an indefinitely small pencil of rays incident at A, and parallel to FA. The object is therefore to determine the locus of the point B.

Let BE and BG be drawn parallel to CK and AF respectively. And let BG $= x$, CG $= y$, AC $= r$, and ACK $= \phi$. By what has been stated, we find

$$\text{AB} = \tfrac{1}{2}r \sin. \phi,$$

$$\text{AE} = \text{AB} . \cos. (\pi - 2\phi) = -\text{AB} . \cos. 2\phi = -\tfrac{1}{2}r \sin. \phi \cos. 2\phi,$$

$$\therefore \text{AE} = -\tfrac{1}{2}r \sin. \phi \cos. 2\phi,$$

$$\text{BE} = \text{AB} . \sin. (\pi - 2\phi) = \text{AB} \sin. 2\phi,$$

$$\therefore \text{BE} = \tfrac{1}{2}r \sin. \phi \sin. 2\phi.$$

But since $x =$ AF $-$ AE, and $y =$ CF $-$ GF, and AF $= r \sin. \phi$, CF $= r \cos. \phi$, $\therefore$

$$x = r \sin. \phi + \tfrac{1}{2}r \sin. \phi \cos. 2\phi,$$
$$y = r \cos. \phi - \tfrac{1}{2}r \sin. \phi \sin. 2\phi;$$

but by trigonometry,

$$\sin. \phi \cos. 2\phi = \tfrac{1}{2}(\sin. 3\phi - \sin. \phi),$$
$$-\sin. \phi \sin. 2\phi = \tfrac{1}{2}(\cos. 3\phi - \cos. \phi).$$

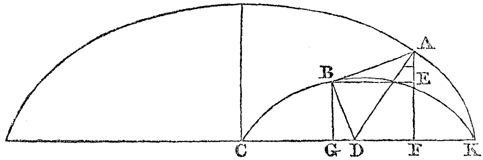
Making these substitutions, we find, after reduction,

$$x = (\tfrac{1}{2}r + \tfrac{1}{4}r) \sin. \phi + \tfrac{1}{4}r \sin. 3\phi$$
$$y = (\tfrac{1}{2}r + \tfrac{1}{4}r) \cos. \phi + \tfrac{1}{4}r \cos. 3\phi.$$

By comparing these equations with those of epicycloids in (507), it is obvious that the *caustic* is an epicycloid, the base of which is concentrical with the given circle, its radius being half that of the given circle, and the radius of the generating circle being one-fourth of the radius of the given circle.

PROP. CCCLIX.

(669.) *To find the caustic by reflection, the reflecting curve being the common cycloid, and the incident rays being parallel to its axis.*



Let C be the centre of the base of the cycloid, and let FA be a ray incident at A, and DA being the normal to the point A, let DAF = DAB, and let DB be drawn perpendicular to DB, the point B will be the focus of reflection of a small pencil of parallel rays incident at A in the direction FA. For, since by the properties of the cycloid already proved, DA is half the radius of the osculating circle (492), the pencil of rays parallel to AF must be brought to a focus at B. The object is therefore to find the locus of the point B.

Let BG and BE be parallel to AF and CK respectively, and let AF $= y'$, CF $= x'$, BG $= y$, and CG $= x$, and DAF $= \phi$. Since BA $=$ AF, and BAF $=$ 2DAF, we have

$$y' - y = y' \cos. 2\phi,$$
$$x' - x = y' \sin. 2\phi;$$

but $\cos. 2\phi = \frac{y'-r}{r}$, and $\sin. 2\phi = \frac{\sqrt{2ry' - y'^2}}{r}$.

Making these substitutions, and finding the values of y' and x', we have

$$y'^2 - 2ry' = -ry,$$
$$\therefore y' = r \pm \sqrt{r^2 - ry},$$
$$\therefore x' = x + \sqrt{ry} \pm \sqrt{ry - y^2}.$$

Making these substitutions in

$$y' - r \cos. \frac{x' - \sqrt{2ry'-y'^2}}{r} - r = 0,$$

which is the equation of the cycloid when the origin is at the middle point of the base, the result is

$$\sqrt{r^2 - ry} - r \cos. \frac{x + \sqrt{ry - y^2}}{r} = 0.$$

Now, if $\frac{x + \sqrt{ry - y^2}}{r} = \omega$, $\cos. \omega = \frac{\sqrt{r^2 - ry}}{r}$, $\therefore$

$$\cos. 2\omega = 2 \cos.^2 \omega - 1 = 1 - \frac{y}{\frac{1}{2}r},$$

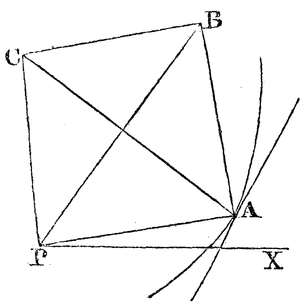
and hence we find

$$y - \tfrac{1}{2}r \cos. \frac{x + \sqrt{ry - y^2}}{\frac{1}{2}r} - \tfrac{1}{2}r = 0,$$

which is evidently the equation of a cycloid described upon half the base of the given one, and which is therefore the *caustic* sought.

PROP. CCCLX.

(670.) *To find the caustic by reflection, the reflecting curve being the logarithmic spiral, and the focus of incident rays being at its pole.*



Let P be the pole of the spiral, and PX the line given in position, from which the angle APX $= \omega$ is measured, PA being a ray incident at A; let AC be the radius of the osculating circle. By the properties of this spiral, P is the middle point of the chord of the osculating circle, and $\because$ CPA is a right angle; and if P be the focus of incident rays by the principles of optics, the focus B of reflected rays is found by drawing BA, making CAB equal to CAP, and CB perpendicular to AB. The angle at which the radius vector is inclined to the tangent being θ, let the equation of the spiral be

$$z' = a^{\omega'}.$$

And let PB $= z$, and BPX $= \omega$, and BPA $= \theta$, we find

$$z' = \tfrac{1}{2} z \sec. \theta,$$

$$\because z = 2 \cos. \theta \, a^{\omega - \theta},$$

which is the equation of a logarithmic spiral similar to the given one, and which is therefore the caustic sought.

SECTION XXIV.

Praxis.

(1.) Given the base and the sum of the sides of a triangle to determine the loci of the points where the angles of the inscribed square meet the sides, and also that of the centre of the inscribed square.

(2.) In a right angled triangle given in magnitude and position, one of the sides containing the right angle, to find the locus of the centre of the inscribed circle.

(3.) Given the base and vertical angle of a triangle, to determine when the sum of the sides is a maximum.

(4.) To determine the greatest ellipse inscribable in a given triangle, and touching one of the sides at a given point.

(5.) To find the greatest triangle inscribable in a circle.

(6.) To find the least triangle which can be circumscribed about a circle.

(7.) To find the greatest rectangle inscribable in a circle.

(8.) To find the least quadrilateral which can be circumscribed round a circle.

(9.) Given in position two sides of a triangle, and a point through which the third side passes, to determine the triangle so that the sum of the sides given in position shall be a minimum.

(10.) On the same conditions, to construct the triangle so that the area shall be a minimum.

(11.) Given the base of a triangle, the ratio of the sum of the squares of the sides to the rectangle under them, to find the locus of the vertex.

(12.) Given the base of a triangle, and the ratio of the sum of the squares of the sides to the difference of their squares, to find the locus of the vertex.

(13.) A triangle of a given area has an angle given in position, to find a curve to which the opposite side is always a tangent.

(14.) The sum of the circumferences of an epicycloid and hypocycloid, described with the same base, is independent of the magnitude of the base when their generating circles are equal.

(15.) To determine when an epicycloid is an algebraic, and when a transcendental curve.

(16.) To determine the nodes of an epitrochoid, the describing point being outside the circumference of the generating circle.

(17.) To determine the inflections of an epitrochoid, the describing point being within the generating circle.

(18.) A circle being described concentrical with the base of an epitrochoid, and with a radius equal to the difference between the sum of the radii of the base and generating circle, and the distance of the describing point from the centre to the generating circle, to determine the points when the epitrochoid meets this circle, and the position of the tangent to the epitrochoid at these points.

(19.) To determine the position of the tangent to an epicycloid at the points where it meets the base.

(20.) To apply investigations similar to the preceding to hypotrochoids and hypocycloids.

(21.) To deduce the equations of cycloids in general from those of epitrochoids.

(22.) To exhibit the different analogies between epitrochoids and cycloids.

(23.) Two lines of the second degree being given, a right line touches one of them and cuts the other, and through the points of section two tangents are drawn to determine the locus of their point of intersection.

(24.) To determine the curve in which the perpendicular from the origin on the tangent is always equal to half the normal through the point of contact.

(25.) To find the curve in which the normal bears a given ratio to the part of the axis of x, intercepted between it and a given point.

(26.) To find the equation of the curve in which the radius of curvature varies as the inclination of the tangent to a line given in position.

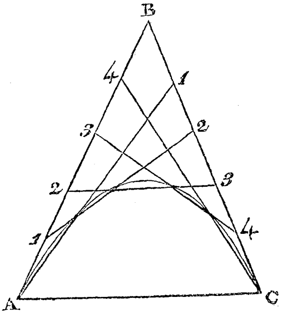
(27.) To determine the locus of the vertex of a triangle constructed on a given base, one of the angles at the base differing from twice the other by a given angle.

(28.) The vertices of two angles given in magnitude are given in position, and the point of intersection of one pair of sides describes a right line, to find the curve described by the point of intersection of the other sides.

(29.) To determine the curve whose tangent is always equal to the part of the axis intercepted between it and the origin.

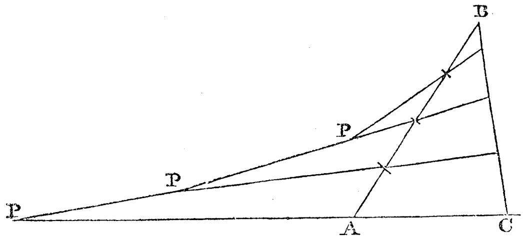
(30.) If the ordinate to the axis of a line of the second degree be produced until the produced part equals the normal, to find the locus of its extremity.

(31.) To inscribe an ellipse in a given parallelogram, so that its area shall be a maximum.



(32.) If two sides of a triangle ABC be divided into the same number of equal parts, and the points of division of one side, beginning from the base, be joined with those of the other side, beginning from the vertex, to find the curve on which the intersections of every pair of consecutive lines taken in this order are placed.

(33.) If on two sides of a triangle ABC, equal parts be assumed, to find the curves on which the points P of intersection of every pair of lines joining the successive points of division are placed.



(34.) To explain and prove the principle of the *Pentagraph.*

For a description of this instrument, see Hutton's Mathematical Dictionary.

(35.) To determine the area and inflection of a curve, in which y varies as the square of the sine of x.

(36.) A circle revolves in its own plane uniformly round a point on its circumference, and at the same time a point on the circumference revolves round the centre with the same angular motion, to find the curve traced by this point.

(37.) The length of a circular arc is given, to find such a radius that the area of the segment may be a maximum.

(38.) To find the point in an ellipse, where the part of the tangent between the point and a perpendicular from the centre is a maximum.

(39.) To find where the intercept of the tangent between a perpendicular from the focus of the ellipse and the point of contact is a maximum.

(40.) From a given point to draw a line intersecting an ellipse, so that the part intercepted within the curve may be a maximum.

(41.) To find the greatest ellipse inscribable in a semicircle.

(42.) An angle of a triangle being given in magnitude and position, and the sum of the containing sides being constant, to find the curve to which the side opposite to it is always a tangent.

(43.) An angle of a triangle being given in magnitude and position, and the difference of the containing sides being given, to find the curve to which the opposite side is always a tangent.

(44.) To find the least triangle which can be included by a tangent to a given curve and the axes of co-ordinates.

(45.) To find the greatest parallelogram which can be included under the co-ordinates of a point in a curve.

(46.) Given the length of the arc of a semicubical parabola, to find when the area included by it, and the co-ordinate of its extremity, shall be a maximum.

(47.) To determine the curve which shall intersect similar and concentric ellipses at right angles.

(48.) To determine the curve which shall cut any number of ellipses or hyperbolas, having the same centre and vertex at right angless.

(49.) To determine the locus of the points of contact of concentric circles, touching similar and similarly posited concentric ellipses.

(50.) To describe a circle with a given centre, and touching a given parabola.

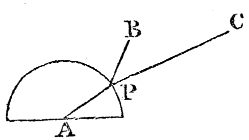
(51.) To draw a tangent to an ellipse, so that the intercept of it between the axes shall be a minimum.

(52.) To express the area of the sector of an ellipse contained by the axis and the radius vector in terms of the eccentric anomaly.

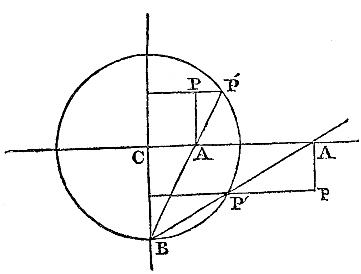
(53.) If a portion be assumed on the ordinate to the axes of an ellipse equal to the semiconjugate diameter, to find the locus of its extremity.

(54.) To circumscribe a given ellipse by a triangle whose area shall be a minimum.

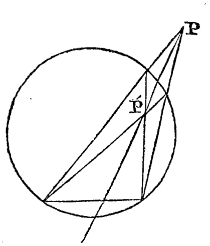
(55.) Two points B and C and a circle being given, to determine a point P on the circle from which right lines being drawn to the centre, and to the two given points, the sines of the angles APC and APB, which the radius forms with the lines to the given points shall be in a given ratio.



(56.) A point and two right lines are given in position, to determine the equation of a circle, which, passing through the given point and touching one of the given lines, will intersect the other, so that the part of it intercepted within the circle shall have a given magnitude.



(57.) Two diameters of a circle being drawn, intersecting perpendicularly, from the extremity B of one of them a right line is drawn, meeting the other at any point A, the given circle at P′, and through the points A and P′ parallels to the diameters are drawn intersecting at P; to find the locus of the point P of intersection of these parallels.



(58.) Two angles are inscribed in the same segment of a line of the second degree, whose sides intersect at the points P and P′; a right line passing through P and P′ always passes through a fixed point; to determine this point.

(59.) Given the base of a triangle, and the ratio of the sum of the squares of the sides to the sum of the squares of the segments of the base, intercepted between the perpendicular and its extremities, to find the locus of the vertex.

(60.) Given the base of a triangle, and the ratio of the rectangle under the sides to the rectangle under the segments of the base intercepted between the perpendicular and its extremities, to find the locus of the vertex.

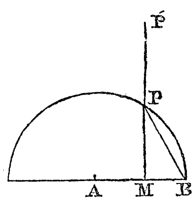
(61.) To determine the axes of co-ordinates, to which a line of the second degree must be related, in order that the sum of squares of the values of y, which correspond to the same value of x, may be invariable.

(62.) To determine the conditions under which the sum of the mth powers of the values of y, corresponding to the same value of x, shall be the same for all points of the curve.

(63.) To determine for all algebraic curves, the condition under which the product of all the values of y, for the same value of x, is invariable.

(64.) To determine for all algebraic curves, the condition under which the sum of the products of every n values of y, for the same value of x shall be invariable, n being an integer less than that which marks the degree of the equation.

(65.) The ordinate (PM) to the diameter of a circle being produced until the produced part (PP′) equals the chord (PB) of the arc intercepted between the ordinate and the extremity of the diameter, to investigate the figure and properties of the locus of the extremity (P′) of the produced part.



(66.) The ordinate to the diameter of a circle being produced until the whole produced ordinate (MP′) equals the tangent of the arc (PB), to find the locus of the extremity P′ of the produced ordinate.

(67.) The ordinate to the diameter of a circle being produced until it becomes equal to the secant of the arc (PB), to find the locus of its extremity.

(68.) The ordinate to the conjugate axis of an ellipse or hyperbola being produced until the produced part is equal to the line connecting the extremity of the ordinate with the focus, to find the locus of the extremity of the produced ordinate.

(69.) To investigate the line or lines represented by the equation

$$y^3 - ay^2x + x^2y - ax^3 + bx^2 + by^2 - cy + cax + bc = 0.$$

(70.) To determine the locus of the equation

$$y^3 - xy^2 + yx^2 - x^3 - 2rxy + 2rx^2 = 0.$$

(71.) To determine the locus of the equation

$$y^4 + y^2x^2 - py^2x - px^3 - r^2y^2 + pr^2x = 0.$$

(72.) To determine the locus of the equation

$$a^3y^3 - a^2by^2x - ab^2x^2y + b^3x^3 + a^3b^2y - a^2b^3x = 0.$$

(73.) To determine the figure and inflections of the curve represented by the equation

$$x^2 - ay + a^{\frac{3}{2}}x^{\frac{1}{2}} = 0.$$

(74.) To determine the figure and inflections of the locus of the equation

$$x^3 - ay^2 + a^2x = 0.$$

(75.) To determine the figure and inflections of the locus of the equation

$$3x^5 - 3ax^4 + 110\,a^2x^3 - 180\,a^3x^2 + a^4y = 0.$$

(76.) To determine the figure and quadrature of the locus of the equation

$$x^2y - a^2y + a^3 = 0.$$

(77.) To determine the inflection of a curve in which

$$y^3 \propto x.$$

(78.) To determine the locus of the vertex of a triangle on a given base, the square of the altitude of which varies as the quote of the segments. And to show the inflections of this locus.

(79.) To find the curve in which the rectangle under perpendiculars from two given points on the tangent is of a constant magnitude.

(80.) To find the curve in which the subnormal is constant.

(81.) To find the curve in which the perpendicular from the origin on the tangent is constant.

(82.) To find the curve in which the perpendicular from the origin on the tangent varies in the subduplicate ratio of the radius vector.

(83.) To find the curve in which the locus of the intersection of the perpendicular through a given point and the tangent is a right line.

(84.) To find the curve in which the locus of the intersection of a perpendicular through a given point with the tangent is a circle.

(85.) To find the curve in which the locus of the extremity of the polar subtangent is a straight line.

(86.) To find the locus of the intersection of tangents to an ellipse or hyperbola which intersect at a given angle.

(87.) Two tangents to an ellipse or hyperbola intersect the transverse axis at angles, the difference of which is given, to determine the locus of their point of intersection.

(88.) Investigate the figure and properties of the class of curves included under the polar equation $z^n = \cos. n\omega$.

(89.) If three right lines in the same plane move angularly round three fixed points, and two of the three points of intersection describe right lines, the third will describe a line of the second degree; to determine its species, centre, axes, &c.

(90.) If three lines revolve round three fixed points in the same manner, and one of the points of intersection describes a line of the mth order, and another one of the nth order, the third will describe a line of the mnth, neither of the lines of the *m*th or *n*th order being supposed to pass through the centres of rotation.

(91.) To determine the locus of the intersection of the rectangular tangents to the cardioide.

(92.) The line joining the points of contact of rectangular tangents to the cardioide passes through a fixed point.

(93.) To determine the equation of the class of curves distinguished by the property, that tangents through the extremities of a chord passing through a given point shall intersect at a given angle.

(94.) To determine the equation of a class of curves in which all chords drawn through a given point are of a given length.

(95.) To determine the inflections of the curve represented by the equation

$$A^3y^3 + B^3x^3 = A^3B^3.$$

(96.) To find the multiple point of the curve represented by

$$ay^2 - x^2(b + x) = 0.$$

(97.) To determine the singular point of the curves represented by the equations

$$y^2 + ax + \frac{c}{x} = 0,$$

$$y = \frac{x^2(a^2 - x^2)}{a^3},$$

$$y^4 = px,$$

$$a^3y = (x^2 - b^2)(x^2 - c^2),$$

$$y^5 = px,$$

$$y^6 = px,$$

$$y^5 = px^3,$$

$$x^4 - ayx^2 + by^2 = 0,$$

$$x^4 + y^4 - 2ay^2 + 2bx^2y = 0,$$

$$x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0,$$
$$y = b + cx^2 + (x - a)^{\frac{5}{2}},$$
$$y^2 = (x - a)^4 (x - b),$$
$$y^2 = (x - a)^6 (x - b)$$
$$y^5 + ax^4 - b^2xy^2 = 0,$$
$$a^3y^2 - 2abx^2y - x^5 = 0,$$
$$z = a \sin. 2\omega.$$

(98.) Given the angle of elevation at which a cannon is fixed, to find the strength of the charge necessary to make the ball strike a given plane perpendicularly.

(99.) To find the elevation which will require the least quantity of powder to cause a ball of a given weight to strike a given object.

(100.) Two circles, described upon the same vertical plane, with their centres in the same horizontal, are given in magnitude and position, to determine a point from which tangents to the two circles will be described by falling bodies in the same time, and to find the locus of all such points.

(101.) To find the point in a planet's elliptical orbit, where its velocity is an harmonic mean between its velocities at the apsides.

(102.) To determine the points in the moon's elliptic orbit, where her angular velocity round the remote focus is accurately equal to her angular velocity on her axis.

(103.) To find the latitude at which the vertical line is most inclined to the line drawn to the centre.

(104.) In all curves described by a body moving round a centre of force, the velocity of the body is equal to that of a body in the equidistant circle at that point at which the angle under the radius vector and tangent, or the angle of projection, is a minimum.

(105.) A body being supposed to fall from any distance towards a centre of force, the law of which is the inverse square of the distance; if a cycloid be described on the line of descent as axis, the ordinate to the axis which passes through the body is always proportional to the time of its fall, and the tangent of the angle at which this ordinate is inclined to the curve represents the acquired velocity.

(106.) In the hyperbolic spiral the centripetal and centrifugal forces are equal.

(107.) To express the times of the successive revolutions of the radius vector passing through a body moving in a logarithmic spiral, and also the time of arriving at the centre.

(108.) To apply the same investigations to a body moving in an hyperbolic spiral.

(109.) A body revolves in an ellipse or hyperbola, the centre of force being at the focus, the place of the body being given, to determine the ratio of its velocity to that of a body moving in a circle at the same distance.

(110.) In the same case, the place of the body being given, to determine the ratio of the centripetal to the centrifugal force.

(111.) In the same case, to determine the maximum paracentric velocity by means of the polar subtangent.

(112.) In the same case, to determine the point at which the velocity is a geometrical mean between the velocities at the apsides.

(113.) In the same case, to determine the point at which the angular velocity equals the mean angular velocity.

(114.) To determine the curve affected by a repulsive force, parallel and proportional to the ordinate.

(115.) To determine the curve in which the attractive force is proportional to the ordinate.

(116.) A body is moved in a cycloid by the attraction of the points of the base, to determine the law of the attraction.

(117.) To determine the period in this case, and show its analogy to the periods of bodies moving in a line of the second degree with the force at the focus.

(118.) Jets of water spout from apertures at the same depth below the surface of the reservoir at different elevations; to determine the locus of their points of greatest ascent without regard to the resistance of the air.

(119.) Jets of water spout from apertures at different depths, but with the same elevation, to determine the locus of their highest points.

(120.) Given the place of the aperture, to determine the direction of the jet, so that the area included by the curve and its chord shall be a maximum.

(121.) Jets of water at the same depth below the surface in the reservoirs spout with different elevations, to determine the locus of their foci.

(122.) Jets of water at different depths spout with the same elevation, to determine the locus of their foci.

(123.) What would happen to the earth if the sun's mass were diminished one half?

(124.) If all the bodies of a system but one be quiescent, and that one describe any given curve, to find the curve which the centre of gravity of the system will describe.

(125.) Rays diverging from a luminous point are refracted by a spherical surface, to find the point at which each refracted ray intersects the diameter of the sphere passing through the luminous point.

(126.) An object is placed between two mirrors, the planes of which are not parallel, to find the line on which all the images are placed.

NOTES ON PART I.

NOTES.

Art. 13.

THE method of determining the number and order of the terms of a general equation is explained in Sect. XXI.

Art. 14.

In the general equation of the right line, and, indeed, in every general equation, the constant co-efficients A, B, &c. must be supposed to represent such quantities as render the entire equation homogeneous; that is, so that all the terms which compose it shall be composed of the same number of linear factors. Thus, in the general equation of the first degree, if C be supposed to represent a line, A and B must represent numbers. In the general equation of the second degree,

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0,$$

if F be supposed a quantity of two linear dimensions, and therefore to represent a surface, all the other terms must also represent surfaces; therefore D and E must represent lines, and A, B, and C, numbers. If F be a quantity composed of three linear factors, D and E must be quantities composed of two, and A, B, and C, of one.

It may be observed, that in the equation of a right line, the inclination to the axes of co-ordinates depends on the value of $\frac{B}{A}$, and the points where it meets the axes on $\frac{C}{A}$ and $\frac{C}{B}$.

Art. 15.

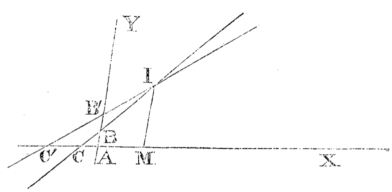
If the axes of co-ordinates be rectangular, sin.ly = cos.lx, and therefore $-.\frac{B}{A} = \tan.\ lx$.

Art. 23, et seq.

It may be interesting to students accustomed to geometrical investigations, to see how some of the formulæ found in this section analytically may be obtained by geometrical principles. The application of geometrical principles to establish formulæ is not, however, a habit in which the young analyst should indulge; it sometimes appears to give greater facility and clearness than the analytic process, but in many more cases it embarrasses and perplexes the student, and *always* contracts and particularises his conclusions.

The result of art. (23) may be found geometrically thus.

Let the equations of the lines be as in the text. We have by the similar triangles CAB and CMI,



$$CA : AB :: CM : MI.$$

And by the similar triangles C'AB' and C'MI,

$$C'A : AB' :: C'M : MI.$$

Hence, by compounding the ratios,

$$CA \times AB' : C'A \times AB :: CM : C'M;$$

but by (17),

$$CA = -\frac{C}{B}, \quad C'A = -\frac{C'}{B'},$$

$$BA = -\frac{C}{A}, \quad B'A = -\frac{C'}{A'}.$$

Hence we find

$$\frac{\text{CC}'}{\text{A}'\text{B}} : \frac{\text{CC}'}{\text{AB}'} :: x - \frac{\text{C}}{\text{B}} : x - \frac{\text{C}'}{\text{B}'}.$$

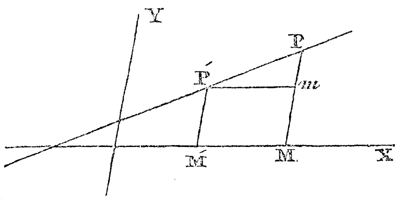
And by *division*,

$$\frac{\text{CC}'}{\text{A}'\text{B}} - \frac{\text{CC}'}{\text{AB}'} : \frac{\text{CC}'}{\text{AB}'} :: \frac{\text{C}'}{\text{B}'} - \frac{\text{C}}{\text{B}} : x - \frac{\text{C}'}{\text{B}'}.$$

From this proportion, $x - \frac{\text{C}'}{\text{B}'}$ is determined geometrically, and thence the value of x.

In a similar way the value of y may be found.

The result of art. 26 may be thus found geometrically:



Let P′ be the given point through which the line is required to pass, and whose co-ordinates are $y'x'$. Let P be any point on the right line, the co-ordinates of which are yx. Since the angles PP′m and P′Pm are always the same, the *species* of the triangle PmP′ is independent of the position of the point P, and therefore the ratio Pm : P′m is independent of it. Let this constant ratio be $a : 1$, ∵

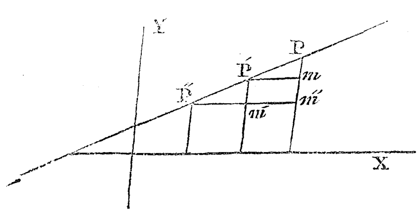
$$\text{P}m : \text{P}'m :: a : 1,$$

$$\text{or } y - y' : x - x' :: a : 1,$$

$$\therefore (y - y') \div a(x - x') = 0;$$

and if $a = -\frac{\text{B}}{\text{A}}$, we have

$$\text{A}(y - y') + \text{B}(x - x') = 0.$$



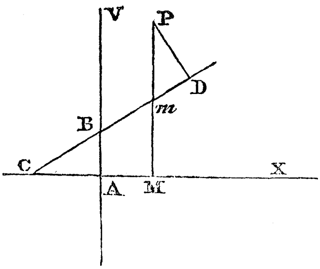
If the right line be required to pass through two points P′P″, we have the ratio Pm'' : P″m'' always the same as P′m' : P″m', because the triangle $\text{P}m''\text{P}''$ is always similar to $\text{P}'m'\text{P}''$: hence

$$\text{P}m'' : \text{P}''m'' :: \text{P}'m' : \text{P}''m',$$

$$\text{or } y - y'' : x - x'' :: y' - y'' : x' - x'',$$

$$\text{or } (x' - x'')(y - y'') - (y' - y'')(x - x'') = 0.$$

The formula in art. 50, for the length of the perpendicular, may be found thus:



Let P be the point from which the perpendicular is supposed to be drawn to the line CD. The triangles $\text{PD}m$ and BAC are manifestly similar; therefore

$$\text{PD} : \text{P}m :: \text{CA} : \text{BC},$$

$$\text{or PD} : \text{PM} - \text{M}m :: \text{CA} : \text{BC}.$$

But $\text{CA} = -\frac{\text{C}}{\text{B}}$, and $\text{BA} = -\frac{\text{C}}{\text{A}}$, and hence

$$\text{BC} = \frac{\text{C}}{\text{AB}}\sqrt{\text{A}^2 + \text{B}^2};$$

but also, since the equation of the line CD is

$$\text{A}y + \text{B}x + \text{C} = 0.$$

If $x = x'$, and $y = \text{M}m$, we have

$$\text{M}m = -\frac{\text{B}x' + \text{C}}{\text{A}},$$

$$\therefore \text{PM} = \frac{\text{A}y' + \text{B}x' + \text{C}}{\text{A}}.$$

And by these substitutions, the proportion becomes

$$\text{PD} : \frac{\text{A}y' + \text{B}x' + \text{C}}{\text{A}} :: -\frac{\text{C}}{\text{B}} : \frac{\text{C}}{\text{AB}}\sqrt{\text{A}^2 + \text{B}^2},$$

$$\therefore \text{PD} = -\frac{\text{A}y' + \text{B}x' + \text{C}}{\sqrt{\text{A}^2 + \text{B}^2}}.$$

The preceding examples show the student how some of the results may be obtained geometrically. They also give striking proofs how inferior both in brevity and facility that method is to the analytic process.

Art. 57.

A formula is said to be symmetrical with respect to any quantities, when their denominations being interchanged the formula remains unchanged. Thus, in the value of R, if the denominations of the three sides be changed, the formula remains the same; for example, let c be changed into c', *vice versa*.

$$R = \frac{c'cc''}{4A},$$

which is the same as before. Or if c be changed into c'', and *vice versa*, the formula still remains the same.

Art. 69.

This proposition, with the various modifications of which it is susceptible, is given by Apollonius (see note on art. 89) in his treatise *De Locis Planis*, a remarkable collection of curious properties of the circle and right line. This treatise was first restored by *Fermat*, though his work was not published until after his death. *Schooten* afterwards published a work on the same subject; but his demonstrations are algebraical in many cases, and not in the spirit of the original. The best work extant on the subject is *Robert Simpson's Apollonii Loca Plana restituta*, an excellent specimen of the style of the ancient geometry.

Art. 80, et seq.

A more general definition of *diameters* of a curve will be found in Section XXI.

We shall here explain the algebraical principles assumed in this and the following articles.

In an equation of the form

$$ax^2 + bx + c = m,$$

the values of a, b, and c, being supposed given, the sign of m will depend on that of a, and the nature of the values of x which render $m = 0$.

1°. Let the values of x, which render $m = 0$, be real and unequal, and be represented by x' and x''. The above equation may, by a well known property of equations, be expressed thus:

$$(x - x')(x - x'') = \frac{m}{a}.$$

See Wood's Algebra, Part II.

If x'' be considered the greater, and x' the lesser root, all values of $x > x'$ and $< x''$ must give the factors $x - x'$ and $x - x''$ *different* signs, and therefore render their product negative; and all values of $x > x''$, or $< x'$, must give these factors the *same* sign, and therefore render their product positive. Hence, for all values of x between the roots x' and x'', $\frac{m}{a}$ must be negative, and therefore m and a must have different signs; and for all values of x beyond the limits x' and x'', the quantity $\frac{m}{a}$ must be positive, and therefore m and a must have the same sign.

2°. If the values of x which render $m = 0$ be impossible, let the equation

$$ax^2 + bx + c = m$$

be solved for x, and we find

$$x = \frac{-b \pm \sqrt{b^2 - 4ac + 4am}}{2a}.$$

Since the value of x is by supposition impossible, when $m = 0$, we have the condition

$$b^2 - 4ac < 0,$$

and therefore $am > 0$, and hence a and m must always have the same sign, and therefore the quantity $\frac{m}{a}$ must be positive for all real values of x, whether positive or negative.

3°. If the values of x, which render $m = 0$, be real and equal. The equation can, in this case, be reduced to

$$a(x - x')^2 = m.$$

Hence, for all values whatever of x, except that which gives $m = 0$, the value of $\frac{m}{a}$ must be positive, and therefore m and a must have the same sign.

In an equation of the form

$$bx + c = m,$$

the sign of m will depend on that of b, and the value of x which renders $m = 0$. Let this be x', and we find

$$x' = -\frac{c}{b}.$$

Hence the equation may be expressed

$$b(x - x') = m.$$

Hence it appears that all values of $x < x'$ render $\frac{m}{a}$ negative; and all values $> x'$ render $\frac{m}{a}$ positive, and therefore in the one case m and a have *different* signs, and in the other the same sign.

These principles may be applied to the cases in the text, by supposing $a = \text{B}^2 - 4\text{AC}$, and $b = 2(\text{BD} - 2\text{AE})$, or $= 2(\text{BE} - 2\text{CD})$, and $m = \text{R}^2$, or $= \text{R}'^2$.

It may be observed, that the condition $\text{B}^2 - 4\text{AC} = 0$ renders the first three terms of the general equation a complete square.

Art. 89.

It should be remembered, that these conditions involve the supposition that A and C are finite.

The curves represented by the general equation of the second degree are the same as those which are produced by the section of a conical surface by a plane, as will be shown in the second part of this work. They have hence been called *Conic Sections*. These curves originated in the Platonic school. Some suppose Plato himself to have first conceived them; others Menechme, a distinguished geometer of that time, and a pupil of Plato. The first properties which were discovered, were those of diameters and their ordinates, the centre and foci, the parallelism of the diameters of a parabola, the value of the subtangent, the properties of the lines from the foci in (217), and the similar property of the parabola in (255), the property proved in (209), and the properties of the asymptotes of the hyperbola.

Even at so early a period as the time of Plato, Menechme displayed a considerable knowledge of the properties of the conic sections in his solutions of the famous problem of the duplication of the cube by the intersection of two parabolæ, and by the intersection of a parabola and hyperbola. Aristæus, the teacher of Euclid, and pupil of Plato, also at that time wrote two works, one consisting of five books on the *Conic Sections*, the other consisting also of five books on *Solid Loci*. Pappus prescribes these books as a study for his son in geometry. These works were unfortunately lost in the general wreck of letters.

The principal treatise which the ancients have handed down on the properties of lines of the second degree is that of Apollonius Pergæus, who flourished about the middle of the third century before the Christian era. He was distinguished among the ancients by the title of The Great Geometer, and was decidedly the second geometer of antiquity. According to Pappus, it would appear that he was not so liberally endowed with the qualities of the heart as those of the head. He represents his character as marked by arrogance, envy, and as seizing with avidity every opportunity of trampling upon the claims and lowering the merits of others. His principal work is the treatise on Conics. He defines the lines of the second degree by the section of the cone. The treatise consists of eight books, the first four of which have been handed down in the original Greek; the fifth, sixth, and seventh, we have through an Arabic version, and the eighth has been lost. Halley has attempted to restore the eighth book in his edition of Apollonius. The last four books contain the principal discoveries of Apollonius, the subject of the first four having been previously known. Among his discoveries are, the first notion of osculating circles and evolutes, the results of prop. LII., prop. LXXIII., and several propositions relating to *maxima* and *minima*.

Art. 132.

The method of determining the equation of a tangent used here is the invention of DESCARTES. It is not confined in its application to curves of the second degree, but is generally applicable to all curves.

Let the equation of any curve be

$$\mathrm{F}(yx) = 0;$$

when $\mathrm{F}(yx)$ means any function of the variables yx. And let the equation of a right line intersecting this curve be

$$a(y - y') + b(x - x') = 0.$$

Eliminating y by these equations, the result will be an equation involving only x, the roots of which will be the values of x for the points where the line intersects the curve. If two of these points unite, two of the roots will be equal, and the line will become a tangent. The method by which DESCARTES determined the condition under which two of the roots would be equal, was by assuming an equation of the same degree, having two equal roots, and comparing with it the proposed equation. In the case of lines of the second degree this ingenious artifice is rendered unnecessary, the solution of the equation being sufficient.

This method of determining the equation of a tangent is that which appears in the letters of DESCARTES. That which he gives in his Geometry is somewhat different, and nearly as follows. Let

$$y^2 + (x - x')^2 - r^2 = 0$$

be the equation of a circle, the centre of which is on the axis of x. Let y be eliminated by means of this equation and that of the curve, and the roots of the resulting equation will be the values of x for the points where the circle meets the curve. The centre of the circle being supposed fixed, and the radius r arbitrary, let it be supposed to have such a value as will render two of the roots of the equation equal; the circle will then touch the curve, and will therefore have the same rectilinear tangent. The value of r, which renders the roots equal, may be found by the artifice mentioned above.

These methods are both founded on the same principle; and though we cannot but admire the ingenuity they display, yet they must in general yield to the more simple and direct method furnished by the *Calculus*. We have used one of them here, as it is thought desirable that the study of a part of algebraic geometry should precede that of the *Calculus*.

Art 154.

The principle on which the solution of this problem depends is, that if the relation between the co-ordinates of any two points upon a right line be expressed by the same equation *of the first degree*, that equation will express the relation between the co-ordinates of any point on the right line, and is therefore the equation of that right line. This principle is evident from the consideration that two points are sufficient to determine a right line.

Art. 162.

The names *ellipse, hyperbola,* and *parabola,* originated from a property expressed by the equation

$$y^2 = px - \frac{p}{2A}x^2 ;$$

$\frac{p}{2A}$ is positive for the *ellipse,* negative for the *hyperbola,* and $= 0$ for the *parabola.* Hence the proper equations of these three curves are

$$y^2 = px - \frac{p}{2A}x^2,$$

$$y^2 = px + \frac{p}{2A}x^2,$$

$$y^2 = px.$$

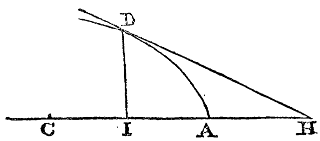
By these equations it appears that the square of the semiordinate to the diameter falls short of the rectangle under the parameter, and absciss in the ellipse, and exceeds it in the hyperbola by the quantity $\frac{p}{2A}x^2$; and in the parabola the square of the ordinate is equal to the rectangle under the parameter and absciss. Hence the names *ellipse* (defect), *hyperbola* (excess), and *parabola* (equality).

The parameter was formerly called the *latus rectum.* The ancients called the focus *punctum comparationis.*

Art. 168.

This beautiful property was discovered by Apollonius, and is one of the propositions of the seventh book of his Conics.

Art. 189.



This corollary points out a method of drawing geometrically a tangent to an *ellipse*, or *hyperbola*, from a point outside it. Let H be the point. Draw HC to the centre, and take CI, a third proportional to HC and CA, and draw DI an ordinate to the diameter CA, the line DH will be the tangent.

Art. 196.

This proposition proves the disc of a planet, except when in opposition or superior conjunction, to be a figure bounded by a semicircle and semiellipse, the ratio of the axes being that of the cosine of the angle subtended by the earth and sun at the planet to radius.

Art. 205.

In the equilateral hyperbola, if A represent the semiaxis, and A′ any semidiameter,

$$\sin. \theta = \frac{A^2}{A'^2}.$$

Art. 207.

The result of this proposition was discovered by Apollonius, and is contained in the seventh book of his conics.

Art. 209.

By an extension of this property, Descartes invented a class of curves, of which the ellipse is a particular instance, and which have been called the *ovals of Descartes*. As in the ellipse, the radii vectores vary so that the increment of one shall always be equal to the simultaneous decrement of the other, and in the hyperbola, the simultaneous increments of both are always equal, so in the *Cartesian ovals* these increments are in an invariable ratio. If z and z' be two lines drawn from the two foci to any point in the curve, the condition

$$dz + mdz' = 0$$

will always be fulfilled; which being integrated, gives

$$z + mz' = 2a,$$

$2a$ being any arbitrary constant. These curves may therefore be defined, the locus of the vertex of a triangle on a given base, one of whose sides bears a given ratio to the sum or difference of a given line and the other side. The equation of this class of curves may easily be determined. Let $2c$ be the distance between the foci, and ω the angle under z and $2c$; hence

$$4cz \cos. \omega = z^2 + 4c^2 - z'^2.$$

By eliminating z' by means of this and the former equation, we find

$$(m^2 - 1)z^2 + 4(a - m^2c \cos. \omega)z + 4(m^2c^2 - a^2) = 0,$$

which is a curve of the fourth order, except when $m = 1$, in which case, after reduction, it becomes

$$z = \frac{a(1-e^2)}{1-e \cos \omega},$$

which is the polar equation of an ellipse or hyperbola.

The circumstance which occasioned the invention of these curves was the investigation of the figure of the surface, which must divide two mediums of given densities, so that rays of light emerging from a given point shall be all refracted accurately to another given point. Descartes proved that the surface must be one generated by the revolution of these ovals upon the line joining their foci. And he showed that if the focus of incident rays be at an infinite distance, or if the pencil of rays be parallel, the oval becomes an ellipse. See art. 665. For a more detailed account of these curves and their optical properties, see Huygens de Lumine, and Rabuel's Commentary on the Geometry of Descartes.

Art. 212.

The polar equations found in this proposition are of considerable use in physical astronomy. The variable z expresses the distance of the planet from the sun, $(\omega - \phi)$ the anomaly, and ϕ the direction of the apsides.

Art. 215.

From this proposition it obviously follows, that the perpendicular from the focus of an hyperbola on the asymptote is equal to the semiconjugate axis; the asymptote being considered as the tangent to a point at an infinite distance.

Art. 227.

This property is used by some geometrical authors to distinguish the species of lines of the second degree. See LESLIE's Geometry of Curve Lines.

Art. 257.

The focal tangents are those which touch the curve at the extremity of the focal ordinate.

The property expressed in this proposition is not peculiar to the parabola, but common to all lines of the second degree. See art. 315.

Art. 281.

This proposition might also be investigated in a manner similar to art. 282.

Art. 298.

This principle has furnished means of describing an ellipse mechanically. Let AB and AC be two fixed rulers, and BC another ruler with rings at B and C capable of running upon the fixed rulers, and a pencil at any point P, which, upon moving the ruler BC, will describe an ellipse.

Art. 302.

This property extends to all similar curves; for if the radii vectores of any curve expressed by a polar equation be all increased or diminished in a given ratio, they will produce a similar curve, and *vice versa.* It should be observed that the curves are supposed to have a common vertex and *one* common axis.

Art. 318.

The equation of a tangent drawn from a given point outside a curve may be found thus: Let $y'x'$ be the point, and $y''x''$ the point of contact. The equation of the tangent is

$$(y - y') - \frac{dy''}{dx''}(x - x') = 0.$$

By means of this equation, the equation of the curve, and its first differential, and by the condition

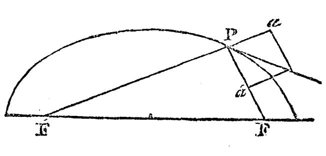
$$\frac{y-y'}{x-x'} = \frac{y'-y''}{x'-x''},$$

the quantities $y''x''$ and their differentials may be eliminated, and the resulting equation will be that of the tangent sought.

The method of drawing tangents to curves explained here, and founded on the principles of the differential calculus, has superseded the other solutions for the same problem given by Descartes, Fermat, Roberval, and others. The methods given by these geometers were either limited to particular classes of curves, or in some cases so incommodious as to amount nearly to impracticability. The determination of the equation of a tangent by the calculus is at once simple and general. It depends merely on differentiating the equation of the curve, and therefore extends to every curve capable of being expressed by an equation, and whose equation is capable of differentiation. The methods of Descartes, which have been explained in the note on art. 132, require that the conditions on which two roots of the equation resulting from the elimination of one of the variables shall be equal should be determined. These methods extend at most only to algebraic curves.

The method of Roberval deserves notice, as well on account of the elegance of the conception on which it is founded, as of its close analogy to the fundamental principle of the Newtonian fluxions. He considered a curve described by a point affected with two motions, the variation in the quantity and direction of which are to be determined by the nature of the curve. At any point of the curve he supposed a parallelogram constructed, the sides of which are proportional to and in the direction of the generating velocities, and laid it down as a principle, that the diagonal which represents the direction of the resultant is the direction of the element of the curve at that point, and therefore the direction of the tangent. There are many instances in which this method may be applied with great clearness and facility; but in most cases its application is either totally impracticable, or attended with very perplexing difficulties, owing to the intricacy of the investigations necessary to determine the component velocities of the generating point. We shall give some examples in which its application is effected with great clearness and beauty.

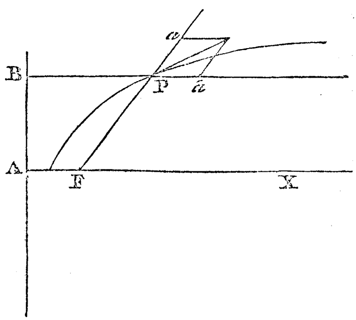
1°. To determine the tangent to a point in an ellipse or hyperbola.



In the ellipse the sum of the distances F′P and FP of the describing point from the foci is invariable; therefore one increases with the same velocity that the other diminishes. Hence the velocity of the describing point in the directions Pa and Pa' are equal; therefore if Pa = Pa', the diagonal is the tangent which bisects the exterior angle.

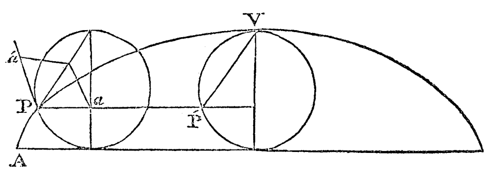
In the hyperbola the difference of the distances from the foci is constant, and therefore the two distances increase with the same velocity. Hence, Pa' should, in this case, be taken as well as Pa on the produced part of the focal distance, and therefore the tangent bisects the angle under the radii vectores from the focus.

2°. To draw a tangent at a given point in a parabola.



Let AB be the directrix, and AX the axis, and F the focus. By the properties of this curve, FP = BF, ∵ the velocities in the directions Pa and Pa' are equal; therefore, as before, the tangent bisects the angle aPa'.

3°. To draw a tangent at a given point in a cycloid.



Let P be the given point. By the definition of the cycloid, the generating point at P has two motions, one in the direction of the tangent Pa' to the generating circle, and the other in the direction Pa parallel to the base; and these two motions are equal, because the generating point moves uniformly round the circumference of the gererating circle in the same time that the circle itself is carried along the base through an equal space. Hence, if Pa and Pa' represent the two motions, $Pa = Pa'$, and therefore the tangent bisects the angle aPa', and is parallel to the corresponding chord P′V of the generating circle described upon the axis.

The method of Roberval is peculiarly applicable to curves which can be described mechanically by motion.

Barrow subsequently invented a method of tangents which approached as near the principle of the differential calculus as Roberval's did to the fluxional principle. He investigates an infinitely small triangle composed of the increments of the absciss and ordinate, and the elementary arc of the curve. The student will readily perceive this to be in effect the spirit of the method used in the text; but both this and the method of Roberval want what constitutes the principal excellence of the methods in the fluxional and differential calculus, that uniform algorithm by which a general formula expresses the equation of a tangent to any curve, and the general rules by which the particular values of the quantities composing this general formula can be found in particular cases.

It should be observed, that the method of Barrow is very nearly the same as that of Fermat.

Art. 328.

The polar subtangent is a line drawn from the pole perpendicular to the radius vector, and terminated in the tangent.

Art. 331.

The value of the second differential co-efficient given by the equation (1) is here understood to be substituted in (4).

Art. 337.

Although the first idea of the evolute of a curve is to be found in Apollonius, yet Huygens must be admitted to be the inventor of the theory of evolutes in general. It forms the third part of his *De Horologium Osculatorium.* He defines the evolute by the property in art. 342, and from this definition deduces its other properties. The first curve to which he applied this theory was the parabola. This consideration led him to the discovery of the property of the cycloid, on which its tautochronism depends.

Art. 338.

The involute of a curve, whose equation is given, may be found by eliminating $y'x'$ and their differential co-efficients by means of the equations of the curve and its differentials combined with the values $y'x'$, and the condition

$$\frac{dy}{dx} = -\frac{dx'}{dy'}.$$

Art. 364

It may be observed, that the condition

$$\frac{d^2y}{dx^2} = \infty$$

also indicates a point of inflection, since the sign of $\frac{d^2y}{dx^2}$ changes in passing through this value.

Art. 368.

A multiple point in general is characterised thus: the equation of the curve

$$\text{F}(yx) = 0$$

being differentiated, let the result be

$$\text{P}dy + \text{Q}dx = 0;$$

if the same values of y and x give more than one value of $\frac{dy}{dx}$, there will necessarily be at the point, whose co-ordinates are these values, an intersection of as many different branches of the curve as there are different values of $\frac{Q}{P}$ for the same value of y and x. In this case there must be always one or more radicals in $\frac{dy}{dx}$, which do not appear in the equation of the curve solved for y. The possibility of this may easily be conceived, when we consider that a variable multiplier of a radical may be removed by differentiation, and, consequently, any value of x, which would render that multiplier $= 0$, will make the radica' disappear from the equation of the curve, but not from $\frac{dy}{dx}$.

A difficulty, however, of a different kind presents itself here. If the equation of the curve be cleared of irrational functions of the variables, and then differentiated, its differential co-efficient $\frac{dy}{dx}$ will be necessarily also a rational function of the variables. How then can this, for one and the same value of x, and one and the same value of y, have different values? This is explained by showing that in this case the value of $\frac{dy}{dx}$ must assume the form $\frac{0}{0}$, which may be proved thus: Let two of the values, of which $\frac{dy}{dx}$ is susceptible, be p, p'. Hence we have the equations

$$Q + Pp = 0,$$
$$Q + Pp' = 0.$$

By subtraction, we find

$$P(p - p') = 0;$$

but since by supposition p and p' are unequal, we infer

$$P = 0,$$

which, substituted in the first equation, gives

$$Q = 0,$$
$$\therefore \frac{dy}{dx} = \frac{0}{0}.$$

The true values of $\frac{dy}{dx}$ may be found in this case by substituting the value of y resulting from the solution of the equation of the curve in the value of $\frac{dy}{dx}$, which then becomes a function of x alone. If it continues of the form $\frac{0}{0}$, its values may be determined by the general method furnished by the calculus for determining the true values of functions of this form, or we may frequently obtain the result without substituting the value of y with greater facility by finding the successive differentials of the equation of the curve until we find one from which all the differential co-efficients, except the first, will disappear by the particular values of y and x which render $\frac{dy}{dx} = \frac{0}{0}$. The roots of this equation will give the true values of $\frac{dy}{dx}$. An example will render these observations easily apprehended. Let the equation of the curve be

$$y^4 + 2ay^2x - ax^3 = 0 \quad (1).$$

By differentiating, we find

$$\frac{dy}{dx} = \frac{a(3x^2 - 2y^2)}{4y(y^2 + ax)} \quad (2).$$

By substituting for y in this, its value in the equation of the curve, it becomes

$$\frac{dy}{dx} = \frac{3ax + 2a^2 \mp 2a\sqrt{a^2 + ax}}{4\sqrt{a^2 + ax} \cdot \sqrt{-ax \pm \sqrt{a^2x^2 + ax^3}}} \quad (3).$$

If y and x be supposed $= 0$, the values of $\frac{dy}{dx}$ become, one infinite and the other $\frac{0}{0}$. The true values of the latter may be determined by differentiating the numerator and denominator of (3), which gives, when $x = 0$,

$$\frac{dy}{dx} = \pm \frac{1}{\sqrt{2}},$$

or more readily by differentiating the equation (2) reduced to the form

$$4y\sqrt{a^2 + ax}\,dy - (3ax + 2a^2 - 2a\sqrt{a^2 + ax})dx = 0,$$

which gives

$$8y(a^2 + ax)^{\frac{3}{2}}d^2y + 8(a^2 + ax)^{\frac{3}{2}}dy^2 - (4a^3 + 3a^2x)dx^2 = 0,$$

which, when y and x both $= 0$, becomes

$$8a^3dy^2 - 4a^3dx^2 = 0,$$

$$\therefore 2dy^2 - dx^2 = 0,$$

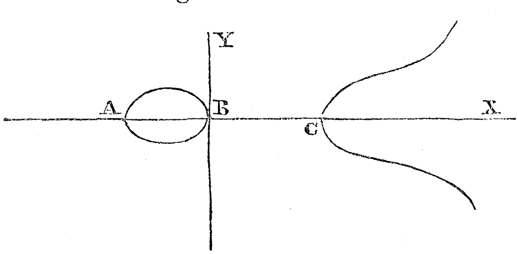
$$\therefore \frac{dy}{dx} = \pm \frac{1}{\sqrt{2}}.$$

Art. 374.

Conjugate points derive their existence from some particular value being given to one of the constants in the equation of a curve, one part of which is an *oval*, which value, rendering the diameter of the *oval* $= 0$, causes it to vanish into a point. Thus the curve represented by the equation

$$ay^2 - x^3 + (b - c)x^2 + bcx = 0$$

has this figure in general. But if c be supposed $= 0$, the oval AB vanishes into a point, and the other branches of the curve continue unchanged.



There is another species of singular point called a *point of undulation*; its nature, and the circumstances which produce it, are explained in the XXth section.

Art. 375, 376.

It appears, from the results of these articles, that the *quadrature* and *rectification* of the circle are two problems depending on each other, so that the solution of either necessarily involves that of the other.

The *quadrature* of the circle is a problem which has exercised and baffled the ingenuity of geometers from the remotest ages of mathematical record. The earliest attempts at its solution were merely tentative. Amongst these are mentioned those of *Anaxagoras*, *Hippocrates*, *Bryson*, and *Antiphon*. The first attempt to ascertain demonstratively the limit of the ratio of the diameter to the circumference was that of *Archimedes*, who proved that the circumference is less than the diameter multiplied by $3\frac{10}{70}$, and greater than the diameter multiplied by $3\frac{10}{71}$. Archimedes might easily have carried his approximation farther, but his object was only to obtain the ratio with sufficient accuracy for practical purposes. A nearer approximation was subsequently made by *Apollonius*.

After the invention of the differential calculus, various mathematicians gave series for calculating the ratio within any proposed limits of accuracy.

Art. 378.

The result of this proposition may be extended. Let a' and b' be any system of semiconjugate diameters, and θ the angle under them, and let two ellipses be constructed, the equal conjugate diameters of each being inclined at the angle θ, and those of one being equal to $2a'$, those of the other to $2b'$; the area of the given ellipse will be a geometrical mean between the areas of these. Thus the proposition will have the same result if the lines in the cut representing the axes be supposed to represent any conjugate diameters, and the two circles to be ellipses, of which these are the equiconjugate diameters respectively.

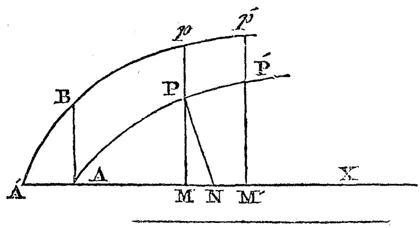
Art. 386.

Amongst the discoveries of *Archimedes*, none is more conspicuous than the quadrature of the parabola. He effected this in two ways; one by mechanical, the other by mathematical principles. He showed by the principles of Statics, independently of any experiment, the relation between the weights of a lamina of matter bounded by right lines, and a parabolic arc, and a rectilinear space. It has been erroneously stated by some that his proof depended on actually weighing the one against the other; but, on the contrary, the demonstration is founded on the abstract principles of Statics, totally independent of any tentative means. He also gave a geometrical solution to the same problem. This solution is memorable for being the first complete one which was given for the quadrature of a curve.

Art. 396.

The *semicubical parabola*, which is here proved to be the evolute of the common parabola, is remarkable for having been the first curve which was rectified. The discovery of this is contended for by *William Neil*, an English geometer, and *Van Huraet*, a Dutch mathematician, who was very active in the cultivation of the Cartesian geometry. They each seem to have a right to the invention, as there is every reason to suppose that neither was aware of the other's discovery. It seems, however, that the English geometer has the priority in publication. The method used by *Van Huraet* merits notice, as it is a general one by which rectification is reduced to quadrature.

Let APP′ be the curve whose rectification is sought, and let PM be an ordinate to any point of the axis AX, and PN the normal, and let m be any given right line.



Assume pM so that

PM : PN : : m : pM,

and all the ordinates being thus produced, let the curve A$'pp'$ be the locus of the extremities of the produced ordinates. The rectangle under the line m and the arc AP is equal to the area ABpM, and therefore if the curve A$'pp'$ is susceptible of quadrature, APP$'$ can be rectified.

Let PM $= y$, and pM $= y'$, $\therefore$ PN $= y \dfrac{\sqrt{dy^2+dx^2}}{dx}$, $\therefore$

$$y' = m \cdot \frac{\sqrt{dy^2+dx^2}}{dx}$$

$$\therefore y'dx = m\sqrt{dy^2 + dx^2},$$

$$\therefore \int y'dx = m\int\sqrt{dy^2 + dx^2}.$$

One side of this equation is the area APM, and the other is the rectangle under the arc AP and the given line m. Hence,

$$\int\sqrt{dy^2 + dx^2} = \frac{\int y'dx}{m}.$$

If the curve be the semicubical parabola represented by the equation

$$py^2 = x^3,$$

by differentiating, we find

$$2pydy = 3x^2dx,$$

$$\therefore \frac{\sqrt{dy^2+dx^2}}{dx} = \frac{\sqrt{9x+4p}}{2\sqrt{p}}.$$

By this substitution, we find

$$y'^2 = m^2 \cdot \frac{9x+4p}{4p},$$

which is the equation of a common parabola, the co-ordinates of whose vertex are

$$y = 0, \; x = -\frac{4}{9}p.$$

Hence the rectification of the semicubical parabola depends on the quadrature of the common parabola, which can be effected geometrically.

If the curve be the common parabola represented by the equation

$$x^2 = py,$$

by differentiating

$$\frac{dy}{dx} = \frac{2x}{p},$$

$$\therefore \frac{\sqrt{dy^2 + dx^2}}{dx} = \frac{p^2 + 4x^2}{p^2}.$$

Hence, by substituting this in the general formula, we find

$$y'^2 = m^2 \cdot \frac{p^2 + 4x^2}{p^2},$$

$$\therefore p^2 y'^2 - 4m^2 x^2 = m^2 p^2,$$

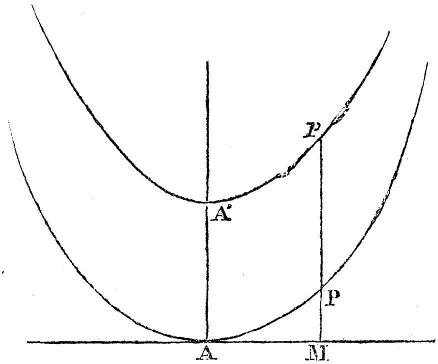
which is the equation of the *hyperbola*. The rectification of the *parabola* therefore depends on the quadrature of the *hyperbola*.

To find the axes 2A and 2B of the *hyperbola*, let x and y successively $= 0$ in the above equation, and we find

$$\text{A}^2 = m^2,$$

$$\text{B}^2 = -\frac{p^2}{4}.$$

Hence, if AP be a *parabola*, and A*p* an *hyperbola*, whose conjugate or second axis is equal to the parameter of the *parabola*, and whose centre is at A, the rectangle under AA$'$, and the arc AP is equal to the area AA$'p$M.



Art. 397.

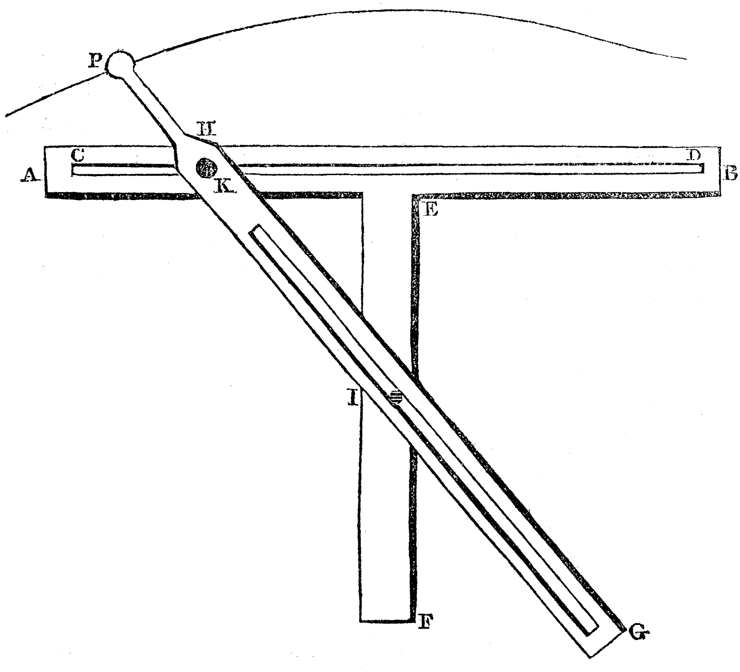
The *logarithmic curve* was first proposed by James Gregory, the celebrated inventor of the reflecting telescope: see his work entitled *Geometriæ Pars Universalis.* Professor Leslie states that Gregory, of St. Vincent, was the inventor. He does not, however, give his authority for this statement, nor does he mention in what work of that geometer the invention is to be found. Its leading properties were very fully investigated by Huygens. It is of considerable importance in its applications to physical science, particularly to the relations and properties of elastic fluids. For example, since the density of the atmosphere decreases geometrically as its height increases arithmetically, its density may be represented by the ordinate to a logarithmic, the altitudes being measured upon the asymptote.

Art. 408.

Nicomedes, a Greek geometer, who lived about two centuries before the Christian era, and shortly after the time of

Archimedes, invented this curve for the solution of the famous problems of the duplication of the cube and the trisection of an angle. He invented also an instrument which has been called the *trammel* for describing it by continuous motion.

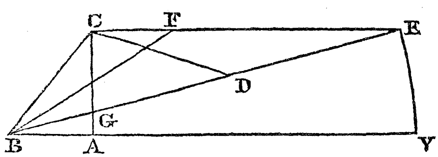
Let AB be a flat ruler in which there is a groove CD. Attached to the middle E of this is another flat ruler EF perpendicular to it, in which at I there is a fixed pin, which is inserted in the groove of a third ruler GH, in which there is also a fixed pin at K, which is inserted in the groove CD. The system being thus adjusted, let a stem of any proposed length HP be attached to H, carrying a pencil at P, and the rectangular rulers AB and EF being fixed, let HG be moved,



so that the pin at K will move in the groove CD, the pin at I continuing in the groove of the moveable ruler, the pencil at P will trace the *superior conchoid.* And if another pencil were fixed to the moveable ruler at the same distance on the other side of the pin K, it would describe the *inferior conchoid.*

To apply the *conchoid* to the bisection of an angle.

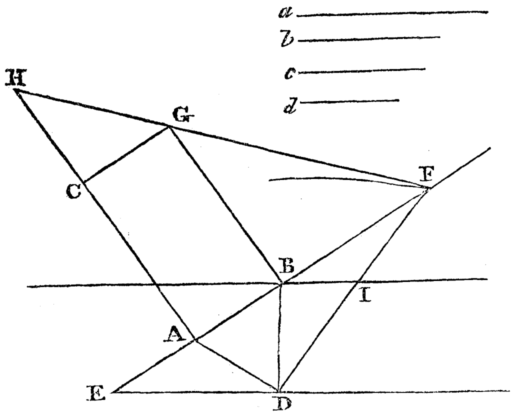
Let ABC be the angle to be trisected. With the vertex B of the given angle as *pole,* and any perpendicular CA to one of its sides BA as *rule,* and a *modulus* AV = 2BC, let a *conchoid* be described, and let CE be drawn parallel to AV meeting the curve in E, and draw BE. Let BF be drawn bisecting the angle CBE. The angle ABC will be trisected by BF and BE.



For let GE be bisected at D, and draw DC. Since GCD is a right angle, CD = DE = GD, ∵ CD = CB. Hence the angle CBD is equal to CDB, which is equal to twice CED. But CED is equal to DBV, therefore CBD is equal to twice DBV; and since CBD is bisected by BF, it is plain that ABC is trisected by BE and BF.

To find two geometrical means (b and c) between two given lines (a and d) by means of the *conchoid.*

Let a rectangle be constructed, the sides of which are equal to the extremes AB = a, AC = d. On AB construct an isosceles triangle BDA, the side of which BD is equal to half of AC. Produce BA so that AE = BA, and connect D and E, and



through B draw BI parallel to DE. Through B produce AB, and with D as *pole*, BI as *rule*, and BD as *modulus*, describe a *conchoid* meeting AB produced in F, and draw FG intersecting AC produced in H. Then BF $= b$, and CH $= c$.

For since BI and DE are parallel, DI : IF : : EB : BF, or DI : $\frac{1}{2}d$: : $2a$: BF. But also on account of the similar triangles, HC : a :: d : BF, ∵ HC = DI. Since BDA is isosceles, the square of DF is equal to the rectangle under AF and FB, together with the square of BD. But also the square of DF is equal the square of the sum of HC, and half of AC, or to the rectangle under AH and HC, together with the square of half of AC. Taking away this last from both, it follows that the rectangle under AH and HC is equal to the rectangle under AF and FB. By this and the similar triangles we have the proportions

AH : AF : : AC : BF,
AH : AF : : BF : HC,
AH : AF : : HC : AB,
∵ AC : BF : HC : AB,
or $a : b : c : d$.

Montucla in his history has fallen into an error in giving the construction for the solution of this problem. He constructs the isosceles triangle in an altitude equal to half the line AC.

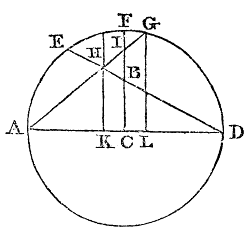
The *conchoid* is the only monument of the labours of *Nicomedes* which has descended to us. In the appendix to his *Universal Arithmetic,* Newton approves of it highly for the construction of the roots of equations of the third and fourth order. For these purposes he prefers it to the means which the lines of the second order present.

The intercept of the superior *conchoid* between the vertex and the point of inflection is sometimes used in architecture as the figure of the shaft of the Corinthian column.

The etymology of the name *conchoid* is from the Greek word κογχος, *a shell.*

Art. 412.

Diocles, a Greek geometer, who lived in the fifth century after Christ, was the inventor of this curve. The occasion of its invention was the solution of the problem of the insertion of two mean proportionals. *Pappus* had previously shown that this problem might be solved by the following construction. Let the extremes AC and CB be placed at right angles, and with C as centre and the greater CA as radius, describe a circle and join DB, and produce it to meet the circle at E, and produce CB to meet the circle at F. Let a chord AG be inflected so that HG shall be bisected by CF, and CI will be the first of the two means.

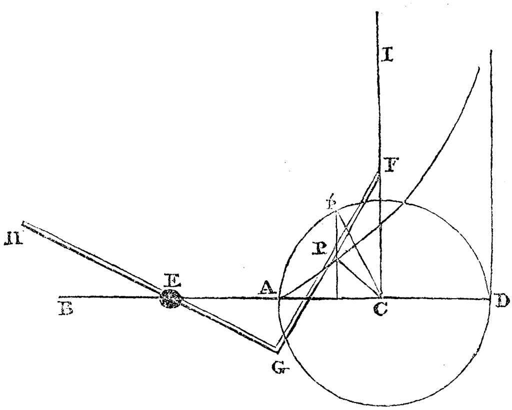


Thus the solution of the problem is made to depend upon the possibility of inflecting AG so as to be bisected by CF.

This led Diocles to investigate the locus of the point H, AC being considered constant and CB variable, and which is evidently equivalent to supposing AK always equal to LD.

To attain the object for which the *cissoid* was invented, it was still, however, necessary to be able to describe this curve mechanically, and here the inventor failed. NEWTON, however, gives a very simple and elegant method of effecting this.

Produce the axis DA of the curve until AE = AC, and through the centre draw CI perpendicular to AB. Let an indefinite fixed ruler be placed upon CI, and let a square, having one leg GF equal to AD, and the other GH indefinite, be so moved, that the indefinite leg GH shall always pass through E, and that the extremity F of the other leg shall move along the indefinite ruler CI; if a pencil be attached to the middle point P of GF, it will trace out the *cissoid*. For the proof of this see (626).



The quadrature of the cissoid has been inadvertently omitted in the text. It is easily effected. By the equation of the curve solved for y,

$$y = \frac{x^{\frac{3}{2}}}{(2r - x)^{\frac{1}{2}}},$$

$$\because ydx = \frac{x^{\frac{3}{2}}dx}{(2r-x)^{\frac{1}{2}}}.$$

If the angle P′CA $= \phi$, $x = (1 - \cos. \phi)r$, where $r =$ AC, and $\sqrt{2rx - x^2} = r \sin. \phi$. Hence

$$\int ydx = \int \frac{x^2dx}{(2rx-x^2)^{\frac{1}{2}}} = \int \frac{-(1-\cos. \phi)^2 d \cos. \phi}{\sin. \phi} . r^2,$$

$$\text{or} \int ydx = r^2(1 - \cos. \phi)^2 . d\phi,$$

$$\because \int ydx = r^2(\tfrac{3}{2}\phi + \tfrac{1}{2} \sin. \phi \cos. \phi - 2 \sin. \phi),$$

which, taken between the limits $\phi = 0$ and $\phi = 2\pi$, gives

$$\int ydx = 3r^2\pi,$$

which shows that the area included between the cissoid and its asymptote is equal to three times the area of the circle, whose radius is CB.

The name of this curve is derived from the Greek word κισσος, *ivy*.

Eutocius attributes to Diocles the solution of the problem "to divide a sphere by a plane into two segments in a given ratio;" a problem which at that time presented considerable difficulty. The solution given proves him to have been a most able and profound geometer. It is however subject to the same objection as most of the ancient solutions, as he employs two conic sections instead of using one and the circle. The work from which Eutocius quotes the solutions was entitled De Pyriis, from which Montucla thinks that Diocles was probably an engineer.

Art. 418.

The properties of this curve were first considered by James Bernouilli, of whom see note on art. 430. It belongs to a general class of curves which are investigated in (631) and (632).

Among the physical properties of this curve, we may observe that a body moving in it, by the influence of a force directed to its centre, would be attracted by a force varying as the inverse seventh power of the distance, and that its velocity would bear to that in a circle at the same distance the invariable ratio $1 : \sqrt{3}$.

The chord of the osculating circle passing through the pole is two-thirds of the distance of the point of osculation from the centre, and hence the locus of the point where the osculating circle intersects the radius vector is the lemniscata represented by the equation

$$z^2 = \tfrac{1}{9}a^2 \cos. 2\omega.$$

Art. 422, *et seq.*

The trigonometrical curves took their origin probably from "the extension of the meridian line by Edward Wright, who computed that line by collecting the successive sums of the secants, which is the same thing as the area of the figure of secants." Hutton, Math. Dict. art. FIGURE.

Under this point of view, the area of the curve of sines determines the sum of all the sines in the semicircle to be equal to twice the square of the radius.

This curve differs only apparently from the companion of the cycloid. See note on art. (497).

If the ordinates of the curve of sines be increased or diminished in the same ratio, the *harmonic curve* will be produced. The areas of these curves will be evidently as their ordinates; and tangents from the extremities of coincident ordinates have the same subtangent. *Taylor de Incrementis.*

If the points of the axis be endued with an attractive power which would cause a material point to move in the *sinusoid*, the force would vary as the ordinates. The curve enjoys this property in common with the *logarithmic*, in which, however, the force must be repulsive.

If a *sinusoid* be described on paper, and the paper wrapped on a cylinder, the radius of which is equal to the axis of the curve, or to the radius with respect to which the curve is constructed, and so that the base of the curve shall coincide with the circumference of a circle made by the section of the cylinder by a plane perpendicular to its axis; all the points of the curve will lie in the same plane intersecting the axis of the cylinder at an angle of 45°, and therefore the curve will be coincident with the ellipse made by the section of that plane with the cylinder. The rectification of the sinusoid depends therefore on that of the ellipse, its length being equal to the circumference of an ellipse, of which the semiconjugate axis is equal to the axis of the *sinusoid*, and the semitransverse axis to the semiconjugate in the ratio of $\sqrt{2} : 1$. This beautiful property is nearly evident from the consideration that the axis, which in the original curve is considered as the circumference of a circle extended into a straight line, is, when wrapped round the cylinder, restored to its proper form, and the sines of the corresponding arcs are equal to the ordinates, being the sides of a right angled triangle having an acute angle of 45°.

I am not aware whether it has been noticed that this property extends to all *harmonic curves*. This class of curves is represented by the equation

$$y = m \sin. x.$$

If a curve of this kind be rolled round a cylinder in the manner before described, it will coincide with an elliptical section of the cylinder by a plane inclined to the axis at an angle whose tangent is $\frac{1}{m}$. Thus the rectification of the harmonic curves and ellipses depend on the same principles.

Art. 430.

The earliest notice we find of this curve is in the works of DESCARTES, who must be considered as possessing all the credit of its invention. In the investigation of the motion of bodies on inclined planes, he observed, that the part of the force of gravity which accelerates a body down an inclined plane cannot be rigorously considered constant, inasmuch as the direction of the force of gravity is continually changing, and the direction of the plane continues unchanged. This suggested the investigation of the figure of a line on which a body would be uniformly accelerated by the influence of an uniform force directed to a fixed point. He therefore inferred the true line of descent to be a spiral described round the centre of the earth. Being afterwards solicited by *Father Mersenne* to give a more explicit account of what he meant, he answered that the characteristic property of the curve was that which has been proved in (433).

Upon this being made known to the mathematicians, the properties proved in arts (431) and (437), and their consequences, were immediately discovered. But a complete discussion of the properties of this curve was reserved for JAMES BERNOUILLI. This great geometer was born at *Basil*, 27th Dec. 1654, and died at the age of 50. He was the first to apply the new calculus to geometrical investigations, and to bring it into general use. One of the first curves which he examined in this manner was the *logarithmic spiral*. He discovered it to be its own *evolute* and *involute*, its own *caustic* and *pericaustic*, both by reflection and refraction, the focus of incident rays being at the pole. His enthusiastic admiration of this curve may be conceived from the following passage from a paper of his published in the *Leipsic* acts, and quoted by Mr. *Peacock* in his excellent collection of examples on the calculus.

"Cum autem ob proprietatem tam singularem tamque admirabilem mire mihi placeat spira hæc mirabilis, sic ut ejus contemplatione satiari vix queam; cogitavi, illam ad varias res symbolice repræsentandas non inconcinne adhiberi posse. Quoniam enim semper sibi similem et eandem spiram gignit, utcunque volvatur, evolvatur radiet; hinc poterit esse vel sobolis parentibus per omnia similis emblema; *simillima filia matri.* Vel (si rem æternæ Veritatis Fidei mysteriis accommodare non est prohibitum) ipsius æternæ generationis Filii, qui Patris veluti imago, et ab illo ut lumen a lumine emanans, eidem ομοουσιος existit qualiscunque adumbratio. Aut, si mavis, quia curva nostra mirabilis in ipsa mutatione semper sibi constantissime manet similis et numero eadem, poterit esse, vel fortitudinis et constantiæ in adversitatibus; vel etiam carnis nostræ post varias alterationes et tandem ipsam quoque mortem, ejusdem numero resurrecturæ symbolum; adeo quidem, ut si Archimedem imitandi hodiernum consuetudo obtineret, libenter spiram hanc tumulo meo juberem incidi cum Epigraphe: *Eadem numero mutata resurgo.*"

It might be further observed, that if a planet moved in a *logarithmic spiral*, the sun being in the pole, the curve of a star's aberration in a plane parallel to the plane of the orbit would be also a logarithmic spiral. Also, if a body moving in a *logarithmic spiral*, the force being at the pole, be stopped at any point, and allowed to descend towards the pole, the locus of the point at which it will acquire the velocity in the curve is a *logarithmic spiral.* See arts. (638) (655).

If the pole of a logarithmic spiral be a centre of force, the law of the force necessary to retain a body in the spiral is the inverse cube of the distance.

PROFESSOR LESLIE has described an instrument for tracing this curve mechanically. See his Geometry of Curve Lines, p. 436.

If a *logarithmic spiral* be described upon the plane of a great circle of a sphere with the centre as pole, and perpendiculars be drawn from every point in it to meet the surface of the sphere, the extremities of those perpendiculars will trace out a *loxodromic curve*, or, in other words, the projection of the *loxodromic curve* on the plane of the equator is a *logarithmic spiral.* The loxodromic curve is the track of a ship which continues sailing towards the same point of the compass, provided that point be not one of the cardinal points. Its distinctive property is that it cuts all the meridians at the same angle. The properties of this curve were fully investigated by James Bernouilli, being one of those on which the powers of the calculus were first tried.

The *loxodromic curve* may be looked upon as a kind of logarithmic spiral described upon the surface of the sphere, of which the *pole* is the *pole* of the sphere, and of which the *radius vector* is the arc of a meridian intercepted between the point and the pole.

The genesis of the *logarithmic spiral* may be derived from the *logarithmic.* Let a logarithmic be represented by the equation

$$y = a^x.$$

With the origin of co-ordinates as centre, and a radius equal to the linear unit, let a circle be described, and a tangent to this circle be drawn parallel to the asymptote. Suppose this tangent a flexible string to which the ordinates y are attached, so as to continue perpendicular to it, and let it be wrapped round the periphery of the circle. The ordinates still retaining the same length, and still continuing to be perpendicular to the string, will all meet in the centre, and the portions of them which before were intercepted between the asymptote and parallel tangent become radii of the circle emanating from the centre, the parts which lay above the tangents being the productions of the radii. The values of x become arcs of the circle with the radius unity, so that, if x be changed into ω, and y into r, the equation of the curve found by the extremities of the ordinates which have now become radii vectores is

$$r = a^{\omega},$$

which is that of the *logarithmic spiral.*

It is plain that this reasoning is not confined to the logarithmic spiral. All curves represented by equations related to rectangular co-ordinates have corresponding spirals, whose equations may be derived in the same manner.

Thus the equations

$$y^2 = px,$$
$$yx = p,$$

give two spirals,

$$r^2 = p\omega,$$
$$r\omega = p,$$

called, for the same reason, the *parabolic* and *hyperbolic spirals.*

Or, as the class of parabolæ and hyperbolæ in general are represented by the equations

$$y^n = px^m,$$
$$y^n x^m = p,$$

where m and n are positive integers, so the general classes of parabolic and hyperbolic spirals are included under the general equations

$$r^n = p\omega^m,$$
$$r^n \omega^m = p.$$

Art. 445.

This spiral was first imagined by *Conon*, a friend of *Archimedes.* A point being supposed to move uniformly towards the centre of a circle, and at the same time the radius passing through it to revolve round the centre with an uniform angular velocity; by the combination of these two motions, the point will describe the spiral. But *Conon* advanced no further than merely to imagine its description. All its properties were discovered by *Archimedes*, and it has been hence called *the spiral of Archimedes*, because, as *Montucla* says, "Celui qui pénètre fort avant dans un pays inconnu, mérite à plus juste titre de lui donner son nom, que celui qui ne fait que le reconnoitre."

Archimedes was a native of Sicily, born about three centuries before Christ. He was the most illustrious of the ancients in both geometry and mechanics. Of the latter, as a science, he may be justly styled the father; for until his time almost no general principles of mechanics were known. Amongst his most remarkable geometrical discoveries may be enumerated the relation between spheres, cylinders, and cones; the approximation to the quadrature of the circle (see note on art. 376); his discoveries of the properties of conoids and spheroids, and his quadrature of the parabola. In mechanics he first established the condition of equilibrium, that the weights must be inversely as their distances from the centre of motion, and the properties of the centre of gravity, and the methods of finding it. His discoveries in Hydrostatics were occasioned by the well known circumstance of the golden crown of Hiero, in which an alloy of silver was supposed to be mixed by the artist, and which fact Archimedes discovered by weighing it in water, and ascertaining the specific gravity by the loss of weight. When Syracuse was besieged by the Romans, he assisted the citizens in defending it by the invention of offensive machines, which struck such horror into the besiegers, that they were obliged to discontinue their attack, and turn the siege into a blockade. The Syracusans slumbering in too great security, left part of the walls unguarded on some occasion, whereby the

Romans were enabled to scale them and possess themselves of the town, and Archimedes fell by the hand of a Roman soldier in the seventy-fifth year of his age.

Art. 454.

The hyperbolic spiral is one of a general class of spirals, in which, if a material point were to move attracted by a force directed to the pole, the law of its variation would be the inverse third power of the distance. One of the most remarkable circumstances attending the motion in this curve is, that the centripetal and centrifugal forces are equal, and therefore the paracentric velocity is uniform.

If the earth's orbit were an hyperbolic spiral, the sun being at the pole, the curve of aberration of a star in a plane parallel to that of the earth's orbit would be the involute of the circle.

Art. 471.

The *cycloid* has been so remarkable for the dissensions it has created amongst those mathematicians who discovered its properties, that it has been called the *Helen* of geometry. It was first imagined by *Galileo*, who long sought its quadrature without success. Upon this failure he attempted to discover its relation to the area of the generating circle by describing a cycloid and its generating circle on a lamina of matter of uniform thickness, and ascertaining their weights. The result of this experiment showed the cycloid to be nearly but not exactly three times the area of the generating circle, and finding several repetitions of the experiment agree, he abandoned the inquiry, concluding that the ratio could not be expressed by rational numbers. This circumstance, which certainly does not reflect much honour on the memory of the great Italian philosopher, has been defended by the example of *Archimedes* in his quadrature of the parabola; but this was effected in a very different manner. *Archimedes* founded his solution on the abstract principles of equilibrium, axioms nearly as general and certain as those of mathematical science itself.

About the year 1630 the same problem was proposed by *Mersenne* to *Roberval*, who, after a period of six years spent in the cultivation of the geometry of the Greeks, and particularly the works of *Archimedes*, gave the solution of it, and proved the area three times that of the generating circle. *Descartes* being apprised by *Mersenne* of the discovery of Roberval, declared that any one tolerably skilled in geometry would have solved the problem, and, on the instant, himself gave a solution for it. This was the foundation of a quarrel between Descartes and Roberval.

Descartes next discovered the method of drawing tangents to the cycloid, and challenged the mathematicians of the day to solve the problem, which was effected by *Fermat*, a geometer who may be considered to rank almost with Descartes himself.

Subsequently Pascal discovered the quadrature of any segment of a cycloid, and the contents of the segments of solids of revolution formed by a cycloid revolving round its base or axis, and their centres of gravity.

The rectification of the cycloid was discovered by Sir Christopher Wren, who also discovered the dimensions of the surfaces of solids of revolution of the cycloid, and the centre of gravity of a cycloidal arc.

The rectification of the curtate and prolate cycloids was proved by Pascal to depend on the rectification of an ellipse.

The evolute of the cycloid was discovered by Huygens, who also discovered the remarkable physical property that a cycloidal pendulum is tautochronous. A cycloid is the line of swiftest descent between two points so placed, that the line which joins them shall be neither vertical nor horizontal.

It is remarkable, that if a material point describe a cycloid by the attraction of a force parallel to the axis, the law of the force is the inverse square of the ordinate; a law analogous to that of universal gravitation. And further, the times of describing different cycloids, whose bases coincide, observe the harmonic law, their squares being proportional to the cubes of the axes.

Art. 477.

The notation $\cos.^{-1} A$ signifies the angle, the cosine of which is A. This very convenient notation is the invention of Mr. J. F. Herschel, of Cambridge. It is used in the valuable collection of examples on the Calculus of Finite Differences, lately published by him.

Art. 497.

The discovery of this curve followed that of the cycloid. Its properties were investigated by *Roberval*, *Wallis*, *Lalouere*, and others. It is the curve of sines presented under another point of view; for the equation

$$y - r \cos. \frac{x}{r} - r = 0$$

being put under the form,

$$y - r = r \cos. \frac{x}{r},$$

and the origin transformed to the middle point of the axis, gives

$$y = r \cos. \frac{x}{r},$$

which may be expressed,

$$y = r \sin. \left(\frac{\pi}{2} - \frac{x}{r}\right),$$

which is the equation of the curve of sines.

It is therefore a species of the more general class called *harmonic curves*. See note on art. 422.

Art. 506.

The invention of epicycloids is due to *Roemer*, the Danish astronomer, illustrious for the discovery of the progressive motion of light by observations on the satellites of Jupiter. He invented this class of curves in Paris, about the year 1674, and showed that an epicycloid is the proper figure for the teeth of wheelwork, so as to prevent as much as possible the friction arising from their action. The first who gave a solution for their rectification was Newton, in the first book of his PRINCIPIA. Their properties were subsequently investigated by John Bernouilli. The epicycloids are remarkable for being among the *caustics* of the circle. See art. 668.

Art. 516.

This curve has a remarkable physical property. If a material point be supposed to move in it attracted by a force directed to its cusp, the law of the force will be the inverse fourth power of the distance. Also, the velocity in it bears to the velocity in a circular orbit, at the same distance, the invariable ratio $\sqrt{2} : \sqrt{3}$.

If a chord be drawn through the cusp of the cardioide, it will always be equal to the axis; for in its equation, if $\pi + \omega$ be substituted for ω, and the two values of z added, their sum will be equal to the axis, since

$$\cos.\ \omega = -\cos.\ (\pi + \omega).$$

Also, if tangents be drawn through the extremities of any of these chords, they will intersect at right angles; for if θ and θ' be the angles under the radius vector and tangent, we have the condition,

$$\theta = \frac{\pi + \omega}{2},$$

$$\theta' = \frac{\omega}{2},$$

$$\theta - \theta' = \frac{\pi}{2},$$

and $\theta - \theta'$ is obviously equal to the angle under the tangents.

This curve derives its name from the similitude of its figure to *a heart.*

Art. 529.

Dinostratus, a pupil of Plato, applied this curve to the quadrature of the circle and the multisection of an angle. Although the curve has been distinguished by the name of this geometer, there is some reason to suppose its invention was antecedent to his time. Proclus mentions the quadratrix as the invention of *Hippias*. Now the only ancient geometer of this name was a contemporary of Socrates, and therefore prior to the time of Dinostratus. The mere circumstance of the curve being named from Dinostratus is no more a proof of its being his invention than the name of the spiral of Archimedes proves it the invention of that geometer.

Art. 539.

This curve is named from its inventor *Tshirnhausen*, a German mathematician of the seventeenth century. He is celebrated for having been one of the first to adopt and apply the modern inventions of the differential calculus and the geometry of Descartes. He was also the inventor of *caustics.*

Art. 545.

For the origin of the name of this curve, see note on art. 652.

Art. 567.

The *tractrix* has been very erroneously identified with the *catenary*. See Hutton's Mathematical Dictionary, Tractrix. The tractrix received its name from a supposition that it is the curve which would be described by a weight drawn on a plane by a string of a given length, the extremity of which is carried along the directrix. Euler has shown that this conclusion is wrong, unless the momentum of the weight which is generated by its motion be every instant destroyed. The real track of the weight he has shown to be a semicycloid with its vertex downwards. See Euler, *Nova Comm. Petrop.*, 1784. The tractrix was invented by Huygens.

An instrument is described by Professor Leslie for describing mechanically the tractrix or its involute the catenary.

Art. 580, *et seq.*

The method of determining the roots of equations was probably suggested to Descartes, who appears to have been the first who used it, by the method of the ancients for solving determinate problems by the intersection of geometric loci, which originated in the Platonic school. In his *Geometrie*, Descartes constructs equations of the third degree by multiplying them by $x = 0$, as in art. 585, and thereby reduces them to the fourth degree, and constructs them by the intersection of the circle and parabola. He also gives similar methods of constructing equations of superior orders by a curve of the third degree, called the *parabolic conchoid* and the circle. Descartes supposed that the most simple mean of constructing equations by the intersection of curves was to select from the different curves capable of fulfilling the required conditions, those whose equations were of the simplest form. Newton, however, was directed in his choice by a different principle. He considered that the principle of Descartes would make the parabola more proper than the circle, since the equation $y^2 = px$ is simpler than any form which the equation of the circle can assume. He therefore selected those curves as fittest for the purpose which were most easily described by continued motion. In this respect he conceived the conchoid of Nicomedes to be the most proper for the purpose next to the circle, as the instrument by which it is described (see note on art. 408) is next in simplicity to the compass. Newton, however, seems to have overlooked the instrument described in the note on art. 298 for tracing the ellipse by continued motion, and which is certainly simpler than the trammel of Nicomedes.

Art. 588.

The resolution of $(x^m \pm a^m)$ into its factors was first effected by Cotes, and published in his *Harmonia Mensurarum*, in the year 1722, being six years after his death. In the same year the more general theorem by which

$$x^{2m} - 2x^m \cos. \phi + 1$$

is resolved into its factors was published by Moivre in the *Philosophical Transactions*. The elegant theorem expressed by the equation

$$(\cos. x + \sqrt{-1} \sin. x)^m = \cos. mx + \sqrt{-1} \sin. mx$$

is also the discovery of *Moivre*, and that on which the former depends. It may be established thus: Let

$$y = \sin. x, \qquad v = \cos. x,$$
$$\because dy = \cos. x dx, \quad dv = - \sin. x dx,$$
$$dy = v dx, \qquad dv = - y dx.$$

If the first be multiplied by $\sqrt{-1}$, and added to the second, the result is

$$dv + \sqrt{-1}\, dy = \{-y + v\sqrt{-1}\} dx,$$
$$\because dv + \sqrt{-1}\, dy = (y\sqrt{-1} + v) dx \sqrt{-1},$$
$$\because \frac{d(v + \sqrt{-1}\, y}{v + \sqrt{-1}\, y} = x \sqrt{-1},$$

which, by integration, becomes

$$\log. \{v + \sqrt{-1}\, y\} = x\sqrt{-1},$$
$$\because v + \sqrt{-1}\, y = e^{x\sqrt{-1}}.$$

No constant is added, because when $x = 0$, $y = 0$, and $v = 1$. The values of v and y being substituted, we find

$$\cos. x + \sqrt{-1} \sin. x = e^{x\sqrt{-1}},$$

and therefore in general,

$$\cos. \phi + \sqrt{-1} \sin. \phi = e^{\phi\sqrt{-1}};$$

let $\phi = mx$, and we find

$$\cos. mx - \sqrt{-1} \sin. mx = e^{mx\sqrt{-1}},$$

and by raising the former equation to the mth power,

$$(\cos. x + \sqrt{-1} \sin. x)^m = e^{mx\sqrt{-1}},$$
$$\because (\cos. x + \sqrt{-1} \sin. x)^m = \cos. mx + \sqrt{-1} \sin. mx.$$

In the whole range of analysis there is probably no formula which exhibits more simplicity and elegance in its form, and extensive utility in its various applications, than this. It may be considered as implicitly involving the whole science of trigonometry. Although it may be somewhat foreign to the subject of the text, we trust the student will excuse us for giving him here a few examples of its fertility.

By multiplying the equations

$$\cos. x + \sqrt{-1} \sin. x = e^{x\sqrt{-1}},$$

$$\cos. y + \sqrt{-1} \sin. y = e^{y\sqrt{-1}},$$

we find

$$\cos. x \cos. y - \sin. x \sin. y + \sqrt{-1}$$

$$(\sin. x \cos. y + \sin. y \cos. x) = e^{(x+y)\sqrt{-1}};$$

but we have also

$$\cos. (x + y) + \sqrt{-1} \sin. (x + y) = e^{(x+y)\sqrt{-1}},$$

$$\because \cos. x \cos. y - \sin. x \sin. y + \sqrt{-1}$$

$$(\sin. x \cos. y + \sin. y \cos. x) = \cos. (x + y) + \sqrt{-1} \sin. (x + y).$$

The real and impossible parts of this equation must be respectively equal, and therefore

$$\cos. x \cos. y - \sin. x \sin. y = \cos. (x + y),$$

$$\sin. x \cos. y + \sin. y \cos. x = \sin. (x + y).$$

From these equations may be deduced all the other formulæ of trigonometry.

We can find expressions for the sine and cosine of an arc in terms of the arc itself from the same formulæ by adding and subtracting the equations

$$\cos. x + \sqrt{-1} \sin. x = e^{x\sqrt{-1}},$$

$$\cos. x - \sqrt{-1} \sin. x = e^{-x\sqrt{-1}},$$

the results of which are

$$\cos. x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2},$$

$$\sin. x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}.$$

From these beautiful formulæ may easily be deduced the series for the sin. x and cos. x in powers of x. *Euler* derived the above expressions from comparing these series with the developement of e^x. See Vol. VII. Misc. Berol. He also deduced them from the series for multiple arcs. We shall pursue the results of this formula no farther, having said enough to excite the young student to further inquiry.

Art. 592.

Curves represented by equations, in which the exponents of the variables, or any of them, are rational numbers, are called by Leibnitz *interscendental*, as holding an intermediate place between algebraic and transcendental curves.

Art. 593.

In the geometrical treatises on curve lines by Professor Leslie, the degree of a line is determined by the greatest number of points in which a right line can intersect it. This, however, is not any criterion for the degree of a curve; for there are many curves of the fourth degree, which no right line can intersect in more than two points. For example, the curve which is discussed in art. 631, when $c < b$. It is true, that in this case the curve has a conjugate point through which a line passing is equivalent to two points of intersection; but the existence of conjugate points cannot be recognised geometrically.

Art. 602.

The principle assumed here, that the equation found by eliminating one of the variables from two equations, one of the mth, and the other of the nth degree, cannot exceed the mnth degree, may be thus established: Let the two proposed equations be

$$x^m + Ax^{m-1} + Bx^{m-2} \quad . \quad . \quad . \quad V = 0 \quad (1),$$
$$x^n + A'x^{n-1} + B'x^{n-2} \quad . \quad . \quad . \quad V' = 0 \quad (2),$$

in which the co-efficients A, B, . . . V and A′, B′, . . . V′ are functions of y of the following forms:

$$A = ay + b,$$
$$B = cy^2 + dy + e,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad .$$
$$. \quad . \quad . \quad . \quad . \quad . \quad .$$
$$V = py^m + qy^{m-1} \quad . \quad . \quad . \quad v,$$
$$A' = a'y + b',$$
$$B' = c'y^2 + d'y + e',$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$V' = p'y^n + q'y^{n-1} \quad . \quad . \quad . \quad v'.$$

Now it is evident that the degree of the final equation will not be diminished, if these co-efficients be supposed only to consist of that term in each, which involves the highest dimension of y. By this condition the several co-efficients are reduced to

$$A = ay,$$
$$B = cy^2,$$
$$. \quad . \quad . \quad .$$
$$. \quad . \quad . \quad .$$
$$V = py^m,$$
$$A' = a'y$$
$$B' = c'y^2,$$
$$. \quad . \quad . \quad .$$
$$. \quad . \quad . \quad .$$
$$V' = p'y^n.$$

And the equations are reduced to the forms,

$x^m + ax^{m-1}y + bx^{m-2}y^2 + cx^{m-3}y^3 \ldots txy^{m-1} + vy^m = 0$ (3),
$x^n + ax^{n-1}y + b'x^{n-2}y^2 + c'x^{n-3}y^3 \ldots txy^{n-1} + v'y^n = 0$ (4).
These equations are less general than (1) and (2); but as far as respects the final equation resulting from the elimination of either of the variables, its degree is not diminished by the deficiency of the terms, including the inferior dimensions of the variables.

Let (3) be divided by y^m, and (4) by y^n, and the results are

$$\left(\frac{x}{y}\right)^m + a\left(\frac{x}{y}\right)^{m-1} + b\left(\frac{x}{y}\right)^{m-2} \ldots t\left(\frac{x}{y}\right) + v = 0,$$
$$\left(\frac{x}{y}\right)^n + a'\left(\frac{x}{y}\right)^{n-1} + b'\left(\frac{x}{y}\right)^{n-2} \ldots t'\left(\frac{x}{y}\right) + v' = 0.$$

Considering $\frac{x}{y}$ as the unknown quantity, let the roots of the first equation $r, r', r'', \ldots$ and those of the second $p, p', p'', \ldots$ and they may be expressed

$$\left(\frac{x}{y} - r\right)\left(\frac{x}{y} - r'\right)\left(\frac{x}{y} - r''\right) \ldots = 0,$$
$$\left(\frac{x}{y} - p\right)\left(\frac{x}{y} - p'\right)\left(\frac{x}{y} - p''\right) \ldots = 0,$$

which being multiplied by y^m and y^n respectively,

$$(x - ry)\ (x - r'y)\ (x - r''y) \ldots = 0,$$
$$(x - py)\ (x - p'y)\ (x - p''y) \ldots = 0.$$

If the values which fulfil the latter be successively substituted in the former, the results are

$$y^m(p - r)\ (p - r')\ (p - r'') \ldots = 0,$$
$$y^m(p' - r)\ (p' - r')\ (p' - r'') \ldots = 0,$$
$$y^m(p'' - r)\ (p'' - r')\ (p'' - r'') \ldots = 0.$$

. . . .

And since the number of roots of the latter cannot exceed n, the number of these equations cannot exceed n. The product of these equations is necessarily the equation which would result from the elimination of x; for it becomes $= 0$ for those roots which make its several factors vanish, and only for those.

It is obvious that the condition which has been introduced affects the numerical values of each system of values of y and x which fulfil the equation, but it does not increase or diminish the number of such values, which is the only object of the present inquiry. This is only a particular case of a much more general algebraical theorem respecting the final equation found by eliminating $(n - 1)$ variables by means of n equations of any proposed degrees. What has been proved here is sufficient for the particular application of the principle made in the text. Students desirous of inquiring further will find extensive information on this subject in *Garnier's Elemens d'Algebre*, Chap. XXV. His *Analyse Algebrique*, Chap. VIII. Also an Essay by *M. Bret*, published in the Journal of the *Polytechnic School*, Cah. 15, Tom. VIII. The extension of the principle to an equation involving any even number of variables has been effected by *M. Poisson:* see *Journal de l'Ecole Polytechnique*, Cah. 11.

Art. 608.

Those who are desirous of further information concerning the general properties of algebraic curves are referred to Cramer, *Int. à l'Analyse de Lignes Courbes.* Euler's *Analysis Infinitorum.* Stirling on Newton's enumeration of lines of the third order. De Gua, *l'Usage de l'Analyse*, &c. We conceived that entering further on the subject in the present treatise would be swelling the volume without offering any adequate advantage.

Art. 629.

This curve is called *the witch.* It is the invention of an

Italian lady, M. Maria Gaetana Agnesi, a celebrated mathematician. She is the author of a work on the algebraic and transcendental analysis, entitled Analytical Institutions. She subsequently, according to Montucla, retired to a convent.

Art. 631.

These articles contain a full investigation of the properties of the general class of curves, of which the lemniscata of Bernouilli is a very particular case. I am not aware whether the properties of these curves have been ever investigated.

Art. 633.

This curve was invented by the celebrated Cassini, and is sometimes called Cassini's ellipse.

Art. 635.

This proposition and its applications were suggested to me by an article in the works of John Bernouilli. He applies it to show the relation between the parabola and the spiral of Archimedes. John Bernouilli derived the idea from a paper published by his brother James in the Leipsic acts, in which he supposes the axis of a parabola wrapped upon the circumference of a circle and its ordinates, therefore to converge towards the centre, and proposes to investigate the spiral so produced.

Art. 636.

This question is solved by Delambre, by considering that as the sun in the course of each day describes a parallel of declination, a ray passing through the top of the perpendicular style will describe a conical surface, the intersection of which with the horizon will be the path of the shadow. The method given in the text, however, seems more analytical.

Art. 638.

In this and the following propositions the effect of the projection of the aberration on the surface of the sphere is not taken into account. The curve whose plane is parallel to the ecliptic being determined, its projection may easily be found.

Art. 646.

The equation of the path of a projectile *in vacuo* may be determined thus: Let the axes of co-ordinates be vertical and horizontal, and the force of gravity be represented by $2m$, the velocity it produces in the unit of time. This force acting parallel to the axis of y, and there being no force acting parallel to the axis of x, we have

$$\frac{d^2y}{dt^2} = -2m, \quad \frac{d^2x}{dt^2} = 0.$$

By integrating these

$$\frac{dy}{dt} = c - 2mt \ (1), \quad dx = c'dt \ (2),$$

c and c' being the arbitrary constants introduced in the integration.

By integrating a second time,

$$y = ct - mt^2, \quad x = c't.$$

By eliminating t by these equations, we find

$$y = \frac{c}{c'}x - \frac{m}{c'^2}x^2,$$

which is the equation of the path of the projectile. Each of the integrations has its peculiar signification. The value of $\frac{dy}{dt}$, determined by the first integration, expresses the vertical velocity, and $\frac{dx}{dt}$ the horizontal velocity of the projectile. The equation (2) shows that the horizontal velocity is uniform, and expressed by the constant c'. The equation (1) shows that the constant c expresses the velocity of projection resolved in the vertical direction.

If H be the height due to the velocity of projection, that velocity is $2\sqrt{\text{H}m}$. Hence we have

$$c = 2\sqrt{\text{H}m} \sin. \varepsilon,$$
$$c' = 2\sqrt{\text{H}m} \cos. \varepsilon.$$

By these substitutions, the equation of the path of the projectile assumes the form given in the text.

No constant has been introduced in the second integration, because y, x, and t, are supposed to vanish together.

The subjects of this and the next two propositions are taken from an introductory essay on Central Forces, published by the author of the present work, for the use of the students in the university of Dublin.

Art. 652, 653.

These demonstrations are taken from Whewell's Mechanics, where a very detailed account of the various species of catenaries is given.

The catenary, which has received its name from the property proved in this proposition, was first solved by James Bernouilli. Long before this, Galileo had directed his attention to the curve into which a perfectly flexible string forms itself, and very inconsiderately, and without any good reasons, concluded it to be a parabola. A German geometer, *Joachim Jungius,* showed by experiment the error of *Galileo,* and proved that it was neither a parabola nor hyperbola. He, however, did not make any attempt at the true solution of the question. Four great geometers share the honour of its solution; the two Bernouillis, James and John, Leibnitz, and Huygens. The remarkable physical properties of this curve are, 1°. Of all curves of the same length joining two given points, the centre of gravity of the catenary lies the lowest. This property is very apparent from the mechanical principle, that every system of particles of matter will move amongst themselves, until they settle themselves into that position in which their centre of gravity will be at the lowest point which the law of their connexion admits. This physical property points out a very remarkable mathematical one, scil., that of all solids of revolution derived from a curve of a given length joining two given points, that derived from the catenary has the greatest surface.

2°. The catenary is the figure in which an infinite number of voussoirs should be placed, in order to form an arch, which would sustain itself by its own weight.

If the wind acted upon a sail by impact instead of pressure, the curvature of the sail would be that of the catenary.

James Bernouilli prosecuted the inquiry further, and assigned the form of catenaries on the supposition that the thickness and weight of the string were different in different parts of its extension, and that it was differently extensible, and also, that the force acting on different parts of it was different, and varied according to any proposed law.

Art. 654, 655.

The subjects of these articles are taken from Lardner on Central Forces.

Art. 657.

This elegant property of the semicubical parabola was proposed for solution to the mathematicians of Europe by Leibnitz. The solution was effected by James Bernouilli.

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THE END.

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A TREATISE

ON

ALGEBRAICAL GEOMETRY.

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ALGEBRAICAL GEOMETRY.

PART I.

APPLICATION OF ALGEBRA TO PLANE GEOMETRY.

CHAPTER I.

INTRODUCTION.

1. The object of the present Treatise is the Investigation of Geometrical Theorems and Problems by means of Algebra.

Soon after the introduction of algebra into Europe, many problems in plane geometry were solved by putting letters for straight lines, and then working the questions algebraically; this process, although of use, did not much extend the boundaries of geometry, for each problem, as heretofore, required its own peculiar method of solution, and therefore could give but little aid towards the investigation of other questions.

It is to Descartes that we owe the first general application of algebra to geometry, and, in consequence, the first real progress in modern mathematical knowledge; in the discussion of a problem of considerable antiquity, and which admitted of an infinite number of solutions, he employed two variable quantities x and y for certain unknown lines, and then showed that the resulting equation, involving both these quantities, belonged to a series of points of which these variable quantities were the co-ordinates, that is, belonged to a curve, the assemblage of all the solutions, and hence called "the Locus of the Equation."

It is not necessary to enter into further details here, much less to point out the immense advantages of the system thus founded. However, in the course of this work we shall have many opportunities of explaining the method of Descartes; and we hope that the following pages will, in some degree, exhibit the advantages of his system.

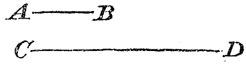
2. In applying algebra to geometry, it is obvious that we must understand the sense in which algebraical symbols are used.

In speaking of a yard or a foot, we have only an idea of these lengths by comparing them with some known length; this known or standard length is called a unit. The unit may be any length whatever: thus, if it is an inch, a foot is considered as the sum of twelve of these units, and may therefore be represented by the number 12; if the unit is a yard, a mile may be represented by the number 1760.

But any straight line A B fig. (1) may be taken to represent the unit of length, and if another straight line C D contains the line A B an exact number (a) of times, C D is equal to (a) linear units, and omitting the words "linear units," C D is equal to (a).

In fig. (1) C D = 3 times A B, or C D = 3.

1



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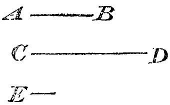


If C D does not contain A B an exact number of times, they may have a common measure E, fig. (2) ; let, then, C D = m times E = m E, and A B = n E, then C D has to A B the same ratio that m E has to n E, or that m has to n, or that $\frac{m}{n}$ has to unity; hence C D = $\frac{m}{n}$ times A B $= \frac{m}{n} = b$.

In fig. (2) C D $= \frac{5}{3}$ of A B $= \frac{5}{3}$

If the lines A B and C D have no common measure, we must recur to considerations analogous to those upon which the theory of incommensurable quantities in arithmetic is founded.

We cannot express a number like $\sqrt{2}$ by integers or fractions consisting of commensurable quantities, but we have a distinct idea of the magnitude expressed by $\sqrt{2}$, since we can at once tell whether it be greater or less than any proposed magnitude expressed by common quantities; and we can use the symbol $\sqrt{2}$ in calculation, by means of reasoning founded on its being *a limit* to which we can approach, as nearly as we please, by common quantities.

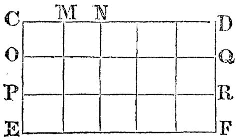
Now suppose E to be a line contained an exact number of times in A B, fig. (2), but not an exact number of times in C D, and take m a whole number, such that m E is less than C D, and $(m + 1)$ E greater than C D. Then the smaller E is, the nearer m E and $(m + 1)$ E will be to C D; because the former falls short of, and the latter exceeds, C D, by a quantity less than E. Also E may be made as small as we please; for if any line measure A B, its half, its quarter, and so on, *ad infinitum*, will measure A B. Hence we may consider C D as a quantity which, though not expressible precisely by means of any unit which is a measure of A B, may be approached as nearly as we please by such expressions. Hence C D is *a limit* between quantities commensurable with E, exactly as $\sqrt{2}$ is *a limit* between quantities commensurable with unity.

We conclude, then, that any line C D may be represented by some one of the letters a, b, c, &c., these letters themselves being the representatives of numbers either integral, fractional, or incommensurable.

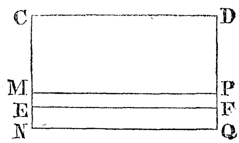
3. If upon the linear unit we describe a square, that figure is called the square unit.

Let C D F E, fig. (1), be a rectangle, having the side C D containing (a) linear units C M, M N, &c., and the side C E containing (b) linear units C O, O P, &c., divide the rectangle into square units by drawing lines parallel to C E through the points M, N, &c., and to C D through the points O, P, &c. Then in the upper row C O Q D there are (a) square units, in the second row O P R Q the same, and there are as many rows as there are units in C E, therefore altogether there are ($b \times a$) square units in the figure, that is, C F contains ($a\,b$) square units, or is equal in magnitude to $(a\,b)$ square units; suppressing the words "square units," the rectangle C F is equal to $a\,b$.

If C D = 5 feet and C E = 3 feet, the area C F contains 15 square feet.



(1.)



(2.)

The above proof applies only to cases where the two lines containing the rectangle can be exactly measured by a common linear unit.

Suppose C D to be measurable by any linear unit, but C E (fig. 2) not to be commensurable with C D; then, as has been shown, we may find lines C M, C N commensurable with C D approaching in magnitude as nearly as we please to C E.

Completing the rectangles C P and C Q, we see, that as C M and C N approach to C E, the rectangles C P and C Q approach to the rectangle C F, that is, the rectangle C E, C D is *the limit* of the rectangle C M, C D, just as C E is *the limit* of C M. Let therefore a and b be respectively the commensurable numbers representing C D and C M, and let c be the incommensurable number expressing C E, then the rectangle C E, C D = the limit of the rectangle C M, M P = the limit of the number $a\,b$, by the first part of this article, = the product of the respective limits of a and $b = a\,c$. *

Hence, generally, the algebraical representative of the area of a rectangle is equal to the product of those of two of its adjacent sides.

If $b = a$, the figure C F becomes the square upon C D, hence the square upon C D is equal to $(a \times a)$ times the square unit $= a^2$.

We are now able to represent all plane rectilineal figures, for such figures can be resolved into triangles, and the area of a triangle is equal to half the rectangle on the same base, and between the same parallel lines.

4. To represent a solid figure, it will be sufficient to show how a solid rectangular parallelopiped may be represented.

Let a, b, c, be, respectively, the number of linear units in the three adjacent edges of the parallelopiped; then, dividing the solid by planes parallel to its sides, we may prove, as in the last article, that the number of solid units in the figure is $a \times b \times c$, and, consequently, the parallelopiped equal to $a \times b \times c$.

The proof might be extended to the case where the edges of the parallelopiped are fractional, or incommensurable with the linear unit.

If $b = c = a$, the solid becomes a cube, and is equal to $a \times a \times a$, or a^3.

5. We proceed, conversely, to explain the sense in which algebraic expressions may be interpreted consistently with the preceding observations.

* That "the product of the limits of two incommensurable numbers is the limit of their product," may be thus shown. Let v and w be incommensurable numbers, and let $v = m + m'$ and $w = n + n'$, m and n being commensurable numbers, and m' and n' diminishable without limit; that is, v and w are the respective limits of m and n, then $vw = mn + mn' + nm' + m'n'$, the right-hand side of this equation ultimately becomes mn, and the left-hand side of the equation is the product of the limits.

Algebraic expressions may be classed most simply under the form of homogeneous equations, as follows :—

$$x = a$$
$$x^2 + ax = bc$$
$$x^3 + ax^2 + bcx = def$$
$$x^4 + ax^3 + bcx^2 + defx = ghkl$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$x^m + ax^{m-1} + bcx^{m-2} + \&c \; . \; . \; . \; . \; . = pqrs \ldots \text{ to } m \text{ terms.}$$

In the first place, each equation may be understood as referring to linear units; thus, if L be put instead of the words 'the linear units,' the equations may be written

x times L $= a$ times L,

x^2 times L $+ ax$ times L, or $(x^2 + ax)$ times L $= bc$ times L,

$(x^4 + ax^3 + bcx^2 + defx)$ times L $= ghkl$ times L, and so on. The solution of each equation gives x times L in terms of $(a, b, c, \ldots)$ times L; and thus the letters $a, b, c, \ldots x$ are merely numbers, having reference to lines, but not to figures.

This will be equally true if L is not expressed, but understood; and it is in this sense that we shall interpret all equations beyond those of the third order.

The same reasoning would equally apply if we assumed L to represent the square or cubic unit, only it would lead to confusion in the algebraic representation of a line.

6. Again, these equations may, to a certain extent, have an additional interpretation.

For if we consider the letters in each term to be the representatives of lines drawn *perpendicular* to each other, the second equation refers to areas, and then signifies that the sum of two particular rectangles is equal to a third rectangle; the third equation refers to solid figures, and signifies, that the sum of three parallelopipeds is equal to a fourth solid.

Moreover we can pass from an equation referring to areas to another referring to lines, without any violation of principle; for, considering the second equation as referring to areas, the rectangles can be exhibited in the form of squares; and if the squares upon two lines be equal, the lines themselves are equal, or the equation is true for linear units.

7. It follows as a consequence of the additional interpretation, that every equation of the second and third order will refer to some geometrical theorem, respecting plane or solid figures; for example, the second equation, when in the form $x^2 = a(a - x)$ is the representation of the well-known problem of the division of a line into extreme and mean ratio.

By omitting the second and third terms of the third equation, and giving the values of $2a$, a, and a to d, e and f, respectively, we obtain the algebraic representation of the ancient problem of the duplication of the cube.

8. The solution of equations leads to various values of the unknown quantity, and there are then two methods of exhibiting these values; first, by giving to a, b, c, &c., their numerical values, and then performing any operation indicated by the algebraic symbols.

Thus, if $a = 4$, $b = 5$, and $c = 9$,

we may have $x = a + c - b = 8$ times the linear unit.

$x = \frac{ab}{c} = \frac{20}{9} = 2\frac{2}{9}$ of the linear unit.

$x = \sqrt{ac} = \sqrt{36} = 6$ times the linear unit.

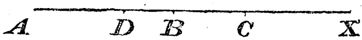
We can then draw the line corresponding to the particular value of x. This is the most practical method.

Again, we may obtain the required line from the algebraical result, by means of geometrical theorems; this method is called 'the Construction of Quantities'; it is often elegant, and is, moreover, useful to those who wish to obtain a complete knowledge of Algebraical Geometry.

THE CONSTRUCTION OF QUANTITIES.

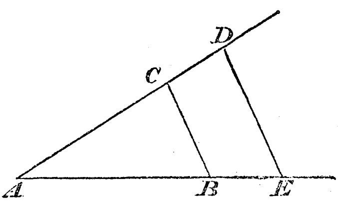
9. Let $x = a + b$.

In the straight line A X, let A be the point from whence the value of x is to be measured; take A B $= a$, and B C $= b$, then A C $=$ A B $+$ B C $= a + b$ is the value of x.



Let $x = a - b$, in B A take B D $= b$, then A D $=$ A B $-$ B D $= a - b$.

Let $x = \frac{ab}{c}$, then $x : a :: b : c$, and x is a fourth proportional to the three given quantities c, b, and a; hence the line whose length is expressed by x, is a fourth proportional to three lines, whose respective lengths are c, b, and a. From A draw two lines A C D, A B E, forming any angle at A; take A B $= c$, B E $= a$, and A C $= b$, join B C, and draw D E parallel to B C; then, A B : A C :: B E : C D, or $c : b :: a :$ C D $\therefore$ C D is the required value of x.



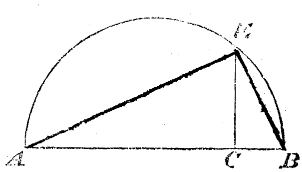
$$\text{Let } x = \frac{abc}{de}; \text{ construct } y = \frac{bc}{e}, \text{ and then } x = \frac{ay}{d};$$

$$\text{similarly for } x = \frac{abc}{d^2}, \text{ or } \frac{ab^2}{d^2}, \text{ or } \frac{a^3}{d^2}, \text{ or } \frac{abcd}{efg}.$$

Let $x = \frac{abc + def}{gh} = \frac{abc}{gh} + \frac{def}{gh}$, construct each term separately, and then the sum of the terms.

10. Let $x = \sqrt{ab}$.

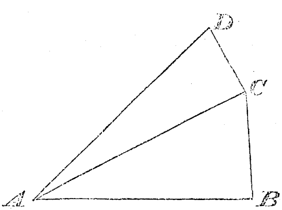
Since $x^2 = ab$, x is a mean proportional between a and b. In the straight line A B take A C $= a$, and C B $= b$; upon A B describe a semicircle, from C draw C E perpendicular to A B, and meeting the circle in E; then C E is a mean proportional to A C and C B, (Euclid, vi. 13, or Geometry, ii. 51,) and therefore C E is the required value of x.



The same property of right-angled triangles may be advantageously employed in the construction of the equation $x = \frac{a^2}{b}$; for, take AC $= b$, and draw C E perpendicular to A C and equal to a, join A E and draw E B perpendicular to A E; then C B $= \frac{a^2}{b}$.

Let $x = \sqrt{ab + cd}$, $x^2 = ab + cd = a\left(b + \frac{cd}{a}\right) = ay$ by substitution; construct y, and then $x = \sqrt{ay}$.

Again, x is a line, the square upon which is equal to the sum of the rectangles ab, cd. This sum may be reduced to a single rectangle, and the rectangle converted into a square, the base of which is the required value of x.—Euclid, i. 45, and ii. 14; or Geometry, i. 57, 58.



Let $x = \sqrt{a^2 + b^2}$; take a straight line A B $= a$, from B draw B C ($= b$) perpendicular to A B; A C is the value of x.

Let $x = \sqrt{a^2 + b^2 + c^2}$, from C draw C D ($= c$) perpendicular to A C, A D is the required value of x.

Let $x = \sqrt{a^2 - b^2} = \sqrt{(a+b)(a-b)}$; x is a mean proportional between $a + b$ and $a - b$; or by taking (in the last figure but one) A B $= a$, and A E $= b$, we have B E $= \sqrt{a^2 - b^2}$.

Let $x = \sqrt{\{a^2 + b^2 - c^2 - d^2\}}$, find $y^2 = a^2 + b^2$ and $z^2 = c^2 + d^2$ and then x.

Let $x = \sqrt{\frac{a^4 + b\,c^3}{b^2 - c^2}}$, find $y^2 = a^2 + \frac{b\,c^3}{a^2}$, and $z^2 = b^2 - c^2$, and then $x = \frac{ay}{z}$.

11. Of course the preceding methods will equally apply, when instead of the letters we have the original numbers, the linear unit being understood as usual.

Thus $x = \sqrt{12} = \sqrt{3.4}$ is a mean proportional between 3 and 4; hence (see last figure but one) take A C equal four times the unit, and C B equal three times the unit, C E is the value of x; or since $\sqrt{12} = \sqrt{16 - 4} = \sqrt{4^2 - 2^2}$, by constructing a right-angled triangle of which the hypothenuse is four times the linear unit, and one side twice that unit: the remaining side $= \sqrt{12}$.

Similarly $x = \sqrt{7} = \sqrt{4 + 4 - 1} = \sqrt{2^2 + 2^2 - 1^2}$, which is of the form $\sqrt{a^2 + b^2 - c^2}$.

Let $x = \sqrt{3} = \sqrt{2+1}$. In the last figure let A B, B C, and C D each be equal to the linear unit, then $AD = \sqrt{3}$.

Let $x = \sqrt{23} = \sqrt{5^2 - 1^2 - 1^2}$.

Let $x = \frac{1}{\sqrt{2}} = \sqrt{\frac{1}{2}} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$,
then x is the hypothenuse of a right-angled triangle, each of whose sides is half the unit.

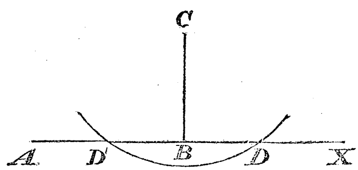
Let $x = \sqrt{\frac{3}{4}}$; this may be constructed as the last.

Let $x = \sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}}$; and so on for all numbers, since any finite number can be decomposed into a series of numbers representing the squares upon lines.

If the letter a be prefixed to any of the above quantities, it must be introduced under the root.

12. In constructing compound quantities, it is best to unite the several parts of the construction in one figure.

Thus if $x = a \pm \sqrt{a^2 - b^2}$, in the line A X take $AB = a$, from B draw B C ($= b$) perpendicular to A B; with centre C and radius a describe a circle cutting A X in D and D′; A D and A D′ are the values required:



$$\text{for } AD = AB + BD = a + \sqrt{a^2 - b^2}.$$

$$AD' = AB - BD' = a - \sqrt{a^2 - b^2}.$$

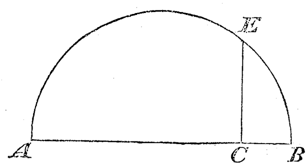
This construction fails when b is greater than a, for then the circle never cuts the line A X; this is inferred also from the impossibility of the roots.

13. Since theorems in geometry relate either to lines, areas, or solids, the corresponding equations must in each case be homogeneous, and will remain so through all the algebraic operations. If, however, one of the lines in a figure be taken as the linear unit and be therefore represented by unity, we shall find resulting expressions, such as $x = \frac{a}{b}$, $x = \sqrt{a}$, $x = \sqrt{a^2 + b}$, &c., in which, prior to construction, the numerical unit must be expressed; thus these quantities must be written $\frac{a}{b} \times 1$, $\sqrt{a \times 1}$, $\sqrt{a^2 + b \times 1}$, and then constructed as above.

CHAPTER II.

DETERMINATE PROBLEMS.

14. GEOMETRICAL Problems may be divided into two classes, Determinate and Indeterminate, according as they admit of a finite or an infinite number of solutions.



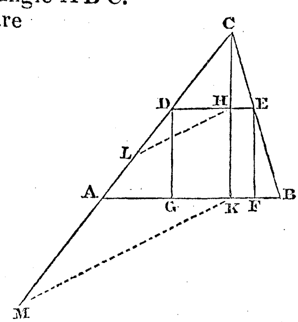
If A B be the diameter of the semicircle A E B, and it be required to find a point C in A B such, that drawing C E perpendicular to A B to meet the circumference in E, C E shall be equal to half the radius of the circle, this is a determinate problem, because there are only *two* such points in A B, each at an equal distance from the centre. Again, if it be required to find a point E out of the line A B such, that joining E A, E B, the included angle A E B shall be a right angle, this is an indeterminate problem, for there are an infinite number of such points, all lying in the circumference A E B.

The determinate class is by no means so important as the indeterminate, but the investigation of a few of the former will lead us to the easier comprehension of the latter; and therefore we proceed to the discussion of determinate problems.

15. In the consideration of a problem, the following rules are useful.

1. Draw a figure representing the conditions of the question.
2. Draw other lines, if necessary, generally parallel or perpendicular to those of the figure.
3. Call the known lines by the letters a, b, c, &c., and some of the unknown lines by the letters x, y, z, &c.
4. Consider all the lines in the figure as equally known, and from the geometrical properties of figures deduce one, two, or more equations, each containing unknown and given quantities.
5. From these equations find the value of the unknown quantities.
6. Construct these values, and endeavour to unite the construction to the original figure.

16. To describe a square in a given triangle A B C.



Let D E F G be the required square
C H K the altitude of the triangle.

The question is resolved into finding the point H, because then the position of D E, and therefore of the square, is determined.

Let $CK = a$, $AB = b$, $CH = x$;
then by the question, $DE = HK$,
and $DE : AB :: CH : CK$,
or $DE : b :: x : a$,

$\therefore DE = \frac{bx}{a}$, and $HK = a - x$

$$\therefore \frac{bx}{a} = a - x$$

$$\therefore x = \frac{a^2}{a+b}.$$

Thus x is a third proportional to the quantities $(a+b)$ and a.

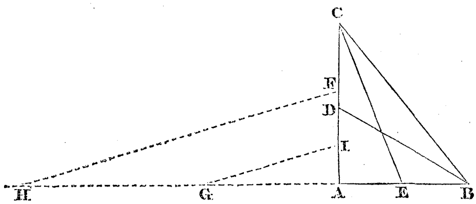
In C A take C L $= a$, produce C A to M so that L M $= b$, join M K, and draw L H parallel to M K ; C H is the required value of x.

17. In a right-angled triangle the lines drawn from the acute angles to the points of bisection of the opposite sides are given, to find the triangle.

Let C E $= a$, B D $= b$, A D $=$ C D $= x$, A E $=$ E B $= y$.

Then the square upon C E $=$ square upon C A $+$ square upon A E,

$$\text{or } a^2 = 4x^2 + y^2$$
$$\text{similarly } b^2 = x^2 + 4y^2$$



whence $y = \pm \sqrt{\dfrac{4b^2 - a^2}{15}}$. Make any right angle A, and on one of the sides take A F $= \dfrac{a}{2}$, with centre F and radii b and $2\,a$, describe circles cutting the other side produced in G and H, respectively; draw G I parallel to F H ; then 2 A I is the required value of y. Hence A D, and therefore A C and A B are found, and the triangle is determined.

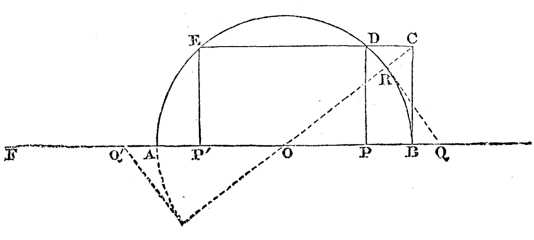
18. To divide a straight line, so that the rectangle contained by the two parts may be equal to the square upon a given line b.

Let A B $= a$

A P $= x$

Then the rectangle A P, P B $= b^2$

$$\text{or } x\,(a - x) = b^2$$
$$\therefore x^2 - ax = -b^2$$
$$\therefore x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b^2}$$



Upon A B describe a semicircle, draw B C ($= b$) perpendicular to A B, through C draw C D E parallel to A B, from D and E draw D P, E P′, perpendicular to A B ; P and P′ are the required points.

If b is greater than $\frac{a}{2}$, the value of x is irrational, and therefore the problem is impossible; but then a point Q may be found in A B produced, such that, the rectangle A Q, Q B $= b^2$.

Let A Q $= x$,

$$\therefore\ x\,(x - a) = b^2$$

$$\therefore\ x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2}.$$

From the centre O draw the line O C cutting the circle in R, from R draw R Q perpendicular to O R, then Q is the required point; for

O Q $=$ O C $= \sqrt{\left(\frac{a^2}{4} + b^2\right)}$, and therefore A Q $= \frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}$.

Let us examine the other root $\frac{a}{2} - \sqrt{\frac{a^2}{4} + b^2}$, which is negative, and may be written in the form $-\left\{\sqrt{\frac{a^2}{4} + b^2} - \frac{a}{2}\right\}$; the magnitude of this quantity, independent of the negative sign, or its *absolute* magnitude, is evidently B Q or A Q′.

Now if the problem had been "to find a point Q in either A B produced, or B A produced, such, that the rectangle A Q, Q B $= b^2$", we might have commenced the solution by assuming the point Q to be in B A produced as at Q′; thus letting A Q′ $= x$, we should have $x\,(a + x) = b^2$, and $x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2}$, of which two roots the first or $-\frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2} = -\left\{\frac{a}{2} - \sqrt{\frac{a^2}{4} + b^2}\right\}$ is the absolute value of the negative root in the last question; hence the negative root of the last question is a real solution of the problem expressed in a more general form, the negative sign merely pointing out the *position* of the second point Q′. Both roots may be exhibited in a positive form by measuring x not from A, but from a point F, A F being greater than b; for letting FA $= c$, and F Q or F Q′ $= x$, we find

$$x = c + \frac{a}{2} \pm \sqrt{b^2 + \frac{a^2}{4}}.$$

The celebrated problem of dividing a given straight line in extreme and mean ratio, is solved in the same manner; letting A P $= x$ we have the rectangle A B, B P $=$ the square upon A P, or $a\,(a - x) = x^2$, whence $x = -\frac{a}{2} \pm \sqrt{a^2 + \frac{a^2}{4}}$; here the negative root, which gives a point to the left of A, is a solution of the problem enunciated more generally*. See Thomson's Euclid; p.p. 52. 53.

19. Through a point M equidistant from two straight lines A A′ and B B′ at right angles to each other, to draw a straight line P M Q, so that the part P Q intersected by A A′ and B B′ may be of a given length b.

From M draw the perpendicular lines M C, M D.

Let $MD = a$, $DQ = x$, $CP = y$,

$$\text{then } PQ = PM + MQ,$$

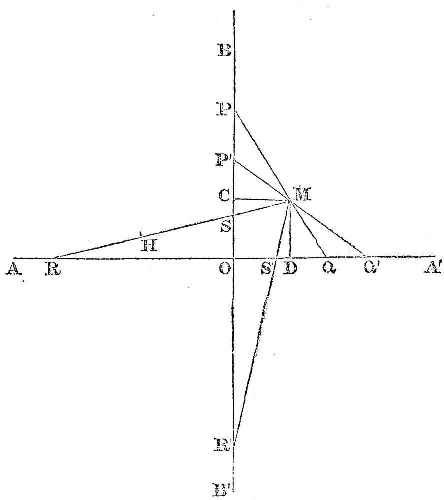
$$\text{or } b = \sqrt{a^2 + y^2} + \sqrt{a^2 + x^2},$$

$$\text{and } \frac{x}{a} = \frac{a}{y} \text{ from the similar triangles } PCM,\ MDQ.$$

$$\therefore b = \sqrt{a^2 + \frac{a^4}{x^2}} + \sqrt{a^2 + x^2}$$

$$= \sqrt{a^2 + x^2}\left(1 + \frac{a}{x}\right);$$

$$\text{whence } x^4 + 2ax^3 + (2a^2 - b^2)\, x^2 + 2a^3 x + a^4 = 0.$$



We might solve this recurring equation, and then construct the four roots, as in the last problems; but since the roots of an equation of four dimensions are not easily obtained, we must, in general, endeavour to avoid such an equation, and rather retrace our steps than attempt its solution. Let us consider the problem again, and examine what kind of a result we may expect.

* Lucas de Borgo, who wrote a book on the application of this problem to architecture and polygonal figures, was so delighted with this division of a line, that he called it the Divine Proportion.

Since, in general, four lines P M Q, P′ M Q′, R S M, R′ S′ M, may be drawn fulfilling the conditions of the question, the two former, in all cases, though not always the two latter, we may conclude that there will be four solutions; but since the point M is similarly situated with respect to the two lines A A′, B B′, we may also expect that the resulting lines will be similarly situated with regard to A A′ and B B′. Thus, if there be one line P M Q, there will be another P′ M Q′ such that O Q′ = OP, and O P′ = O Q.

Again O S will be equal to O S′, and O R to O R′. Hence, if we take the perpendicular from O upon the line S R for the unknown quantity (y), we can have only two different values of this line, one referring to the lines S R and S′ R′, the other to P Q and P′ Q′; hence the resulting equation will be of *two* dimensions only. In this case the equation is

$$by^2 + 2a^2y - ba^2 = 0.$$

Again, since M R = M R′ we may take M H, H being the point of bisection of the line S R, for the unknown quantity, and then also we may expect an equation, either itself of two dimensions, or else reducible to one of that order.

$$\text{Let MH} = x; \therefore \text{MR} = x + \frac{b}{2}, \text{ MS} = x - \frac{b}{2},$$

$$\text{and MR} : \text{MD} :: \text{RS} : \text{SO} = \frac{ab}{x + \frac{b}{2}}$$

$$\text{MS} : \text{OD} :: \text{RS} : \text{RO} = \frac{ab}{x - \frac{b}{2}};$$

but the square upon R S = square upon R O + square upon S O,

$$\therefore b^2 = \left(\frac{ab}{x - \frac{b}{2}}\right)^2 + \left(\frac{ab}{x + \frac{b}{2}}\right)^2$$

$$\therefore x^4 - \left(2a^2 + \frac{b^2}{2}\right)x^2 + \frac{b^4}{16} - \frac{a^2b^2}{2} = 0$$

$$\therefore x = \pm\sqrt{\left\{a^2 + \frac{b^2}{4} \pm a\sqrt{a^2 + b^2}\right\}}$$

an expression of easy construction; the negative value of x is useless: of the remaining two values that with the positive sign is always real, and refers to the lines M S R, M S′ R′; the other, when real, gives the lines P M Q, P′ M Q′; it is imaginary if b^2 is less than $8a^2$, that is, joining O M and drawing P M Q perpendicular to O M, if b is less than P M Q.

This question is taken from Newton's Universal Arithmetic, and is given by him to show how much the judicious selection of the unknown quantity facilitates the solution of problems. The principal point to be attended to in such questions is, to choose that line for the unknown quantity which must be liable to the least number of variations.

20. Through the point M in the last figure to draw P M Q so that the sum of the squares upon P M and M Q shall be equal to the square upon a given line b.

Making the same substitutions as in the former part of the last article, we shall obtain the equations

$$x^2 + a^2 + y^2 + a^2 = b^2,\ xy = a^2,$$

$$\therefore\ x^2 + y^2 + 2xy = b^2, \text{ and } x + y = \pm b,$$

$$\text{or } x + \frac{a^2}{x} = \pm b, \text{ whence } x = \pm \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}.$$

To construct these four values describe a circle with centre M and radius $\frac{b}{2}$, cutting A A′ in two points L, L′; with centres L, L′ and radius $\frac{b}{2}$ describe two other circles cutting A A′ again in four points: these are the required points.

21. To find a triangle A B C such that its sides A C, C B, B A, and perpendicular B D, are in continued geometrical progression.

Take any line A B $= a$ for one side, let B C $= x$,

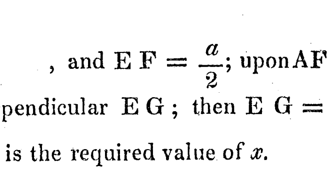
$$AC : CB :: CB : BA :: BA : BD;$$

hence the triangles A C B, A D B, are equiangular, (Eucl. vi. 7, or Geometry, ii. 33,) and the angle A B C is a right angle; also A C $= \frac{x^2}{a}$, then

the square upon A C = the square upon B C + the square upon A B;

$$\therefore\ \frac{x^4}{a^2} = x^2 + a^2, \text{ or } x^4 - a^2x^2 - a^4 = 0,$$

$$\text{whence } x = \pm \sqrt{\frac{a^2 \pm \sqrt{5a^4}}{2}};$$



of these roots two are impossible, since $a^2\sqrt{5}$ is greater than a^2; and of the remaining two the negative one is useless.

In A B produced take B E $= a\sqrt{5}$, and E F $= \frac{a}{2}$; upon AF describe a semicircle, and draw the perpendicular E G; then E G $= \sqrt{\left\{\frac{a}{2}(a + a\sqrt{5})\right\}} = \sqrt{\frac{a^2 + a^2\sqrt{5}}{2}}$ is the required value of x.

CHAPTER III.

THE POINT AND STRAIGHT LINE.

22. Determinate problems, although sometimes curious, yet, as they lead to nothing important, are unworthy of much attention. It was, however, to this branch of geometry that algebra was solely applied for some time after its introduction into Europe. Descartes, a celebrated French

philosopher, who lived in the early part of the seventeenth century, was the first to extend the connexion. He applied algebra to the consideration of curved lines, and thus, as it were, invented a new science.

Perhaps the best way of explaining his method will be by taking a simple example. Suppose that it is required to find a point P without a given line A B, so that the sum of the squares on A P and P B shall be equal to the square upon A B.

Let P be the required point, and let fall the perpendicular P M on A B.

Let A M $= x$, M P $= y$, and A B $= a$; then by the question, we have

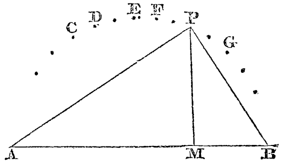
The square on A B = the square on A P + the square on P B.

= the squares on A M, M P + the squares on P M, M B,

or $a^2 = (x^2 + y^2) + y^2 + (a - x)^2$

$= 2y^2 + 2x^2 - 2ax + a^2$

$\therefore y^2 = ax - x^2$.



Now this equation admits of an infinite number of solutions, for giving to x or A M any value, such as $\frac{a}{2}$, $\frac{a}{3}$, $\frac{a}{4}$, &c., we may, from the equation, find corresponding values of y or M P, each of them determining a separate point P which satisfies the condition of the problem.

Let C, D, E, F, &c., be the points thus determined. The number of the values of y may be increased by taking values of x between those above-mentioned and this to an infinite extent, thus we shall have an infinite number of points C, D, E, F, &c., indefinitely near to each other, so that these points ultimately form a line which geometrically represents the assemblage of all the solutions of the equation. This line A C D E F, whether curved or straight, is called the locus of the equation.

In this manner all indeterminate problems resolve themselves into investigations of loci; and it is this branch of the subject which is by far the most important, and which leads to a boundless field for research*.

23. For the better investigation of loci, equations have been divided into two classes, algebraical and transcendental.

An algebraical equation between two variables x and y is one which can be reduced to a finite number of terms involving only integral powers of x, y, and constant quantities: and it is called complete when it contains all the possible combinations of the variables together with a constant term, the sum of the indices of these variables in no term exceeding the degree of the equation; thus of the equations

$$ay + bx + c = 0$$

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

the first is a complete equation of the first order, and the next is a complete equation of the second order, and so on.

Those equations which cannot be put into a finite and rational algebraical form with respect to the variables are called transcendental, for

* For the definition and examples of Loci, see Geometry, iii. § 6; and the Index, article Locus.

they can only be expanded into an infinite series of terms in which the power of the variable increases without limit, and thus the order of the equation is infinitely great, or transcends all finite orders.

$y = \sin. x$, and $y = a^x$, are transcendental equations.

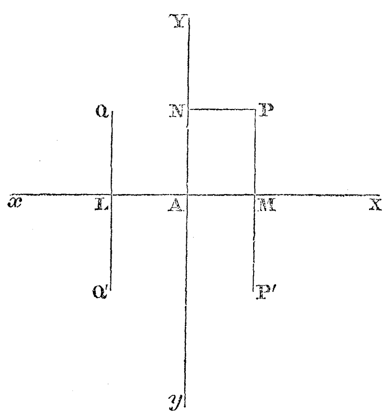
24. The loci of equations are named after their equations, thus the locus of an equation of the first order is a line of the first order; the locus of an equation of the second order is a line of the second order; the locus of a transcendental equation is a transcendental line or curve.

Algebraical equations have not corresponding loci in all cases, for the equation may be such as not to admit of any real values of both x and y; the equation $y^2 + x^2 + a^2 = 0$ is an example of this kind, where, whatever real value we give to x, we cannot have a real value of y: there is therefore no locus whatever corresponding to such an equation.

THE POSITION OF A POINT IN A PLANE.

25. The position of a point in a plane is determined by finding its situation relatively to some fixed objects in that plane; for this purpose suppose the plane of the paper to be the given plane, and let us consider as known the intersection A of two lines x X and y Y of unlimited length, and also the angle between them; from any point P, in this plane, draw P M parallel to A Y, and P N parallel to A X, then the position of the point P is evidently known if A M and A N are known. For it may be easily shown, *ex absurdo*, that there is but one point within the angle Y A X such that its distance from the lines A Y and A X is P N and P M respectively.

A M is called the abscissa of the point P; A N, or its equal M P, is called the ordinate; A M and M P are together the co-ordinates of P; X x is called the axis of abscissas, Y y the axis of ordinates. The point A where the axes meet is termed the origin.



The axes are called oblique or rectangular, according as Y A X is an oblique or a right angle. In this treatise rectangular axes as the most simple will generally be employed.

Let the abscissa A M $= x$, and the ordinate M P $= y$, then if on measuring these lengths A M and M P we find the first equal to a and the second equal to b, we have, to determine the position of this point P, the two equations

$$x = a, y = b$$

and as they are sufficient for this object, we call them, when taken together, the equations to this point.

The same point may also be defined by the equation

$$(y - b)^2 + (x - a)^2 = 0$$

for this equation can only be satisfied by the values $x = a$ and $y = b$.

And in general any equation which can only be satisfied by a single real value of each variable quantity x and y, refers to a point whose situation is determined by the co-ordinates corresponding to these values.

26. In this manner the position of any point in the angle Y A X can be determined, but in order to express the positions of points in the angle Y A x, some further considerations are necessary.

In the solution of the problem, article (18), we observed that negative quantities may be geometrically represented by lines drawn in a certain direction. An extension of this idea leads to the following reasoning.

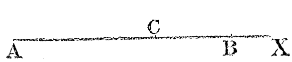
When we affix a negative sign to any quantity, we do not signify any change in its magnitude, but merely the way in which the quantity is to be used, or the operation to be performed on it. Thus the absolute magnitude of -5 is just as great as that of $+5$; but -5 means that 5 is to be subtracted, and $+5$ that it is to be added. As the sign $+$ is applied to quantities variously estimated, the sign $-$ will have in each of these various cases a corresponding meaning, necessarily following from that of the sign $+$. Whatever $+$ means, we must always have $-a+a=0$. Hence we may define $-a$ to be a quantity estimated in such a manner that the altering it by the operation indicated by $+a$ reduces the result to nothing. This is properly the meaning of the sign $-$; it depends entirely on that of the sign $+$ in every case.

The symbol of positive quantity is used in a variety of ways; but in every instance the above principle shows in what way the negative quantity must, as a necessary consequence of the meaning of the positive quantity, be used.

Thus, if we placed a mark on a pole stuck vertically into the ground, at some point in the pole which was bare at low water and covered at high water, and scored upwards the inches from that mark, we might express the height of the surface by the number of inches above the mark, positively, when the surface was above the mark; but at low water when the surface is below the mark, 11 inches for instance, we should call the height -11; because when 11 inches were added to the height, (that is, when the surface of the water was advanced 11 inches upwards, which is the direction in which the positive quantities are supposed to be reckoned,) the surface would be just at the mark, and would be *no inches* in height reckoning from the mark.

Suppose a man to advance directly from a given point p miles in the first 6 hours of a day, and to go back in the next 6 hours q miles; at the end of the 12 hours his advance from the given point would be $(p-q)$ miles. Thus, suppose $p=10$, and $q=6$, he will advance $(10-6)$ or 4 miles. But suppose he recedes 10 miles, then his advance will in the 12 hours be $(10-10)$ or 0: he will be just where he was at first. Suppose he recedes 15 miles, at the end of the 12 hours he will be 5 miles *behind* the original point. Here we say *behind*, because the movement in the direction of the original advance was considered to be *forward*. And it is clear that in this case, from an advance of 10 miles, and a recess of 15, the advance is -5; that is, it requires a further advance of 5 miles to make the man exactly as forward as he was at starting.

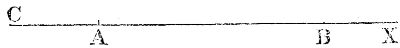
Now let us consider a fixed point A, and a line measured from it by positive quantities in the direction A X. Suppose the line to be described by the motion of a point from A along A X; and after the point has been carried *forward* (that is, towards X) m linear units, as to B, let it be carried



back n linear units, as to C; then altogether the advance of the point or the length of A C will be $(m - n)$ linear units.

Again, suppose n to become $= m$; that is, let the point be carried back exactly to A; then the advance of the point along A X will still be measured by $(m - n) = m - m = 0$.

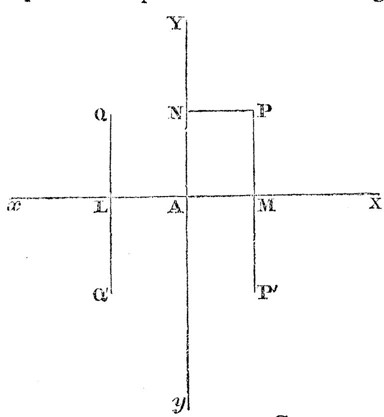
Once more, let n exceed m; that is, let B C exceed A B; the advance of the point will be expressed by $(m - n)$ still; but this will now be a negative number, showing by how many linear units the point must be advanced in order to bring it forward to the original starting point A. Now any line A C may be considered to be determined by the motion of a point either simply along A C, or along first A B and then B C. We see, therefore, if we begin by reckoning distances from A in the direction A X as positive quantities, we are compelled to consider distances from A in the opposite direction as negative quantities.



Conversely again, having designated positive quantities by lines in one direction from a given point, suppose the calculation produces a negative result, what meaning are we to assign to it? The negative result shows how much positive quantity is required to bring the whole result to nothing. Now positive quantity, by the hypothesis, is distance measured in the original direction; therefore the negative quantity shows how much distance measured in the original direction is required to bring the result to nothing. But if there be a distance from A, such that a linear units in the original direction must be subjoined to bring the result to nothing, (that is, to reduce to nothing the distance from A,) it is clear that this distance must be that of a linear units measured in a direction from A opposite to the original direction. That is, the negative quantity must be represented by lines drawn in the direction opposite to that in which the lines representing the positive quantities are drawn.

It is immaterial in which direction the line is drawn which we consider positive: but when chosen, negative quantities of the same kind must be taken in the opposite direction.

27. We are now able to express the position of points in the remaining angles formed by the axes, by considering all lines in the direction A X to be positive and those in A x to be negative: and similarly all those drawn in the direction A Y will be considered positive, and therefore those in A y will be negative.



We have then the following table of co-ordinates.

P in the angle X A Y, $+ x$, $+ y$,
Q in the angle Y A x, $- x$, $+ y$,
Q′ in the angle x A y, $- x$, $- y$,
P′ in the angle X A y, $+ x$, $- y$.

Hence the equations to a given point P are $x = a,\ y = b$

. . . . Q . . $x = -a,\ y = b$

. . . . Q′ . . $x = -a,\ y = -b$

. . . . P′ . . $x = a,\ y = -b$

28. If, the abscissa AM remaining the same, the ordinate MP diminishes, the point P approaches to the axis AX; and when MP is nothing, P is situated on that axis; in this case the equations to the point P are

$$x = a,\ y = 0: \text{ or } y^2 + (x - a)^2 = 0.$$

Similarly when the point P is situated on the axis AY, its equations are

$$x = 0,\ y = b: \text{ or } (y - b)^2 + x^2 = 0.$$

If both AM and MP vanish, we have the equations to the origin A,

$$x = 0,\ y = 0: \text{ or } y^2 + x^2 = 0.$$

Ex. 1. The point whose equations are $x = 4$, $y = -2$, is situated in the angle X A y, at a distance AM $= 4$ times the linear unit from the axis of y, and MP′ $=$ twice that unit from the axis of x.

Ex. 2. The point whose equation is $(y + 3)^2 + (x + 2)^2 = 0$ is situated in the angle x A y, at distances AL $= 2$, LQ′ $= 3$, from the axes.

Ex. 3. The point whose equations are $x = 0$, $y = -3$ is in the line A y, at a distance $= 3$ times the linear unit.

Ex. 4. The point whose equation is $y^2 + (x + a)^2 = 0$, is in A x, at a distance a from the origin.

The preceding articles are true if the co-ordinate axes be oblique.

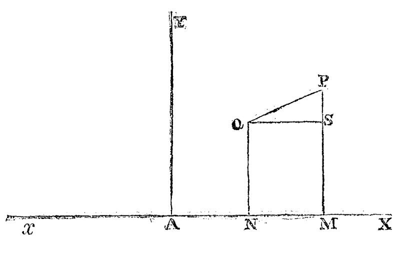
29. To find an expression for the distance D between two points P and Q.

Let the axes be rectangular and let the equations to

P be $x = a,\ y = b$

Q $\quad x = a',\ y = b'$;

or in other words, let the co-ordinates of P be AM $= a$, MP $= b$, and those of Q be AN $= a'$, NQ $= b'$, draw QS parallel to AX.



Then the square upon QP $=$ the square upon QS $+$ the square upon PS;

and QS $=$ NM $=$ AM $-$ AN $= a - a'$

also PS $=$ PM $-$ QN $= b - b'$

$$\therefore D^2 = (a - a')^2 + (b - b')^2$$

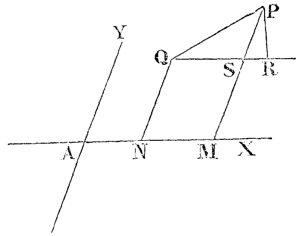
If Q be in the angle Y A x we have AN $= -a'$,

$$\therefore D^2 = (a + a')^2 + (b - b')^2.$$

If Q be at the origin we have $a' = 0$ and $b' = 0$

$$\therefore D^2 = a^2 + b^2, \text{ or } D = \sqrt{a^2 + b^2}.$$

30. If the angle between the axes be oblique and $= \omega$, draw PM and QN parallel to AY, and QS parallel to AX; also let PR be drawn perpendicular to QS; then the square upon QP $=$ the square on QS $+$ the square on PS $+$ twice the rectangle QS, SR;



and, $QS = a - a'$,

$PS = b - b'$,

$SR = PS \cos. PSR$

$= PS \cos. YAX$

$= (b - b') \cos. \omega$;

$$\therefore D^2 = (a - a')^2 + (b - b')^2 + 2(a - a')(b - b') \cos. \omega;$$

and when the point Q is at the origin, and therefore $a' = 0$, and $b' = 0$,

$$D^2 = a^2 + b^2 + 2ab \cos. \omega.$$

THE LOCUS OF AN EQUATION OF THE FIRST DEGREE.

31. To find the locus of an equation of the first degree between two unknown quantities.

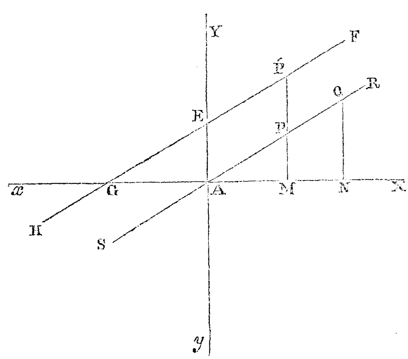
The most general form of such an equation is,

$Ay + Bx + C = 0$, or $y = -\frac{B}{A}x - \frac{C}{A}$, or $y = \alpha x + b$ if $-\frac{B}{A} = \alpha$ and $-\frac{C}{A} = b$; we will in the first place consider the equation in its most simple form $y = \alpha x$.

Let AX, AY be the rectangular axes, then a point in the locus will be determined by giving to x a particular value as 1, 2, 3, &c.; let AM, MP and AN, NQ be the respective co-ordinates of two points P and Q thus determined;

$$\text{since } y = \alpha x, \text{ we have}$$
$$MP = \alpha . AM$$
$$\text{and } NQ = \alpha . AN$$
$$\therefore MP : AM :: NQ : AN;$$

therefore the triangles AMP, ANQ are similar, and the angles MAP, NAQ, equal to one another: hence the two lines AP, AQ coincide. If a third point R be taken in the locus, then, as before, AR will coincide with AP and AQ. Consequently all the lines drawn from A to the several points of the locus coincide; that is, all the points P, Q, R, &c., are in the same straight line AR, and by giving negative values to x we can determine the point S, &c., to be in the same straight line RA produced. Hence the straight line RAS produced both ways indefinitely, being the assemblage of all the points determined by the equation $y = \alpha x$, is the *locus* of that equation.



In considering the equation $y = \alpha x + b$, we observe that the new ordinate y always exceeds the former by the quantity b; hence taking A E in the axis A Y equal to b, and drawing the line H E F parallel to S R, the line H E F is the locus required.

Hence the equation of the first order belongs to the straight line.

32. To explain the nature of the equation more clearly, we will take the converse problem. To find the equation to a straight line H F, that is, to find the relation which exists between the co-ordinates, x and y, of each of its points.

Let A be the origin of co-ordinates, A X, A Y the axes; from A draw A R parallel to H F, and from any point P′ in the given line draw P′ P M perpendicular to A X and cutting A R in P.

$$\text{Let } AM = x,\ MP' = y, \text{ and } AE = b;$$

$$\begin{aligned} \text{then } MP' &= PM + PP' \\ &= AM \tan. PAM + AE \\ &= x \tan. FGX + b; \end{aligned}$$

$$\text{or } y = \alpha x + b, \quad \text{if } \tan. FGX = \alpha.$$

If $AG = a$, we have $AE = AG. \tan. EGA$, or $b = a\,\alpha$, and therefore the equation to the straight line may be written under the form $y = \alpha x + \alpha a$.

33. In general, therefore, the equation to the straight line contains two constant quantities b and α; the former is the distance A E or is the ordinate of the point in which the line cuts the axis of y, the latter is the tangent of the angle which the line makes with the axis of x, for the angle F G A = the angle P A M: hence

$$\tan. FGA = \tan. PAM = \frac{y - b}{x} = \alpha.$$

It is to be particularly observed that, in calling α the tangent of the angle which the line makes with the axis of x, we understand the angle F G X and not F G x.

34. In the equation $y = \alpha x + b$, the quantities α and b may be either both positive, or both negative, or one positive and the other negative; let us then examine the course of the line to which the equation belongs in each case. Now it is clear that the knowledge of two points in a straight line is sufficient to determine the position of that line; hence we shall only find the points where it cuts the axes since they are the most easily obtained.

1. α and b positive; $\therefore y = \alpha x + b$;

Let $x = 0$; $\therefore y = b$; in A Y take $AD = b$;

$y = 0$; $\therefore x = -\dfrac{b}{\alpha}$; in A x take $AB = \dfrac{b}{\alpha}$;

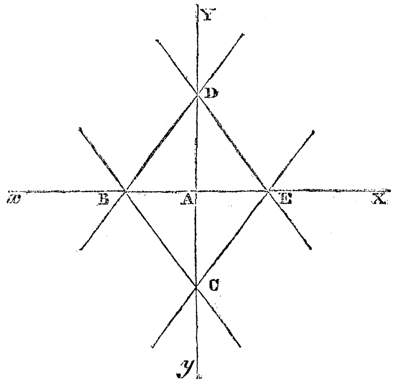
join B D; B D produced is the required locus.

2. α positive and b negative; $\therefore y = \alpha x - b$;

Let $x = 0$; $\therefore y = -b$, in A y take $AC = b$;

$y = 0$; $\therefore x = \dfrac{b}{\alpha}$; in A X take $AE = \dfrac{b}{\alpha}$;

join C E; C E produced is the required locus.



3. a negative and b positive; $\therefore y = -ax + b$;
Let $x = 0$; $\therefore y = b$; in A Y take AD $= b$;
$y = 0$; $\therefore x = \frac{b}{a}$; in A X take A E $= \frac{b}{a}$;
join D E; D E produced is the required locus.

4. a negative and b negative; $\therefore y = -ax - b$;
Let $x = 0$; $\therefore y = -b$; in A y take A C $= b$;
$y = 0$; $\therefore x = -\frac{b}{a}$; in A x take AB $= \frac{b}{a}$;
join B C; B C produced is the required locus.

35. The quantities a and b may also change in absolute value.

Let $b = 0$; $\therefore y = \pm a x$; and the loci are two straight lines passing through the origin and drawn at angles with the axis of x whose respective tangents are $\pm a$.

Let $a = 0$; $\therefore y = 0x \pm b$; $\therefore y = \pm b$ and $x = \frac{0}{0}$; the former of these results shows that every point in the locus is equidistant from the axis of x, and the latter (or $0x = 0$) that every value of x satisfies the original equation; hence the loci are two straight lines drawn through D and C both parallel to the axis of x.

It has been stated (28), that the system of equations $y = b, x = 0$ refers to a point; we here see that the system $y = b$, $x = \frac{0}{0}$ refers to a straight line; hence, although the equation $x = \frac{0}{0}$ is generally omitted, yet it must be considered as essential to the locus.

Let $a = \frac{1}{0}$; referring to article 32, the equation to the straight line may be written $y = ax \pm a a$ or $\frac{y}{a} = x \pm a$, which when $a = \frac{1}{0}$ becomes $0y = x \pm a$; hence, as before, the system $x = \pm a$, $y = \frac{0}{0}$, or more simply the equation $x = \pm a$ denotes two straight lines parallel to the axis of y and at a distance $\pm a$ from that axis.

Again let both $\alpha = 0$, and $b = 0$; and $\therefore$ the equation $y = \alpha x + b$ becomes $y = 0\,x + 0$; and hence, $y = 0$, $x = \frac{0}{0}$, and the locus is the axis of x.

If $\alpha = \frac{1}{0}$, and $b = 0$, the equation becomes $0\,y = x + 0$; $\therefore x = 0$ and $y = \frac{0}{0}$. Hence the equation $x = 0$ denotes the axis of y.

36. By the above methods the line to which any equation of the first order belongs may be drawn.

In the following examples reference is made to parts of the last figure.

Ex. 1. $3\,y - 5\,x - 1 = 0$; let $x = 0$, $\therefore y = \frac{1}{3}$; on the axis A Y take A D one-third of the linear unit, then the line passes through D: again let $y = 0$, $\therefore x = -\frac{1}{5}$; on the axis A x take A B $= \frac{1}{5}$ of the unit, then the line passes through B; hence the line joining the points B and D is the locus required.

Ex. 2. $10\,y - 21\,x + 6 = 0$; a line situated like C E.

Ex. 3. $y - x = 0$; let $x = 0$ $\therefore y = 0$, and the line passes through the origin; also α or the tangent of the angle which the line makes with the axis of $x = 1$, therefore that angle $= 45°$; hence the straight line drawn through the origin and bisecting the angle Y A X is the required locus.

Ex. 4. $5\,y - 2\,x = 0$. The line passes through the origin as in the last example, but to find another point through which the line passes, let $x = 5$; $\therefore y = 2$: hence take A E $= 5$, and from E draw E P ($= 2$) perpendicular to A X; then the line joining the points A, P is the locus required.

Ex. 5. $ay + bx = 0$; a line drawn through A, and parallel to B C.

Ex. 6. $y^2 - 3\,x^2 = 0$; two straight lines making angles of 60° with the axis of x.

Ex. 7. $3y - 4 = 0$; take A D $= \frac{4}{3}$ of the unit, a line through D drawn parallel to A X is the locus.

Ex. 8. $x^2 + x - 2 = 0$; take A E $= 1$, and A B $= 2$, the lines drawn through E and B parallel to A Y are the required loci.

Ex. 9. $y + 2\,x = 4$. The equation to a straight line may be put under the convenient form $\frac{y}{b} + \frac{x}{a} = 1$, and since when $y = 0$, $x = a$, and when $x = 0$, $y = b$, the quantities b and a are respectively the distances of the origin from the intersection of the line with the axes of y and x. Thus Ex. 9. in this form is $\frac{y}{4} + \frac{x}{2} = 1$, take A D $= 4$, and A E $= 2$, join D E, this line produced is the required locus.

37. If the equation involve the second root of a negative quantity its locus will not be a straight line, but either a point or altogether imaginary :

thus the locus of the equation $y + 2x\sqrt{-1} - a = 0$ is a point whose co-ordinates are $x = 0$ and $y = a$, for no other real value of x can give a real value to y; but the locus of the equation $y + x + a\sqrt{-1} = 0$ is imaginary, for there are no corresponding real values of x and y. (24)

38. We have thus seen that the equation to a straight line is of the form $y = \alpha x + b$, and that its position depends entirely upon α and b.

By a given line we understand one whose position is given, that is, that α and b are given quantities; when we seek a line we require its position, so that assuming $y = \alpha x + b$ to be its equation, α and b are the two indeterminate quantities to be found by the conditions of the question: if only one can be found the conditions are insufficient to fix the position of the line.

By a given point we understand one whose co-ordinates are given; we shall generally use the letters x_1 and y_1 for the co-ordinates of a given point, and to avoid useless repetition, the point whose co-ordinates are x_1 and y_1 will be called "the point x_1, y_1." Similarly the line whose equation is $y = \alpha x + b$ will be called "the line $y = \alpha x + b$."

If in the same problem we use the equations to two straight lines as $y = \alpha x + b$ and $y = \alpha' x + b'$, it must be carefully remembered that x and y are not the same quantities in both equations; we might have used the equations $y = \alpha x + b$, and $Y = \alpha' X + b'$, X and Y being the variable coordinates of the second line, but the former notation is found to be the more convenient.

39. We regret much that in the following problems on straight lines we cannot employ an homogeneous equation as $\frac{y}{b} + \frac{x}{a} = 1$. In algebraical geometry the formulas most in use are very simple, much more so indeed than they would be if homogeneous: moreover the advantage of a uniform system of symbols and formulas is so great to mathematicians that it should not be violated without very strong reasons. To remedy in some degree this want of regularity, the student should repeatedly consider the meaning of the constants at his first introduction to the subject of straight lines.

PROBLEMS ON STRAIGHT LINES.

40. To find the equation to a straight line passing through a given point.

The point being given its co-ordinates are known; let them be $x_1 y_1$, and let the equation to the straight line be $y = \alpha x + b$; we signify that this line passes through the point $x_1 y_1$, by saying that when the variable abscissa x becomes x_1, then y will become y_1: hence the equation to the line becomes

$$y_1 = \alpha x_1 + b$$
$$\therefore b = y_1 - \alpha x_1$$

substituting this value for b in the first equation, we have

$$y = \alpha x + y_1 - \alpha x_1$$
$$\text{or } y - y_1 = \alpha (x - x_1)$$

The shortest method of eliminating b is by subtracting the second equation from the first, and this is the method generally adopted.

Since α, which fixes the direction of the line, is not determined, there may be an infinite number of straight lines drawn through a given point; this is also geometrically apparent.

If the given point be on the axis of x, $y_1 = 0$, and $\therefore y = \alpha(x - x_1)$; and if it be on the axis of y, $x_1 = 0$ $\therefore y - y_1 = \alpha x$.

If either or both of the co-ordinates of the given point be negative, the proper substitutions must be made: thus if the point be on the axis of x and in the negative direction from A, its co-ordinates will be $-x_1$ and 0; therefore the equation to the line passing through that point will be

$$y = \alpha(x + x_1).$$

41. To find the equation to a straight line passing through two given points x_1, y_1 and x_2, y_2.

Let the required equation be $y = \alpha x + b$ (1)
then since the line passes through the given points, we have the equations

$$y_1 = \alpha x_1 + b \quad (2)$$

$$y_2 = \alpha x_2 + b \quad (3)$$

Subtracting (2) from (1)

$$y - y_1 = \alpha(x - x_1) \quad (4)$$

Subtracting (3) from (2)

$$y_1 - y_2 = \alpha(x_1 - x_2)$$

$$\therefore \alpha = \frac{y_1 - y_2}{x_1 - x_2}$$

Substituting this value of α in (4), we have finally

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

The two conditions have sufficed to determine α and b, and by their elimination the position of the line is fixed, as it ought to be, since only one straight line can be drawn through the same two points.

This equation will assume different forms according to the particular situation of the given points.

Thus if the point x_2, y_2 be on the axis of x, we have $y_2 = 0$;

$$\therefore y - y_1 = \frac{y_1}{x_1 - x_2}(x - x_1);$$

if it be on the axis of y, $x_2 = 0$; $\therefore y - y_1 = \frac{y_1 - y_2}{x_1}(x - x_1)$;

and if it be at the origin both y_2 and $x_2 = 0$;

$$\therefore y - y_1 = \frac{y_1}{x_1}(x - x_1) = \frac{y_1}{x_1}x - y_1$$

$$\therefore y = \frac{y_1}{x_1}x.$$

This last equation is also thus obtained; the line passing through the origin, its equation must be of the form $y = \alpha x$ (31) where α is the tangent of the angle which the line makes with the axis of x, and this line passing through the point x_1, y_1, α must be equal to $\frac{y_1}{x_1}$ $\therefore y = \frac{y_1}{x_1} x$.

If a straight line pass through three given points, the following relation must exist between the co-ordinates of those points:

$$(y_1 x_2 - x_1 y_2) - (y_1 x_3 - x_1 y_3) + (y_2 x_3 - x_2 y_3) = 0.$$

42. To find the equation to a straight line passing through a given point x_3, y_3, and bisecting a finite portion of a given straight line.

Let the portion of the straight line be limited by the points $x_1\, y_1$ and $x_2\, y_2$, and therefore the co-ordinates of the bisecting point are $\frac{x_1 + x_2}{2}$, $\frac{y_1 + y_2}{2}$; hence the required equation is

$$y - y_3 = \alpha\,(x - x_3) = \frac{y_1 + y_2 - 2\,y_3}{x_1 + x_2 - 2\,x_3}(x - x_3)$$

43. To find the equation to a straight line parallel to a given straight line.

Let $y = \alpha x + b$ (1) be the given line

$y = \alpha' x + b'$ (2) . . . . required line.

then since the lines are parallel they must make equal angles with the axis of x or $\alpha' = \alpha$ $\therefore$ the required equation is

$$y = \alpha x + b' \quad (3).$$

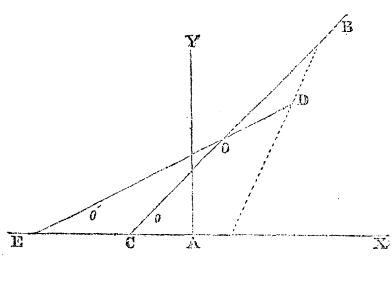
Of course b' could not be determined by the single condition of the parallelism of the lines, since an infinite number of lines may be drawn parallel to the given line; but if another condition is added, b' will be then determined: thus if the required line passes also through a given point $x_1\, y_1$, equation (2) is

$$y - y_1 = \alpha'\,(x - x_1)$$

$$\therefore (3) \text{ becomes } y - y_1 = \alpha\,(x - x_1)$$

44. To find the intersection of two given straight lines C B, E D.

This consists in finding the co-ordinates of the point O of intersection. Now it is evident that at this point they have the same abscissa and ordinate; hence if in the equations to two lines we regard x as representing the same abscissa and y the same ordinate, it is in fact saying that they are the co-ordinates X, Y of the point of intersection O.



Let $y = \alpha x + b$ be the equation to C B

and $y = \alpha' x + b'$ . . . . E D

then at O we have $Y = \alpha X + b = \alpha' X + b'$

$$\therefore X = \frac{b' - b}{\alpha - \alpha'}$$

$$\text{and } Y = \alpha X + b = \frac{\alpha b' - \alpha b}{\alpha - \alpha'} + b = \frac{\alpha b' - \alpha' b}{\alpha - \alpha'}.$$

Ex. 1. To find the intersection of the lines whose equations are

$y = 3x + 1$ and $y - 2x - 4 = 0$. $X = 3$ and $Y = 10$.

Ex. 2. To find the intersection of the lines whose equations are

$y - x = 0$ and $3y - 2x = 1$. $X = 1$ and $Y = 1$.

If a third line, whose equation is $y = \alpha'' x + b''$, passes through the point of intersection, the relation between the coefficients is

$$(\alpha b' - \alpha' b) - (\alpha b'' - \alpha'' b) + (\alpha' b'' - \alpha'' b') = 0.$$

45. To find the tangent, sine and cosine of the angle betwen two given straight lines.

Let $y = \alpha x + b$ be the equation to C B

$y = \alpha' x + b'$ . . . . E D

θ and θ' the angles which they make respectively with the axis of x; then

$$\text{tan. D O B} = \text{tan. E O C} = \text{tan. } (\theta - \theta') = \frac{\text{tan. } \theta - \text{tan. } \theta'}{1 + \text{tan. } \theta \text{ tan. } \theta'} = \frac{\alpha - \alpha'}{1 + \alpha \alpha'}$$

$$\text{also cos. D O B} = \frac{1}{\text{sec. D O B}} = \frac{1}{\sqrt{1 + (\text{tan. D O B})^2}} = \frac{1 + \alpha \alpha'}{\sqrt{(1+\alpha^2)(1+\alpha'^2)}}$$

$$\text{and sine D O B} = \text{tan. D O B} \times \text{cos. D O B} = \frac{\alpha - \alpha'}{\sqrt{1+\alpha^2} \, . \, \sqrt{1+\alpha'^2}}.$$

46. To find the equation to a straight line making a given angle with another straight line.

Let $y = \alpha x + b$ be the given line C B,

$y = \alpha' x + b'$ required line E D,

β = tangent of the given angle D O B.

Then α' = tan. D E C = tan. (B C X — B OD)

$$= \frac{\text{tan. B C X} - \text{tan. B O D}}{1 + \text{tan. B C X . tan. B O D}} = \frac{\alpha - \beta}{1 + \alpha\beta}.$$

Substituting this value for α' in the second equation,

$$y = \frac{\alpha - \beta}{1 + \alpha\beta} x + b'.$$

If the required line passes also through a given point x_1, y_1, the equation is

$$y - y_1 = \frac{\alpha - \beta}{1 + \alpha\beta}(x - x_1).$$

If D be considered the given point x_1, y_1, then not only the line D O E but another (the dotted line in the figure) might be drawn, making a given angle with B C, and its equation is found, as above, to be

$$y - y_1 = \frac{\alpha + \beta}{1 - \alpha\beta}(x - x_1);$$

so that both lines are included in the equation

$$y - y_1 = \frac{\alpha \mp \beta}{1 \pm \alpha\beta}(x - x_1).$$

For example, the two straight lines which pass through the point D and cut B C at an angle of 45° are given by the equations

$$y - y_1 = \frac{\alpha - 1}{\alpha + 1}(x - x_1),$$

$$y - y_1 = \frac{1 + \alpha}{1 - \alpha}(x - x_1).$$

Also the equation to the straight line passing through D and cutting the axis of x at an angle of 135° is

$$y - y_1 = \beta(x - x_1) = \tan. 135^\circ (x - x_1) = -(x - x_1),$$
$$\text{or } y + x = y_1 + x_1.$$

47. If the required line is to be perpendicular to the given line, β is infinitely great; therefore the fraction $\dfrac{\alpha - \beta}{1 + \alpha\beta} = \dfrac{\frac{\alpha}{\beta} - 1}{\frac{1}{\beta} + \alpha} = -\dfrac{1}{\alpha}$, or $\alpha' = -\dfrac{1}{\alpha}$; hence the equation to a straight line perpendicular to a given line $y = \alpha x + b$, is $y = -\dfrac{1}{\alpha}x + b'$.

This may be also directly proved, for drawing O E perpendicular to B C, as in the next figure, we have $\alpha' = \tan.$ O E X $= -\tan.$ O E C $=$ $-\cot.$ O C X $= -\dfrac{1}{\alpha}$: hence in the equations to two straight lines which are perpendicular to one another we have $\alpha\alpha' + 1 = 0$; and, conversely, if in the equations to two straight lines, we find $\alpha\alpha' + 1 = 0$, these lines are perpendicular to one another.

If the perpendicular line pass also through a given point $x_1 y_1$, its equation is

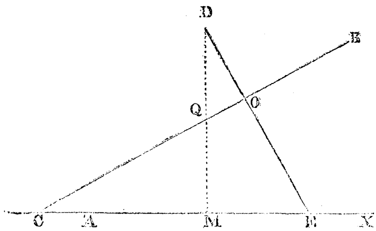
$$y - y_1 = -\frac{1}{\alpha}(x - x_1);$$

and, of course, this equation will assume various forms, agreeing with the position of the point $x_1 y_1$; thus, for example, the line drawn through the origin perpendicular to the line $y = \alpha x + b$, is one whose equation is $y = -\dfrac{1}{\alpha}x$, for here both x_1 and $y_1 = 0$.

48. To find the length of a perpendicular from a given point D $(x_1 y_1)$ on a given straight line C B.

Let $y = a x + b$ (1) be the equation to C B,

then $y - y_1 = - \frac{1}{a}(x - x_1)$ (2) is the equation to the perpendicular line D O E,



Let $p =$ D O; then if X and Y be the co-ordinates of O determined from (1) and (2) we have $p^2 = (X - x_1)^2 + (Y - y_1)^2$; (29)

from (2) $Y = y_1 - \frac{1}{a}(X - x_1) = aX + b$ from (1)

$$= a(X - x_1) + a x_1 + b,$$

$$\therefore (a + \frac{1}{a})(X - x_1) = y_1 - a x_1 - b,$$

$$\therefore X - x_1 = \frac{a}{1 + a^2}(y_1 - a x_1 - b), \text{ also } Y - y_1 = - \frac{1}{a}(X - x_1)$$

$$\therefore p^2 = (X - x_1)^2 + (Y - y_1)^2$$

$$= (X - x_1)^2 + \frac{1}{a^2}(X - x_1)^2$$

$$= \frac{1 + a^2}{a^2}(X - x_1)^2$$

$$= \frac{1 + a^2}{a^2} \frac{a^2}{(1 + a^2)^2} \cdot (y_1 - a x_1 - b)^2 = \frac{1}{1 + a^2}(y_1 - a x_1 - b)^2$$

$$\therefore p = \pm \frac{y_1 - a x_1 - b}{\sqrt{1 + a^2}}.$$

The superior sign is to be taken when the given point is above the given straight line, and the inferior in the contrary case.

If the given line pass through the origin $b = 0$; $\therefore p = \pm \frac{y_1 - a x_1}{\sqrt{1 + a^2}}$.

If the origin be the given point, $x_1 = 0$ and $y_1 = 0$; $\therefore p = \frac{\pm b}{\sqrt{1 + a^2}}$.

There is another way of obtaining the expression for p.

Since the equation $y = a x + b$ applies to all points in C O B, it must to Q, where M D or y_1 cuts C O B; $\therefore$ M Q $= a x_1 + b$.

Now D O = D Q sin. D Q O,

but DQ = D M − M Q $= y_1 - a x_1 - b$,

and sin. D Q O = sin. C Q M = cos. Q C M = $\frac{1}{\text{sec. Q C M}}$ =

$$\frac{1}{\sqrt{1 + (\tan.\ Q\,C\,M)^2}} = \frac{1}{\sqrt{1+\alpha^2}};$$

$\therefore$ D O or $p = \dfrac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}}$, or $\dfrac{\alpha x_1 + b - y_1}{\sqrt{1 + \alpha^2}}$, if D was below the line.

49. If the line D E had been drawn making a given angle whose tangent was β with the given line C O, the distance D O might be found; for instead of equation (2) we shall have

$$y - y_1 = \frac{\alpha - \beta}{1 + \alpha\beta}(x - x_1) \quad (46);$$

hence, following the same steps as above, we shall find

$$p = \pm \frac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}} \, \frac{\sqrt{1 + \beta^2}}{\beta}.$$

This expression is also very easily obtained trigonometrically.

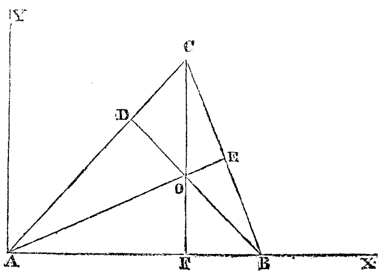
Let γ = sine of the given angle, then

$$D\,O = D\,Q \,.\, \frac{\sin.\ D\,Q\,O}{\sin.\ D\,O\,Q}$$

$$= \frac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}} \cdot \frac{1}{\gamma}.$$

50. The equation to the straight line may be used with advantage in the demonstration of the following theorem :—

If from the angles of a plane triangle perpendiculars be let fall on the opposite sides, these perpendiculars will meet in one point.



In the triangle A B C, let A E, B D, C F be perpendiculars from A, B and C on the opposite sides; let O be the point where A E and B D meet, then the theorem will be established by showing that the abscissa to the point O is A F.

Let A be the origin of co-ordinates,
A B the axis of x,
and A Y, perpendicular to A B, the axis of y.

Let the co-ordinates of C be x_1, y_1
.... B x_2, 0 :

we have then the following equations,

$$\text{to A C} \quad y = \frac{y_1}{x_1}x \quad (41)$$

$$\therefore \text{ to B D, } y = \alpha(x - x_2) = -\frac{x_1}{y_1}(x - x_2) \quad (47)$$

$$\text{to B C, } y - y_2 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_2) \quad (41)$$

$$\text{or } y = \frac{y_1}{x_1 - x_2}(x - x_2) \text{ since } y_2 = 0.$$

$$\therefore \text{ to A E, } y = \alpha x = -\frac{x_1 - x_2}{y_1} x;$$

for the intersection O of B D and A E we have, by equating the values of y,

$$-\frac{x_1}{y_1}(x - x_2) = -\frac{x_1 - x_2}{y_1} x,$$

$$\therefore x_1 x - x_1 x_2 = x_1 x - x_2 x; \quad \therefore x_2 x = x_1 x_2 \text{ and } x = x_1,$$

that is, the abscissa of the point O is found to be that of C.

In the same manner it may be proved that if perpendiculars be drawn from the bisections of the sides, they will meet in one point.

Similarly we may prove that the three straight lines F C, K B, and A L, in the 47th proposition of Euclid, meet in one point within the triangle A B C.

51. We have hitherto considered the axes as rectangular, but if they be oblique, the coefficient of x, in the equation to a straight line, is not the tangent of the angle which the line makes with the axis of x.

Let ω = the angle between the axes,

θ = the angle which the line makes with the axis of x;

$$\text{then } \alpha = \frac{y-b}{x} = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)} \ (33);$$

b remains, as before, the distance of the origin from the intersection of the line with the axis of y: hence the equation to a straight line referred to oblique axes is

$$y = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)} x + b.$$

Since this equation is of the form $y = \alpha x + b$ all the results in the preceding articles which do not affect the ratio of $\frac{\sin.\ \theta}{\sin.\ (\omega - \theta)}$ will be equally true when the axes are oblique.

Thus, articles 40, 41, 42, 43, and 44, require no modification.

To find the tangent (β) of the angle between two given straight lines.

Let $\left.\begin{matrix} y = \alpha x + b \\ y = \alpha' x + b' \end{matrix}\right\}$ be the equations;

from $\alpha = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)}$ we have $\tan.\ \theta = \frac{\alpha \sin.\ \omega}{1 + \alpha \cos.\ \omega}$; and, similarly,

$$\tan.\ \theta' = \frac{\alpha' \sin.\ \omega}{1 + \alpha' \cos.\ \omega}; \text{ hence } \beta = \tan.\ (\theta - \theta') = \frac{(\alpha - \alpha') \sin.\ \omega}{1 + \alpha \alpha' + (\alpha + \alpha') \cos.\ \omega}$$

To find the equation to a straight line passing through a given point $x_1\ y_1$, and making a given angle with a given straight line.

Let β be the tangent of the given angle,

$y = \alpha x + b$, the given line,

$y - y_1 = \alpha' (x - x_1)$, the required line.

From the last formula we have

$$\alpha' = \frac{\alpha \sin.\ \omega - \beta\ (1 + \alpha \cos.\ \omega)}{\sin.\ \omega + \beta\ (\alpha + \cos.\ \omega)}$$

and the required equation is

$$\therefore y - y_1 = \frac{\alpha \sin.\ \omega - \beta\ (1 + \alpha \cos.\ \omega)}{\sin.\ \omega + \beta\ (\alpha + \cos.\ \omega)} (x - x_1).$$

If the lines be perpendicular to each other $\beta = \frac{1}{0}$;

$$\therefore \alpha' = -\frac{1+\alpha \cos. \omega}{\alpha + \cos. \omega}.$$

and the required equation is

$$y - y_1 = -\frac{1+\alpha \cos. \omega}{\alpha + \cos. \omega}(x - x_1).$$

To find the length of the perpendicular from a given point upon a given straight line.

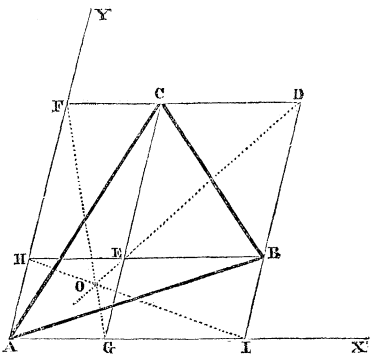
Instead of equation (2), in article 48, we must use the equation just found, and then proceeding as usual we shall find

$$p = \pm \frac{(y_1 - \alpha x_1 - b) \sin. \omega}{\sqrt{\{1 + 2 \alpha \cos. \omega + \alpha^2\}}}.$$

It will be concluded from an observation of these formulas, that oblique axes are to be avoided as much as possible; they may be used with advantage where points and lines, but not angles, are the subjects of discussion. As an instance, we shall take the following theorem.

52. If, upon the sides of a triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of the parallelograms will intersect each other in the same point.

Let A B C be the triangle, A X, A Y the given lines, E B D C, C F A G,



H A I B the parallelograms, the opposite diagonals D E, F G, and H I will meet in one point O.

Let A be the origin, A X, A Y the oblique axes

$x_1 y_1$ the co-ordinates of B

$x_2 y_2$ . . . . C.

To find the equation to D E;

let it be $y = \alpha x + b$

$y_2 = \alpha x_1 + b$ at D

$\therefore y - y_2 = \alpha (x - x_1)$

$y_1 - y_2 = \alpha (x_2 - x_1)$ at E

$$\therefore y - y_2 = \frac{y_1 - y_2}{x_2 - x_1}(x - x_1) \quad (1).$$

To find the equation to F G;

$$y = \alpha x + b$$

$$y_2 = 0 + b \text{ at F}$$

$$\therefore y - y_2 = \alpha x$$

$$0 - y_2 = \alpha x_2 \text{ at G}$$

$$\therefore y - y_2 = -\frac{y_2}{x_2} x \quad (2).$$

To find the equation to H I;

$$y = \alpha x + b$$

$$y_1 = 0 \quad + b \text{ at H}$$

$$\therefore y - y_1 = \alpha x$$

$$0 - y_1 = \alpha x_1 \text{ at I}$$

$$\therefore y - y_1 = -\frac{y_1}{x_1} x \quad (3).$$

Equating the values of y in (1) and (2) we find $X = \frac{x_1 x_2 (y_1 - y_2)}{y_1 x_2 - x_1 y_2}$; also equating the values of y in (2) and (3) we find the same value for X; hence the abscissa for intersection being the same for any two of the lines, they must all three intersect in the same point.

Similarly we may prove that if from the angles of a plane triangle straight lines be drawn to the bisections of the opposite sides, they will meet in one point.

CHAPTER IV.

THE TRANSFORMATION OF CO-ORDINATES.

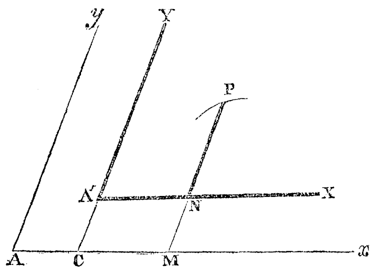
53. Before we proceed to the discussion of equations of higher orders, it is necessary to investigate a method for changing the position of the co-ordinate axes.

The object is to place the axes in such a manner that the equation to a given curve may appear in its most simple form, and conversely by the introduction of indeterminate constants into an equation to reduce the number of terms, so that the form and properties of the corresponding locus may be most easily detected.

An alteration of this nature cannot in the least change the form of the curve, but only the algebraical manner of representing it; thus the general equation to the straight line $y = \alpha x + b$ becomes $y = \alpha x$ when the origin is on the line itself. Also on examining articles 46 and 51 we see that the simplicity of an equation depends very much on the angle between the axes.

Hence in many cases not only the position of the origin but also the direction of the axes may be altered with advantage. The method of performing these operations is called the transformation of co-ordinates.

54. To transform an equation referred to an origin A, to an equation referred to another origin A′, the axes in the latter case being parallel to those in the former.



Let A x, A y be the original axes

A′ X, A′ Y the new axes

$\left.\begin{array}{l} AM = x \\ MP = y \end{array}\right\}$ original co-ordinates of P

$\left.\begin{array}{l} A'N = X \\ NP = Y \end{array}\right\}$ new co-ordinates of P

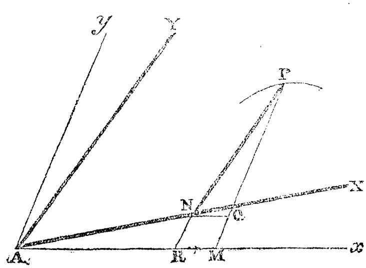
$\left.\begin{array}{l} AC = a \\ CA' = b \end{array}\right\}$ the co-ordinates of the new origin A′,

then M P = M N + N P, that is, $y = b + Y$,

A M = A C + C M, ... $x = a + X$;

substituting these values for y and x in the equation to the curve, we have the transformed equation between Y and X referred to the origin A′.

55. To transform the equation referred to oblique axes, to an equation referred to other oblique axes having the same origin.



Let A x, A y be the original axes,

A X, A Y be the new axes,

$\left.\begin{array}{l} AM = x \\ MP = y \end{array}\right\}$ original co-ordinates of P.

$\left.\begin{array}{l} AN = X \\ NP = Y \end{array}\right\}$ new co-ordinates of P.

Let the angle $x A y = \omega$, $x A X = \theta$, $x A Y = \theta'$;

Draw N R parallel to P M, and N Q parallel to A M,

then $y = \text{M P} = \text{M Q} + \text{Q P} = \text{N R} + \text{Q P}$

$$= \text{A N}\,\frac{\sin.\ \text{N A R}}{\sin.\ \text{N R A}} + \text{P N}\,\frac{\sin.\ \text{P N Q}}{\sin.\ \text{P Q N}}$$

$$= \text{X}\,\frac{\sin.\ \theta}{\sin.\ \omega} + \text{Y}\,\frac{\sin.\ \theta'}{\sin.\ \omega};$$

and $x = \text{A M} = \text{A R} + \text{R M} = \text{A R} + \text{N Q}$

$$= \text{A N}\,\frac{\sin.\ \text{A N R}}{\sin.\ \text{A R N}} + \text{P N}\,\frac{\sin.\ \text{N P Q}}{\sin.\ \text{N Q P}}$$

$$= \text{X}\,\frac{\sin.\ (\omega - \theta)}{\sin.\ \omega} + \text{Y}\,\frac{\sin.\ (\omega - \theta')}{\sin.\ \omega}$$

$$\therefore y = \frac{\text{X} \sin.\ \theta + \text{Y} \sin.\ \theta'}{\sin.\ \omega}$$

$$x = \frac{\text{X} \sin.\ (\omega - \theta) + \text{Y} \sin.\ (\omega - \theta')}{\sin.\ \omega}.$$

56. Let the original axes be oblique, and the new rectangular, or $\theta' - \theta = 90°$.

$$\therefore y = \frac{\text{X} \sin.\ \theta + \text{Y} \cos.\ \theta}{\sin.\ \omega}$$

$$x = \frac{\text{X} \sin.\ (\omega - \theta) - \text{Y} \cos.\ (\omega - \theta)}{\sin.\ \omega}.$$

57. Let the original axes be rectangular, or $\omega = 90°$.

$$\therefore y = \text{X} \sin.\ \theta + \text{Y} \sin.\ \theta',$$
$$x = \text{X} \cos.\ \theta + \text{Y} \cos.\ \theta'$$

58. Let both systems be rectangular, or $\omega = 90°$ and $\theta' - \theta = 90°$

$$\therefore y = \text{X} \sin.\ \theta + \text{Y} \cos.\ \theta,$$
$$x = \text{X} \cos.\ \theta - \text{Y} \sin.\ \theta$$

59. These forms have been deduced from the first, but each of them may be found by a separate process. The first and last pairs are the most useful. Perhaps they may be best remembered if expressed in the following manner.

Both systems oblique, the formulas (55) become

$$y = \{\text{X} \sin.\ \text{X} A x + \text{Y} \sin.\ \text{Y} A x\}\,\frac{1}{\sin.\ x A y}$$

$$x = \{\text{X} \sin.\ \text{X} A y + \text{Y} \sin.\ \text{Y} A y\}\,\frac{1}{\sin.\ x A y}.$$

Both systems rectangular, the formulas (58) become

$$y = \text{X} \cos.\ \text{X} A y + \text{Y} \cos.\ \text{Y} A y$$
$$x = \text{X} \cos.\ \text{X} A x + \text{Y} \cos.\ \text{Y} A x.$$

If the situation of the origin be changed as well as the direction of the axes, we have only to add the quantities a and b to the values of x and y respectively; however, in such a case, it is most convenient to perform each transformation separately.

If the new axis of X falls below the original axis of x, the angle θ must be considered as negative, therefore its sine will be negative and its cosine positive. Hence the formulas of transformation will require a slight alteration before applied to this particular case.

Since the values of x and y are in all cases expressed by equations of the first order, the degree of an equation is never changed by the transformation of co-ordinates.

60. Hitherto we have determined the situation of a point in a plane by its distance from two axes, but there is also another method of much use. Let S be a fixed point, and S B a fixed straight line; then the point P is also evidently determined if we know the length S P and the angle P S B.

If S P $= r$ and P S B $= \theta$, r and θ are called the polar co-ordinates of P. S is called the pole, and S P the radius vector, because a curve may be supposed to be described by the extremity of the line S P revolving round S, the length of S P being variable. The fixed straight line S B is also called the axis.

To transform an equation between co-ordinates x and y into another between polar co-ordinates r and θ,

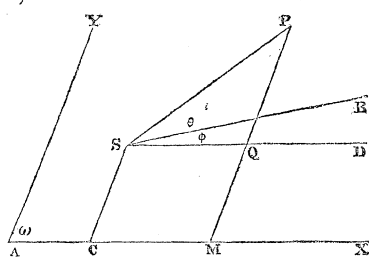
Draw S D parallel to A X, and let the angle B S D $= \phi$, and the angle Y A X $= \omega$.

Let A M $= x$, M P $= y$, A C $= a$, C S $= b$,

$$\left.\begin{aligned} \text{then } y &= \text{M P} = \text{M Q} + \text{Q P} = b + r\frac{\sin.\ (\theta+\phi)}{\sin.\ \omega} \\ x &= \text{A M} = \text{A C} + \text{S Q} = a + r\frac{\sin.\{\omega-(\theta+\phi)\}}{\sin.\ \omega.} \end{aligned}\right\} \quad (1)$$

Let S B coincide with S D, or $\phi = 0$;

$$\left.\begin{aligned} \therefore\ y &= b + r\,\frac{\sin.\ \theta}{\sin.\ \omega} \\ x &= a + r\,\frac{\sin.\ (\omega-\theta)}{\sin.\ \omega} \end{aligned}\right\} \quad (2)$$



61. Let the original axes be also rectangular, or $\omega = \frac{\pi}{2}$;

$$\left.\begin{aligned} \therefore\ y &= b + r\sin.\ \theta \\ x &= a + r\cos.\ \theta \end{aligned}\right\} \quad (3)$$

and if the origin A be the pole, we have $a = 0$ and $b = 0$.

$$\left.\begin{aligned} \therefore\ y &= r\sin.\ \theta \\ x &= r\cos.\ \theta \end{aligned}\right\} (4).$$

Of these formulas (3) and (4) are the most useful.

62. Conversely, to find r and θ in terms of x and y: from (1) we have

$$\frac{x-a}{y-b} = \frac{\sin.\{\omega-(\theta+\phi)\}}{\sin.\ (\theta+\phi)} = \sin.\ \omega\cot.\ (\theta+\phi) - \cos.\ \omega;$$

$$\therefore\ \tan.\ (\theta+\phi) = \frac{(y-b)\sin.\ \omega}{x-a+(y-b)\cos.\ \omega}$$

$$\therefore\ \theta+\phi = \tan.^{-1}\left\{\frac{(y-b)\sin.\ \omega}{x-a+(y-b)\cos.\ \omega}\right\},$$

where the symbol $\tan.^{-1}a$ is equivalent to the words "a circular arc whose radius is unity, and tangent a."

$$\text{also } r^2 = (x-a)^2 + (y-b)^2 + 2\,(x-a)\,(y-b)\cos\ \omega \ldots (30)$$

63. If the axes are rectangular, or $\omega = \frac{\pi}{2}$, the pole at the origin, and therefore $a = 0$ and $b = 0$, and also $\phi = 0$, we have $\frac{y}{x} = \tan.\theta$, and therefore

$$\theta = \tan.^{-1}\frac{y}{x}$$

$$\text{and } r^2 = x^2 + y^2 \ldots. (29)$$

and these are the formulas generally used.

From $\tan.\theta = \frac{y}{x}$ we have $\cos.\theta = \frac{1}{\sqrt{1+(\tan.\theta)^2}} = \frac{1}{\sqrt{1+\frac{y^2}{x^2}}} = \frac{x}{\sqrt{y^2+x^2}}$ and $\sin.\theta = \cos.\theta \times \tan.\theta = \frac{y}{\sqrt{y^2+x^2}}$; hence the value of θ may also be expressed by the equations

$$\theta = \sin.^{-1}\frac{y}{\sqrt{y^2+x^2}} \text{ or } \theta = \cos.^{-1}\frac{x}{\sqrt{y^2+x^2}}.$$

CHAPTER V.

ON THE CIRCLE.

64. Following the order of this treatise, our next subject of discussion would be the loci of the general equation of the second degree; but there is one curve among these loci, remarkable for the facility of its description and the simplicity of its equation: this curve, we need scarcely say, is the circle; and as the discussion of the circle is admirably fitted to prepare the reader for other investigations, we proceed to examine its analytical character.

The common definition of the circle states, that the distance of any point on the circumference of the figure from the centre is equal to a given line called the radius.

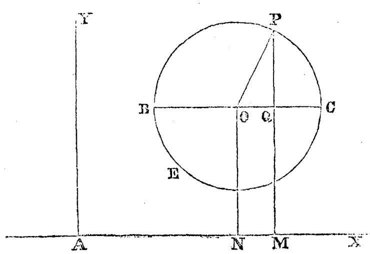
If a and b be the co-ordinates of the centre, x and y those of any point on the circumference, and r the radius, the distance between those points is $\sqrt{\{(y-b)^2+(x-a)^2\}}$ (29): hence the equation to the circle is

$$(y-b)^2+(x-a)^2 = r^2$$

65. To obtain this equation directly from the figure,
let A be the origin,
A X, A Y the rectangular axes,
$\left.\begin{array}{l}\text{A N} = a\\ \text{N O} = b\end{array}\right\}$ the co-ordinates of the centre,

$\left.\begin{array}{l}AM = x \\ MP = y\end{array}\right\}$ those of any point P on the circumference;

B O Q C a line parallel to the axis of x.



Then the square upon O P = the square upon P Q + the square upon O Q,

and $PQ = PM - QM = y - b$

also $OQ = AM - AN = x - a$;

$\therefore r^2 = (y - b)^2 + (x - a)^2$

or $(y - b)^2 + (x - a)^2 = r^2$ (1)

If the axis of y or that of x passes through the centre, the equation (1) becomes respectively

$$\left.\begin{array}{l}(y - b)^2 + x^2 = r^2 \\ \text{or } y^2 + (x - a)^2 = r^2\end{array}\right\} (2)$$

If the origin be at any point of the circumference as E, we have then the equation of condition $a^2 + b^2 = r^2$; expanding (1) and reducing it by means of this condition, we have

$$y^2 - 2by + x^2 - 2ax = 0 \ (3).$$

If the origin is at B, B O being the axis of x, we have $b = 0$ and $a = r$;

$$\therefore y^2 + x^2 - 2rx = 0$$

$$\text{or } y^2 = 2rx - x^2 \ (4).$$

Again, placing the origin at the centre O, we have $b = 0$ and $a = 0$;

$$\therefore y^2 + x^2 = r^2 \ (5).$$

The above equations are all useful, but those most required are (1), (4), and (5).

66. Equation (1), if expanded, is

$$y^2 + x^2 - 2by - 2ax + a^2 + b^2 - r^2 = 0.$$

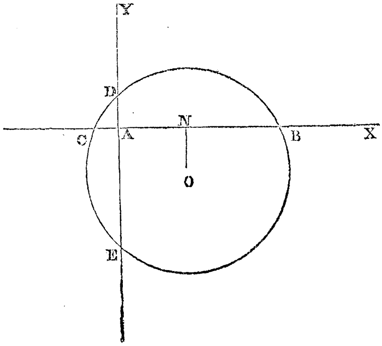
This differs from the complete equation of the second order (23) in having the coefficients of x^2 and y^2 unity, and by having no term containing the product xy.

Any equation of this form being given, we can, by comparing it with the above equation, determine the situation of its locus, that is, find the position of the centre, and the magnitude of the corresponding circle.

Ex. 1. $y^2 + x^2 + 4y - 8x - 5 = 0.$

here $b = -2$, $a = 4$, and $a^2 + b^2 - r^2 = -5$;

$$\therefore r^2 = a^2 + b^2 + 5 = 25.$$



Let A be the origin of co-ordinates, A X, A Y the axes.

In AX take AN = 4 times the linear unit, from N draw NO perpendicular to A X, but downwards, and equal to 2; then O is the centre of the circle. With centre O and radius 5 describe a circle; this is the locus required. The points where it cuts the axis of x are determined by putting $y = 0$;

$$\therefore x^2 - 8x - 5 = 0;$$
$$\therefore x = 4 \pm \sqrt{21};$$
$$\text{hence } AB = 4 + \sqrt{21} \text{ and } AC = 4 - \sqrt{21}.$$

Similarly putting $x = 0$, we find A D = 1 and A E = 5.

67. The shortest way of describing the locus is to put the equation into the form $(y - b)^2 + (x - a)^2 = r^2$.

For example, the equation

$$y^2 + x^2 + cy + dx + e = 0,$$

becomes, by the addition and subtraction of $\frac{c^2}{4}$ and $\frac{d^2}{4}$,

$$y^2 + cy + \frac{c^2}{4} + x^2 + dx + \frac{d^2}{4} + e - \frac{c^2}{4} - \frac{d^2}{4} = 0,$$

$$\text{or } (y + \frac{c}{2})^2 + (x + \frac{d}{2})^2 = \frac{c^2 + d^2}{4} - e,$$

where we observe directly that $-\frac{d}{2}$ and $-\frac{c}{2}$ are the co-ordinates of the centre, and that $\sqrt{\{\frac{c^2 + d^2}{4} - e\}}$ is the radius of the required locus.

Ex. 2. $y^2 + x^2 + 4y - 4x - 8 = 0$

add and subtract 8, and the equation becomes

$$y^2 + 4y + 4 + x^2 - 4x + 4 - 16 = 0$$
$$\text{or } (y + 2)^2 + (x - 2)^2 = 16$$

hence the co-ordinates of the centre are $a = 2$ and $b = -2$, and the radius is 4.

Ex. 3. $2y^2 + 2x^2 - 4y - 4x + 1 = 0$; $a = 1$, $b = 1$, $r = \sqrt{\frac{3}{2}}$.

4. $y^2 + x^2 - 6y + 4x - 3 = 0$; $a = -2$, $b = 3$, $r = 4$.

5. $6y^2 + 6x^2 - 21y - 8x + 14 = 0$; $a = -\frac{2}{3}$, $b = \frac{7}{4}$, $r = \frac{13}{12}$.

6. $y^2 + x^2 + 4y - 3x = 0$; $a = \frac{3}{2}$, $b = -2$, $r = \frac{5}{2}$.

7. $y^2 + x^2 - 4y + 2x = 0$; $a = -1$, $b = 2$, $r = \sqrt{5}$.

In these last two examples there is no occasion to calculate the length of the radius, for the circumference of the circle passes through the origin of co-ordinates, as do the loci of all equations which want the last or constant term.

8. $y^2 + x^2 - 4y = 0$; $a = 0$, $b = 2$, $r = 2$.

9. $y^2 + x^2 + 6x = 0$; $a = -3$, $b = 0$, $r = 3$.

10. $y^2 + x^2 - 6x + 8 = 0$; $a = 3$, $b = 0$, $r = 1$.

In the last three examples the centre of the circle is on the axes.

68. We have seen that the equation to the circle referred to rectangular axes does not contain the product xy, and also that the coefficients of y^2 and x^2 are each unity; we have, moreover, seen that generally an equation of the second degree of this form has a circle for its locus, but there are some exceptions to this last rule.

For example, the equation $y^2 + x^2 - 8y - 12x + 52 = 0$ is apparently of the circular form; its locus, however, is not a circle, but a point whose co-ordinates are $x = 6$ and $y = 4$, for it may be put under the form $(y - 4)^2 + (x - 6)^2 = 0$, the only real solution of which is $x = 6$ and $y = 4$; and this will always be the case when $r^2 = 0$, hence a point may be considered as a circle whose radius is indefinitely small.

Again, the equation $y^2 + x^2 - 4y + 2x + 9 = 0$, may be put under the form $(y-2)^2 + (x+1)^2 = -4$; but there are no possible values of x and y that can satisfy this equation, therefore the locus is imaginary. (24).

69. To find the equation to the tangent to a circle.

Let the origin of co-ordinates be at the centre, and x', y' any point on the circumference.

Then the equation to the straight line through x', y' is

$$y - y' = \alpha (x - x');$$

the equation to the radius through x', y' is $y = \frac{y'}{x'} x$;

but the tangent being perpendicular to the radius, we have $\alpha = -\frac{x'}{y'}$ (47.)

$$\therefore y - y' = -\frac{x'}{y'}(x - x'),$$

$$\text{or } yy' - y'^2 = -xx' + x'^2;$$

$$\therefore yy' + xx' = y'^2 + x'^2$$

$$= r^2.$$

The equation $yy' + xx' = r^2$, thus found, may be easily remembered, from the similarity of its form to that of the equation to the circle, it being obtained at once from $y^2 + x^2 = r^2$ by changing y^2 or yy into yy', and x^2 or xx into xx'.

If we take the general equation to the circle, $(y - b)^2 + (x - a)^2 = r^2$, the equation to the radius is

$$y - b = \frac{y' - b}{x' - a}(x - a) \ldots. (41)$$

$$\therefore \alpha = -\frac{x' - a}{y' - b},$$ and the equation to the tangent is

$$y - y' = \alpha (x - x') = -\frac{x' - a}{y' - b}(x - x')$$

The equation $(y' - b)^2 + (x' - a)^2 = r^2$ enables us finally to reduce the equation to the tangent to the form

$$(y - b)(y' - b) + (x - a)(x' - a) = r^2$$

70. To find the equation to the tangent of a circle parallel to a given straight line.

Let $y = \alpha x + b$ be the given line,
and $yy' + xx' = r^2$ the required tangent, in which x', y' are unknown.

Since these lines are parallel, $-\frac{x'}{y'}=\alpha$, (43) or $-\frac{\sqrt{r^2-y'^2}}{y'}=\alpha$,

$\therefore\ y'=\pm\frac{r}{\sqrt{1+\alpha^2}}$.

Hence by substitution in the equation $y=-\frac{x'}{y'}x+\frac{r^2}{y'}$, we have

$$y=\alpha x\pm r\sqrt{1+\alpha^2};$$

consequently two tangents can be drawn parallel to the given line.

71. To find the intersection of a straight line and circle:

Let the centre of the circle be the common origin, and let the equations be $y=\alpha x+b$, and $y^2+x^2=r^2$; at the point of intersection, y and x must be the same for both. $\therefore\ r^2-x^2=(\alpha x+b)^2$,

$$\text{whence } x=\frac{-\alpha b\pm\sqrt{\{r^2(1+\alpha^2)-b^2\}}}{1+\alpha^2};$$

there being two values of x, we have two intersections; these values may be constructed, and the points of intersection found.

If $r^2(1+\alpha^2)=b^2$ the two values of x are equal, and the line will touch the circle. If $r^2(1+\alpha^2)$ is less than b^2 the line will not meet the circle.

Ex. 1. $y^2+x^2=25$, $y+x=1$; $x=4$ and -3, $y=-3$ and 4

Ex. 2. $y^2+x^2=25$, $y+x=5$; $x=5$ and 0, $y=0$ and 5

Ex. 3. $y^2+x^2=25$, $4y+3x=25$; The line touches the circle.

We may observe that the combination of an equation of the first order with any equation of two dimensions will, as above, give an equation of the second order for solution; and hence there can be only two intersections of their loci.

72. If the axes be oblique and inclined to each other at an angle ω, the equation to the circle is

$$(y-b)^2+(x-a)^2+2(y-b)(x-a)\cos.\omega=r^2, \qquad (30)$$

and $y^2+x^2+2xy\cos.\omega=r^2$, if the origin be at the centre;
hence the equation $y^2+cxy+x^2+dy+ex+f=0$, belongs to the circle in the particular case where the co-ordinate angle is one whose cosine $=\frac{c}{2}$.

Comparing it with the general equation to the circle, we find

$$2\cos.\omega=c, \qquad -2b-2a\cos.\omega=d,$$
$$-2a-2b\cos.\omega=e, \qquad a^2+b^2+2ab\cos.\omega-r^2=f;$$

whence, by elimination, we obtain $a=\frac{2e-cd}{c^2-4}$, $b=\frac{2d-ce}{c^2-4}$,

$$\text{and } r^2=\frac{ced-e^2-d^2}{c^2-4}-f;$$

hence the co-ordinates of the centre and the radius being known, the locus can be drawn.

Ex. 1. $y^2+xy+x^2+y+x-1=0$;
here $2\cos.\omega=1$; $\therefore\ \omega=60°$; hence this equation will give a circle if

the axes be inclined at an angle of 60°; the co-ordinates of the centre are $a = -\frac{1}{3}$, $b = -\frac{1}{3}$; and the radius $= \frac{2}{\sqrt{3}}$

The equation to this circle, when referred to the centre as origin, and to rectangular axes, is obviously $y^2 + x^2 = r^2 = \frac{4}{3}$.

Ex. 2. $y^2 + \sqrt{2}\,.\,xy + x^2 - 9 = 0.$

This will give a circle if the axes be inclined at an angle of 45°, the centre is at the origin of co-ordinates, and the radius $= 3$.

Of course c must never be equal to, or greater than, ± 2, for cos. ω must be less than unity.

If the circle be referred to oblique co-ordinates, the equation to the radius is $y - b = \frac{y' - b}{x' - a}(x - a) \ldots . (41)$

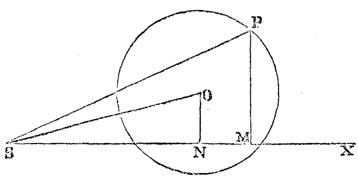
and the equation to the tangent is

$$y - y' = -\frac{(x' - a) + (y' - b)\cos.\omega}{(y' - b) + (x' - a)\cos.\omega}(x - x') \ldots . (51)$$

and reducing as in article 69 we have the equation to the tangent

$$(y - b)(y' - b) + (x - a)(x' - a) + (x - a)(y' - b)\cos.\omega + (x' - a)(y - b)\cos.\omega = r^2.$$

73. To find the polar equation to the circle.



Let the pole be at the origin S, and the angle P S M ($= \theta$) be measured from the axis of x.

Let $\left.\begin{array}{l} SM = x \\ MP = y \end{array}\right\}$ be rectangular co-ordinates of P

and $\left.\begin{array}{l} SN = a \\ NO = b \end{array}\right\}$ .... O

Let $SP = u$, $SO = c$, and angle $OSX = \alpha$; then by the formulas (61) or by the figure $y = u \sin.\theta$, $x = u \cos.\theta$, $a = c \cos.\alpha$, and $b = c \sin.\alpha$.

Substituting these values of x and y in the equation to the circle,

$$y^2 + x^2 - 2by - 2ax + a^2 + b^2 - r^2 = 0,$$

we have

$$u^2 (\sin.\theta)^2 + u^2 (\cos.\theta)^2 - 2cu \sin.\alpha \sin.\theta - 2cu \cos.\alpha \cos.\theta + c^2 (\cos.\alpha)^2 + c^2 (\sin.\alpha)^2 - r^2 = 0,$$

$$\text{or} \quad u^2 - 2cu \{ \sin.\theta \sin.\alpha + \cos\theta \cos.\alpha \} + c^2 - r^2 = 0,$$

$$\text{or} \quad u^2 - 2cu \cos.(\theta - \alpha) + c^2 - r^2 = 0.$$

74. If a and b are not expressed in terms of the polar co-ordinates c and α, the polar equation is then of the form

$$u^2 - 2\{ b \sin.\theta + a \cos.\theta \} u + a^2 + b^2 - r^2 = 0.$$

If the origin be on the circumference we have $a^2 + b^2 = r^2$, and therefore the equation to the circle becomes

$$u = 2\ (b \sin.\ \theta + a \cos.\ \theta)$$

If the axis of x passes through the centre, $b = 0$, and the equation is

$$u^2 - 2\, a\, u \cos.\ \theta + a^2 - r^2 = 0.$$

$$\text{Whence } u = a \cos.\ \theta \pm \sqrt{r^2 - a^2\,(\sin.\theta)^2}\,;$$

which equation may also be directly obtained from the triangle S P O.

CHAPTER VI.

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND ORDER.

75. The most general form in which this equation appears is

$$a y^2 + b\, x\, y + c\, x^2 + d\, y + e\, x + f = 0\,;$$

where a, b, c, &c., are constant coefficients.

Let the equation be solved with respect to y and x separately, then

$$y = -\frac{b x + d}{2\, a} \pm \frac{1}{2\, a} \sqrt{\{(b^2 - 4\, a c)\, x^2 + 2\,(b\, d - 2\, ae)\, x + d^2 - 4\, a f\}}; \quad (1)$$

$$x = -\frac{by + e}{2\, c} \pm \frac{1}{2\, c} \sqrt{\{(b^2 - 4 ac)\, y^2 + 2\,(b\, e - 2\, c\, d) \quad + - 4\, cf\}} \quad (2).$$

On account of the double sign of the root in (1), there are, in general, two values of y; hence there are two ordinates corresponding to the same abscissa: these ordinates may be constructed whenever the values of x render the radical quantity real; but if these values render it nothing, there is only one ordinate, and if they make it imaginary, no corresponding ordinate can be drawn, and therefore there is no point of the curve corresponding to such a value of x. Hence, to know the extent and limits of the curve, we must examine when the quantity under the root is real, nothing, or imaginary.

This will depend on the algebraical sign of the quantity

$$(b^2 - 4\, a\, c)\, x^2 + 2\ (b\, d - 2\, a\, e)\ x + d^2 - 4\, a f.$$

In an expression of this form, a value may be given to x, so large that the sign of the whole quantity depends only upon that of its first term, or upon that of its coefficient $b^2 - 4\, a\, c$, since x^2 is always positive for any real value of x.

For, writing the expression in the form $m\,(x^2 + \frac{n}{m}\, x + \frac{p}{m})$ let q be the absolute value of the greater of the two quantities $\frac{n}{m}$ and $\frac{p}{m}$; then substituting $r = \pm\ (q + 1)$ for x, the expression becomes

$$m\left\{q^2+2q+1\pm\frac{n}{m}q\pm\frac{n}{m}+\frac{p}{m}\right\},$$

which, whatever be the values of $\frac{n}{m}$ and $\frac{p}{m}$, is positive, and the same is true for any magnitude greater than $\pm r$; hence the sign of the expression depends upon that of m.

When b^2-4ac is negative, real values may be given to x, either positive or negative, greater than $\pm r$, which will render y imaginary. The curve will then be limited in both the positive and negative directions of x.

When b^2-4ac is positive, all values of x not less than $\pm r$ will render y real, and therefore the curve is of infinite extent in both directions of x.

Lastly, when b^2-4ac is nothing, the quantity under the root becomes

$$2(bd-2ae)x+d^2-4af.$$

If $bd-2ae$ be positive, real positive values may be given to x, which shall render y real; but if a negative value be given to x greater than $\frac{d^2-4af}{2(bd-2ae)}$, y is imaginary; therefore the curve will be of indefinite extent in the direction of x positive and limited in the opposite direction.

But if $bd-2ae$ be negative, exactly opposite results will follow, that is, the curve will be of indefinite extent in the direction of x negative and limited in the opposite direction.

Taking equation (2) we should find similar results.

The curves corresponding to the equation of the second degree, may therefore be divided into three distinct classes.

1. b^2-4ac negative, curves limited in every direction.
2. b^2-4ac positive, curves unlimited in every direction.
3. b^2-4ac nothing, curves limited in one direction, but unlimited in the opposite direction.

76. *First class b^2-4ac negative.*

Let $\quad -\frac{b}{2a}=\alpha,\quad -\frac{d}{2a}=l,\quad \frac{b^2-4ac}{4a^2}=-\mu,$

and let x_1 and x_2 be the roots of the equation

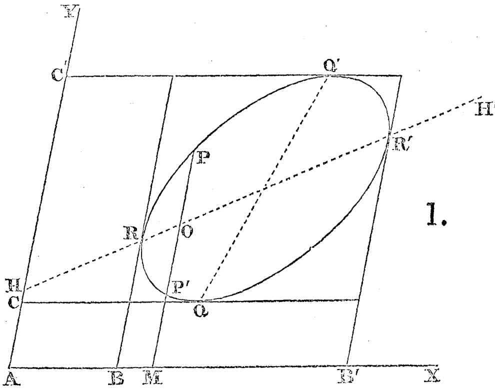
$$(b^2-4ac)x^2+2(bd-2ae)x+d^2-4af=0.$$

Then equation (1) or

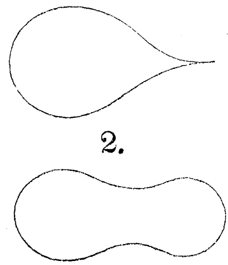
$$y=-\frac{bx+d}{2a}\pm\sqrt{\left\{\frac{b^2-4ac}{4a^2}\left(x^2+2\frac{bd-2ae}{b^2-4ac}x+\frac{d^2-4af}{b^2-4ac}\right)\right\}}$$

becomes by substitution

$$y=\alpha x+l\pm\sqrt{\{-\mu(x-x_1)(x-x_2)\}}$$



1.



2.

Let A be the origin of co-ordinates, A X, A Y the oblique axes.

Let H H′ be the line represented by the equation $y = \alpha x + l$, M O one of its ordinates corresponding to any value of x between x_1 and x_2.

Along the line M O take O P and O P′ each equal to $\sqrt{\{-\mu (x-x_1)(x-x_2)\}}$, then P and P′ are two points in the curve, for

$$MP = MO + OP = \alpha x + l + \sqrt{\{-\mu (x-x_1)(x-x_2)\}}$$
$$MP' = MO - OP' = \alpha x + l - \sqrt{\{-\mu (x-x_1)(x-x_2)\}}$$

If we repeat this construction for all the real values of x which render the root real we obtain the different points of the curve.

The line H H′ is called a diameter of the curve, for it bisects all the chords P P′ which are parallel to the axis of y.

The reality of y depends on the reality of the radical quantity, which last depends on the form of the factors $(x - x_1)$ and $(x - x_2)$, that is, on the roots x_1 and x_2. Now these may enter the equation in three forms—real and unequal—real and equal—or both imaginary.

Case 1. Let x_1 and x_2 be real and unequal, take $AB = x_1$, $AB' = x_2$, then if $x = x_1$ or x_2 the quantity $-\mu (x-x_1)(x-x_2)$ vanishes, and the ordinate to the curve coincides with the ordinate to the diameter, therefore drawing through B and B′ two lines B R and B′ R′ parallel to A Y the curve cuts the diameter in R and R′.

For all values of x between x_1 and x_2 there are two real values of y, for $x - x_1$ is positive and $x - x_2$ is negative, and therefore $-\mu (x-x_1)(x-x_2)$ is positive.

For all values of $x > x_2$ or $< x_1$, $-\mu (x-x_1)(x-x_2)$ is negative, the root being impossible cannot be constructed, hence there is no real value of y corresponding to such values of x, and therefore the curve is entirely confined between the two lines B R and B′ R′.

Similarly by taking equation (2) in (75), we shall find that a straight line Q Q′ is a diameter; that the curve cuts it in two points Q, Q′: that drawing lines parallel to A X through Q and Q′ the curve is confined between those parallels.

We have thus determined that the curve exists and only exists between certain parallel lines: its form is not yet ascertained. We might by giving a variety of values to x between x_1 and x_2 determine a variety of points P, Q, &c., and thus arrive at a tolerably exact idea of its course, but independently of this method, its form cannot much differ from that in the figure, for supposing it to be such as in fig. (2) a straight line could be drawn cutting it in more points than two which is impossible (71).

This oval curve is called the Ellipse.

If we require the points where the curve cuts A X, put $y = 0$, then the roots of the equation $cx^2 + ex + f = 0$ are the abscissas of the points of intersection, and the curve will cut the axis in two points, touch it in one, or never meet it, according as these roots are real and unequal, real and equal, or imaginary. Similarly putting $x = 0$ we find the points, if any, where the curve meets the axis of y.

Case 2. Let the roots x_1 and x_2 be real and equal,

$$\therefore y = \alpha x + l \pm (x - x_1)\sqrt{-\mu}$$

which is imaginary except when $x = x_1$, therefore the locus is the point whose co-ordinates are x_1 and $\alpha x_1 + l$, or $\dfrac{2ae - bd}{b^2 - 4ac}$ and $\dfrac{2cd - be}{b^2 - 4ac}$.

Case 3. Let x_1 and x_2 be impossible, then no real value can be given to x to make $(x - x_1)(x - x_2)$ negative, for the roots are of the form $\pm p + q\sqrt{-1}$ and $\pm p - q\sqrt{-1}$ $\therefore (x - x_1)(x - x_2) = x^2 \pm 2px + p^2 + q^2 = (x \pm p)^2 + q^2$ which quantity is always positive for a real value of x. Hence in this case the radical quantity being impossible there is no locus.

We have not examined the equation of x in terms of y at length, for the results of the latter are dependent on those of the former. By comparing equations (1) and (2) in (75), we see that c stands in one equation where a stands in the other, and therefore that the radical quantities are contemporaneously possible, equal, or impossible, provided that a and c have the same sign, which is the case when $b^2 - 4ac$ is negative.

In discussing a particular example reduce it to the forms

$$y = \alpha x + l \pm \sqrt{\{-\mu(x - x_1)(x - x_2)\}}$$
$$x = \alpha' y + l' \pm \sqrt{\{-\mu'(y - y_1)(y - y_2)\}}$$

there are then three cases.

Case 1. x_1 and x_2 real and unequal. The locus is called an ellipse, its boundaries are determined from x_1, x_2, y_1 and y_2, its diameters are drawn from $y = \alpha x + l$ and $x = \alpha' y + l'$, and its intersections with the axes found by putting x and y successively $= 0$ in the original equation.

Case 2. x_1 and x_2 real and equal: the locus is a point.

Case 3. x_1 and x_2 impossible: the locus is imaginary.

Ex. 1. $y^2 - 2xy + 2x^2 - 2y - 4x + 9 = 0$. Case 1. Fig. 1.

A B $= 2$, A B$' = 4$, A C $= 4 - \sqrt{2}$, A C$' = 4 + \sqrt{2}$, A H $= 1$

Ex. 2. $y^2 + xy + x^2 + y + x - 5 = 0$. Case 1.

The curve cuts the diameters when A B $= 2\frac{1}{3}$, A B$' = -3$, A C $= 2\frac{1}{3}$, A C$' = -3$, and it cuts the axes at distances $1\cdot 7$ and $-2\cdot 7$ nearly;

These six points are sufficient to determine its course.

Ex. 3. $y^2 + 2xy + 3x^2 - 4x = 0$. Case 1.

Ex. 4. $y^2 - 2xy + 2x^2 - 2y + 2x = 0$. Case 1.

Ex. 5. $y^2 + 2x^2 - 10x + 12 = 0$. Case 1.

Ex. 6. $y^2 - 2xy + 3x^2 - 2y - 10x + 19 = 0$. Case 2. The intersection of the diameters in Ex. 1.

Ex. 7. $y^2 - 4xy + 5x^2 + 2y - 4x + 2 = 0$. Case 3.

It is to be observed that no accurate form of the curve is here found, that will be hereafter ascertained, all that we can at present do, is to obtain an idea of the situation of the locus.

77. *Second class, $b^2 - 4ac$ positive.*

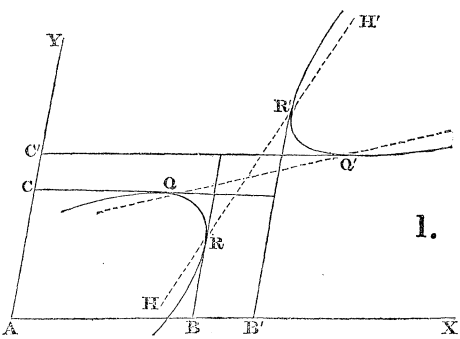
Arranging and substituting as in (76) the equation becomes

$$y = \alpha x + l \pm \sqrt{\{\mu(x - x_1)(x - x_2)\}}$$

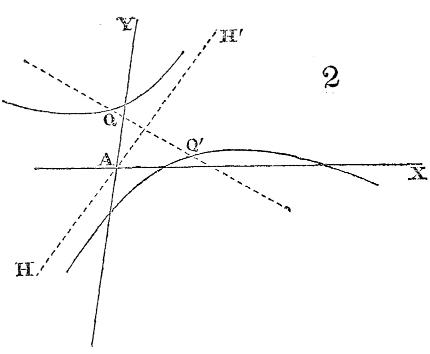
Let H H′ be the diameter whose equation is $y = \alpha x + l$.

Then as before there are three forms of the roots x_1 and x_2.

Case 1. Let x_1 and x_2 be real and unequal, let A B $= x_1$ and A B′ $= x_2$, draw B R, B′ R′ parallel to A Y, the curve meets the diameter in R and



1.



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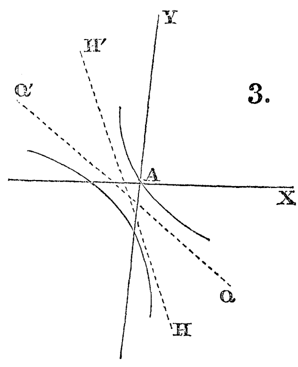


3.

R′. The radical quantity is imaginary for all values of x between x_1 and x_2 but real beyond these limits, hence no part of the curve is between the parallels B R, B′ R′, but it extends to infinity beyond them.

Taking the equation for x in terms of y, we may draw the diameter QQ' and determine the lines CQ, $C'Q'$ parallel to AX between which no part of the curve is found, and beyond which x is always possible.

From this examination it results that the form of the locus must be something like that in fig. 1, consisting of two opposite arcs with branches proceeding to infinity.

This curve is called the Hyperbola.

We must observe that the second diameter does not necessarily meet the curve, for the contemporaneous possibility or impossibility of the radical quantities depends on the signs of a and c, and these may be different in the hyperbola; so that one radical quantity may have possible and the other impossible roots.

Case 2. x_1 and x_2 real and equal.

$$y = \alpha x + l \pm (x - x_1)\sqrt{\mu}$$

this is the equation to two straight lines.

Case 3. x_1 and x_2 imaginary; whatever real values be given to x the radical quantity is real, and therefore there must be four infinite branches. Also since $\mu(x-x_1)(x-x_2)$ can never vanish, (76, Case 3.) the diameter HH' never meets the curve, but we may draw the other diameter as in the first case.

If neither diameter meets the curve, yet they will at least determine where the curve does not pass, we must then find the intersections with the axes. If these will not give a number of points sufficient to determine the locality of the curve we must have recourse to other methods to be explained hereafter.

In discussing a particular example reduce it to the forms

$$y = \alpha x + l \pm \sqrt{\{\mu(x-x_1)(x-x_2)\}}$$
$$x = \alpha' y + l' \pm \sqrt{\{\mu(y-y_1)(y-y_2)\}}$$

there are then three cases.

Case 1. x_1 and x_2 real and unequal. The locus is an hyperbola, its boundaries determined from x_1, x_2, y_1 and y_2, the diameters are drawn from $y = \alpha x + l$ and $x = \alpha' y + l'$, and its intersections with the axes found by putting x and y separately equal nothing.

Case 2. x_1 and x_2, real and equal. The locus consists of two straight lines which intersect each other.

Case 3. x_1 and x_2 impossible. The locus is an hyperbola, draw the diameters, and find the intersection, if any, of the curve with either diameter and with the axes.

Ex. 1. $y^2 - 3xy + x^2 + 1 = 0$. Case 1. Fig. 1. The origin being at the intersection of the dotted lines.

The equations to the diameters are $y = \dfrac{3x}{2}$ and $y = \dfrac{2x}{3}$, $AB' = \dfrac{2}{\sqrt{5}}$.

$AB = -\dfrac{2}{\sqrt{5}}$, $AC' = \dfrac{2}{\sqrt{5}}$, $AC = -\dfrac{2}{\sqrt{5}}$.

Ex. 2. $y^2 - 2xy - x^2 + 2 = 0$. Case 1. The two diameters pass through the origin and make an angle of 45° with the axes, the second QQ' never meets the curve, $AB' = 1$ and $AB = -1$; the curve intersects the axis of x at distances $\pm\sqrt{2}$.

Ex. 3. $4y^2 - 4xy - 3x^2 + 8y + 4x + 16 = 0.$ Case 1.

Ex. 4. $y^2 - 4xy - 5x^2 - 2y + 40x - 26 = 0.$ Case 1.

Ex. 5. $y^2 - 6xy + 8x^2 + 2x - 1 = 0.$ Case 2. The equations to the two straight lines are $y - 4x + 1 = 0$, and $y - 2x - 1 = 0$.

Ex. 6. $y^2 + 3xy + 2x^2 + 2y + 3x + 1 = 0.$ Case 2.

Ex. 7. $y^2 - 4xy - x^2 + 10x - 10 = 0.$ Case 3. Fig. 2.

Ex. 8. $y^2 + 3xy + x^2 + y + x = 0.$ Case 3. Fig. 3.

Here neither diameter meets the curve; but the curve passes through the origin and cuts the axis of x at a distance -1, and that of y also at a distance -1.

Ex. 9. $y^2 - x^2 - 2y + 5x - 3 = 0.$ Case 3.

The diameters are parallel to the axes, but the curve never meets that diameter whose equation is $x = \frac{5}{2}$.

Ex. 10. $y^2 - x^2 - y = 0.$ Case 3.

78. *Third Class.* $b^2 - 4ac = 0.$

In this case the general equation becomes

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{\{2(bd - 2ae)x + d^2 - 4af\}} \quad (1)$$

$$\text{Let } -\frac{b}{2a} = \alpha, \; -\frac{d}{2a} = l, \; \frac{bd - 2ae}{2a^2} = \nu$$

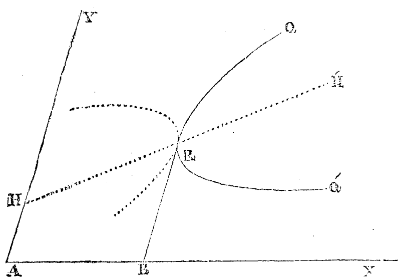
And let x_1 be the root of the equation

$$2(bd - 2ae)x + d^2 - 4af = 0.$$

Substituting equation (1) becomes

$$y = \alpha x + l \pm \sqrt{\{\nu(x - x_1)\}}$$

The locus of $y = \alpha x + l$ is a diameter H H′ as before.



Let ν be positive, then if $x = x_1$ the root vanishes; or if $AB = x_1$ and BR be drawn parallel to AY, the curve cuts the diameter in R. As x increases from x_1 to ∞, y increases to ∞, hence there are two arcs RQ, RQ′ extending to infinity. If x be less than x_1, y is impossible, or no part of the curve extends to the negative side of B.

Let ν be negative, then the results are contrary, and the curve only extends on the negative side of B; this case is represented by the dotted curve.

This curve is called the Parabola.

$$\text{If } bd - 2ae = 0, \quad y = \alpha x + l \pm \sqrt{\left\{\frac{d^2 - 4af}{4a^2}\right\}},$$

and the locus consists of two parallel straight lines; and, according as $d^2 - 4af$ is positive, nothing, or negative, these lines are both real, or unite into one, or are both imaginary.

In discussing a particular example, reduce it to the form

$$y = \alpha x + l \pm \sqrt{\{\nu(x - x_1)\}}$$

Case 1. ν positive or negative. The locus is called a parabola; draw the diameter and find the points where the curve cuts the axes and diameter.

Case 2. $\nu = 0$. The locus consists of two parallel straight lines, or one straight line, or is imaginary.

Ex. 1. $y^2 - 2xy + x^2 - 2y - 1 = 0$. Case 1.

Ex. 2. $y^2 - 2xy + x^2 - 2y - 2x = 0$. Case 1.

Ex. 3. $y^2 + 2xy + x^2 + 2y + x + 3 = 0$. Case 1.

Ex. 4. $y^2 - 2xy + x^2 - 1 = 0$. Case 2. Two parallel straight lines.

Ex. 5. $y^2 - 2xy + x^2 + 2y - 2x + 1 = 0$. Case 2. One straight line.

Ex. 6. $y^2 + 2xy + x^2 + 1 = 0$. Case 2. Imaginary locus.

79. Before we leave this subject, it may be useful to recapitulate the results obtained from the investigation of the general equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0.$$

If $b^2 - 4ac$ be negative, the locus is an ellipse admitting of the following varieties :—

1. $c = a$, and $\frac{b}{2a}$ = cosine of the angle between the axes, locus a circle. (72.)

2. $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$. Locus a point.

3. $(bd - 2ae)^2$ less than $(b^2 - 4ac)(d^2 - 4af)$. Locus imaginary.

If $b^2 - 4ac$ be positive, the locus is an hyperbola admitting of one variety.

1. $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$. Locus two straight lines.

Lastly, if $b^2 - 4ac = 0$, the locus is a parabola admitting of the following varieties,—

1. $bd - 2ae = 0$. Locus two parallel straight lines.

2. $bd - 2ae = 0$, and $d^2 - 4af = 0$. Locus one straight line.

3. $bd - 2ae = 0$, and d^2 less than $4af$. Locus imaginary.

Apparently another relation between the coefficients would be obtained in each variety, by taking the equation of x in terms of y; but on examination, it will be found that in each case the last relation is involved in the former.

CHAPTER VII

REDUCTION OF THE GENERAL EQUATION OF THE SECOND ORDER

80. In order to investigate the properties of lines of the second order more conveniently, we proceed to reduce the general equation to a more simple form, which will be effected by the transformation of co-ordinates. Taking the formulas in (54.)

$$y = y' + n, \text{ and } x = x' + m$$

and substituting in the general equation, we have

$$a(y' + n)^2 + b(x' + m)(y' + n) + c(x' + m)^2 + d(y' + n) + e(x' + m) + f = 0;$$

or arranging

$$a y'^2 + b x' y' + c x'^2 + (2 a n + b m + d) y' + (2 c m + b n + e) x' + a n^2 + b m n + c m^2 + d n + e m f = 0.$$

As we have introduced two indeterminate quantities, m and n, we are at liberty to make two hypotheses respecting the new co-efficients in the last equation; let, therefore, the co-efficients of x' and y' each $= 0$.

$$\therefore\ 2 a n + b m + d = 0, \text{ and } 2 c m + b n + e = 0;$$

whence we find by elimination, $m = \dfrac{2 a e - b d}{b^2 - 4 a c}$, and $n = \dfrac{2 c d - b e}{b^2 - 4 a c}$.

The value of the constant term, or f', may be obtained from the equation,

$$f' = a n^2 + b m n + c m^2 + d n + e m + f = 0;$$

or, since $2 a n + b m + d = 0$, and $2 c m + b n + e = 0$,

Multiply the first of these two equations by n, and the second by m; and, adding the results, we have

$$2 a n^2 + 2 b m n + 2 c m^2 + d n + e m = 0;$$

$$\therefore\ a n^2 + b m n + c m^2 = -\frac{d n + e m}{2};$$

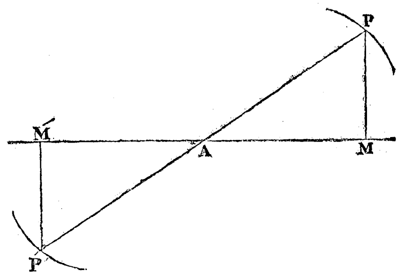
hence $f' = -\dfrac{d n + e m}{2} + d n + e m + f = \dfrac{d n + e m}{2} + f$, which, by the substitution of the values of m and n, becomes

$$f' = \frac{a e^2 + c d^2 - b d e}{b^2 - 4 a c} + f.$$

The reduced equation is now of the form

$$a y'^2 + b x' y' + c x'^2 + f' = 0.$$

81. The point A, which is the new origin of co-ordinates, is called the centre of the curve, because every chord passing through it is bisected in that point. For the last equation remains the same when $-x$ and $-y$ are substituted for $+x$ and $+y$: hence, for every point P in the curve, whose co-ordinates are x and y, there is another point P′, whose co-ordinates are $-x$ and $-y$, or A M′ and M′ P′; hence, by comparing the right angled triangles, A M P, A M′ P′, we see that the vertical angles at A are equal, and therefore, the line P A P′ is a straight line bisected in A.



Whenever, therefore, the equation remains the same on the substitution of $-x$ and $-y$ for $+x$ and $+y$ respectively, it belongs to a locus referred to its centre.

If the equation be of an even order, this condition will be satisfied if the sum of the exponents of the variables in every term be even; thus, in the general equation of the second order, $a y^2 + b x y + c x^2 + d y + e x + f = 0$, the sum of the exponents in each of the three first terms is 2, and in the two next terms is 1; changing the signs of x and y, the equation does not remain the same; or for one point P, there is not another point P′ opposite and similarly situated with respect to the origin; hence that origin is not the centre of the curve. But the equation $a y^2 + b x y + c x^2 + f' = 0$, refers just as much to the point P′ as to P, and thus the origin is here the centre of the curve.

If the equation be of an odd order, the sum of the exponents in each term must be odd, and the constant term also must vanish; for if both these conditions are not fulfilled, the equation would be totally altered by putting $-x$ and $-y$ for $+x$ and $+y$ respectively. Therefore a locus may be referred to a centre if it be expressed by an equation which, by transformation, can be brought under either of the two following conditions:—

(1) Where the sum of the indices of every term is even, whether there be a constant or not, as $a y^2 + b x y + c x^2 + f = 0$.

(2) Where the sum of the indices in every term is odd, and there is no constant term, as $a y^3 + b x y^2 + c x^2 y + d x^3 + e y + f x = 0$.

Now it has been stated (59) that no equation can be so transformed that the new equation shall be of a lower or higher degree than the original one. Hence, if the original equation be of an even degree, the transformed equation will be so too, and the locus can be transferred to a centre only where the equation can be brought under the first condition; but if the original equation be of an odd degree, the transformed equation also will be of an odd degree, and the locus can only be transferred to a centre when the equation can be brought under the second condition. Hence we have a test, whether a locus with a given equation can be referred to a centre or not. If the axes can be transferred so that (1) The original equation being of an even degree, the co-efficients of all the terms, the sum of whose exponents is odd, vanish. (2) The original equation being of an odd degree, the co-efficients of all the terms, the sum of whose exponents is even, and also the constant term, vanish, then the locus may bé referred to a centre, and not otherwise.

Now in the transformation which we effect by making $y = y' + n$, and $x = x' + m$, we can destroy only two terms; we cannot therefore bring, by any substitution, an equation of higher dimensions than the second under the necessary conditions, unless from some accidental relation of the original co-efficients of that equation. But in the case of equations of the second degree, we can always bring them under the first condition, unless the values of the indeterminate quantities, m and n, are found to be impossible or infinite.

In curves of the second order, we see that the values of m and n are real and finite, unless $b^2 - 4ac = 0$; consequently the ellipse and hyperbola have a centre and the parabola has not; hence arises the division of these curves into two classes, central and non-central.

In the case where $b^2 - 4ac = 0$, and at the same time $2ae - bd$ or $2cd - be$ vanish, the equation becomes that to a straight line, as appears on inspecting the equations (1) and (2) in (75).

If by the transformation the term f' should vanish, the equation becomes of the form $ay^2 + bxy + cx^2 = 0$; whence

$y = \{-b \pm \sqrt{b^2 - 4ac}\}\frac{x}{2a}$; and the curve is reduced to two straight lines which pass through the centre; or if $b^2 - 4ac$ is negative, the locus is the centre itself (25).

82. The central class may have their general equation still further reduced by causing the term containing the product of the variables to vanish, which is done by another transformation of co-ordinates. Taking the formulas in (58) let

$$\text{Let } y' = x'' \sin.\theta + y'' \cos.\theta,$$

$$x' = x'' \cos.\theta - y'' \sin.\theta,$$

substituting in the equation $ay'^2 + bx'y' + cx'^2 + f' = 0$, we have

$$a(x''\sin.\theta + y''\cos\theta)^2 + b(x''\sin.\theta + y''\cos.\theta)(x''\cos.\theta - y''\sin.\theta) + c(x''\cos.\theta - y''\sin.\theta)^2 + f' = 0$$

$$\therefore y''^2\{a(\cos.\theta)^2 - b\sin.\theta\cos.\theta + c(\sin.\theta)^2\} + x''^2\{a(\sin.\theta)^2 + b\sin.\theta\cos.\theta + c(\cos.\theta)^2\}$$

$$+ x''y''\{2a\sin.\theta\cos.\theta + b(\cos.\theta)^2 - b(\sin.\theta)^2 - 2c\sin.\theta\cos.\theta\} + f' = 0.$$

Let the co-efficient of $x''y'' = 0$,

$$\therefore 2a\sin.\theta\cos.\theta + b(\cos.\theta)^2 - b(\sin.\theta)^2 - 2c\sin.\theta\cos.\theta = 0,$$

$$\text{or } (a - c)2\sin.\theta\cos.\theta + b\{(\cos.\theta)^2 - (\sin.\theta)^2\} = 0,$$

$$\therefore (a - c)\sin.2\theta + b\cos.2\theta = 0;$$

and dividing by cos. 2θ, we have

$$\tan.2\theta = \frac{-b}{a - c}.$$

Here θ is the angle which the new axis of x makes with the original one (58); hence, if the original rectangular axes be transferred through an angle θ, such that $\tan.2\theta = \frac{-b}{a - c}$, the transformed equation will have no term containing the product $x'' y''$, that is, the equation, when referred to its new rectangular axes, will be reduced to the simple form

$$a' y''^2 + c' x''^2 + f' = 0.$$

83. As a tangent is capable of expressing all values from 0 to ∞, positive or negative, it follows that the angle θ has always a real value, whatever be the values of a, b, and c, and thus it is always possible to destroy the term containing $x y$.

The values of sin. 2θ and cos. 2θ are thus obtained from that of tan. 2θ;

$$\cos. 2\theta = \frac{1}{\pm\sqrt{1 + (\tan. 2\theta)^2}} = \frac{1}{\pm\sqrt{1 + \left(\frac{b}{a-c}\right)^2}} = \frac{a-c}{\pm\sqrt{(a-c)^2 + b^2}}$$

$$\text{And } \sin. 2\theta = \cos. 2\theta . \tan. 2\theta = \frac{-b}{\pm\sqrt{(a-c)^2 + b^2}}.$$

Since θ must be less than 90°, and therefore sin. 2θ positive, the sign of the radical quantity must be taken positive or negative, according as b is itself negative or positive.

84. To express the co-efficients a' and c' of the transformed equation in terms of the co-efficients in the original equation.

Taking the expressions for the co-efficients in article (82) we have

$$a' = a (\cos. \theta)^2 - b \sin. \theta \cos. \theta + c (\sin. \theta)^2$$

$$c' = a (\sin. \theta)^2 + b \sin. \theta \cos. \theta + c (\cos. \theta)^2,$$

$$\therefore a' - c' = a\{(\cos.\theta)^2 - (\sin.\theta)^2\} - 2b \sin.\theta\cos.\theta + c\{(\sin.\theta)^2 - (\cos.\theta)^2\}$$

$$= a \cos. 2\theta - b \sin. 2\theta - c \cos. 2\theta$$

$$= (a - c) \cos. 2\theta - b \sin. 2\theta;$$

$$\text{but } \cos. 2\theta = \frac{a-c}{\pm\sqrt{(a-c)^2 + b^2}}, \text{ and } \sin. 2\theta = \frac{-b}{\pm\sqrt{(a-c)^2 + b^2}};$$

hence substituting, we have

$$a' - c' = \frac{(a-c)^2}{\pm\sqrt{(a-c)^2 + b^2}} + \frac{b^2}{\pm\sqrt{(a-c)^2 + b^2}}$$

$$= \frac{(a-c)^2 + b^2}{\pm\sqrt{(a-c)^2 + b^2}},$$

$$\text{or } a' - c' = \pm\sqrt{(a-c)^2 + b^2}$$

$$\text{Also } a' + c' = a + c,$$

$$\therefore a' = \tfrac{1}{2}\{a + c \pm \sqrt{(a-c)^2 + b^2}.\}$$

$$c' = \tfrac{1}{2}\{a + c \mp \sqrt{(a-c)^2 + b^2}.\}$$

Hence the final equation is

$$\tfrac{1}{2}\{a + c \pm \sqrt{(a-c)^2 + b^2}\} y''^2 + \tfrac{1}{2}\{a + c \mp \sqrt{(a-c)^2 + b^2}\} x''^2 + \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = 0.$$

The upper or lower sign to be taken all through this article, according as the sign of b in the original equation is negative or positive

85. Hitherto in this chapter we have been making a number of alterations in the *form* of the original equation: the following figures will show the corresponding alterations which have been made in the *position* of the curve. The ellipse is used in the figure, in preference to the hyperbola solely on account of its easier description.

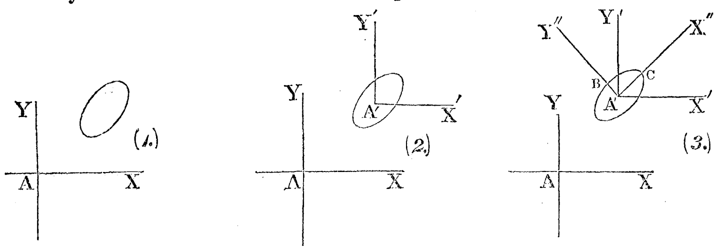


Fig. 1. We have here the original position of the curve referred to rectangular axes A X and A Y, and the corresponding equation is

$$a\,y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0.$$

Fig. 2. The origin is here transferred from A to the centre of the curve A′, the co-ordinates of which are $m = \dfrac{2\,a\,e - b\,d}{b^2 - 4\,a\,c}$, and $n = \dfrac{2\,c\,d - b\,e}{b^2 - 4\,a\,c}$.

The new axes A′ X′ and A′ Y′ are parallel to the former axes, and the equation to the curve is

$$a\,y'^2 + b\,x'\,y' + c\,x'^2 + f' = 0.$$

Fig. 3. The origin remains at A′, but the curve is referred to the new rectangular axes A′ X″ and A′ Y″, instead of the former ones A′ X′ and A′ Y′. The axis A′ X′ has been transferred through an angle X′ A′ X″ into the position A′ X″, the angle X′ A′ X″, or θ, being determined by the equation $\tan.\ 2\,\theta = \dfrac{-\,b}{a - c}$, and the equation to the curve is now

$$a'\,y''^2 + c'\,x''^2 + f' = 0.$$

86. In the ellipse and hyperbola the word "axis" is used in a limited sense to signify that portion of the central rectangular axis which is bounded by the curve.

To find the lengths of the axes, put x'' and y'' successively $= 0$, we then obtain the points where the curve cuts the axes, or, in other words, we have the lengths of the semi-axes.

In the equation

$$a'\,y''^2 + c'\,x''^2 + f' = 0,$$

$$\text{Let } y'' = 0,\ \therefore\ c'\,x''^2 + f' = 0, \text{ and } x'' = \sqrt{\frac{-f'}{c'}}.$$

$$\text{Let } x'' = 0,\ \therefore\ a'\,y''^2 + f' = 0, \text{ and } y'' = \sqrt{\frac{-f'}{a'}}.$$

In fig. 3, the semi-axes are A′ C and A′ B, so that A′ C $= \sqrt{\dfrac{-f'}{c'}}$, and A′ B $= \sqrt{\dfrac{-f'}{a'}}$; putting for a', c', and f', their values in terms of the original co-efficients (80, 84) we have the squares upon the semi-axes both comprehended in the formula,

$$-\frac{2}{a+c\pm\sqrt{(a-c)^2+b^2}}\left(\frac{a\,e^2+c\,d^2-b\,d\,e}{b^2-4\,a\,c}+f\right).$$

Let the equation $a'\,y''^2+c'\,x''^2+f'=0$, be written in the form $\left(-\frac{a'}{f'}\right)y''^2+\left(-\frac{c'}{f'}\right)x''^2=1$.

Then, if the curve is an ellipse, we must have $b^2-4\,a\,c$ negative, or, since $b=0$ in the present case, we must have $-4\left(-\frac{a'}{f'}\right)\left(-\frac{c'}{f'}\right)$ negative, and therefore $-\frac{a'}{f'}$ and $-\frac{c'}{f'}$ both positive; thus both axes meet the curve, (the case where both $-\frac{a'}{f'}$ and $-\frac{c'}{f'}$ are negative, would give an imaginary locus). If the curve is an hyperbola, $b^2-4\,a\,c$ is positive, and therefore $-4\left(-\frac{a'}{f'}\right)\left(-\frac{c'}{f'}\right)$ must be positive, or one of the values, $\left(-\frac{a'}{f'}\right)$ positive, and the other $\left(-\frac{c'}{f'}\right)$ negative; hence, one of the axes in the hyperbola has an impossible value, and therefore does not meet the curve.

The relative lengths of the axes will depend entirely on the magnitude o. $\frac{a'}{f'}$ and $\frac{c'}{f'}$.

87. Hitherto the original co-ordinates have been rectangular, but if they were oblique, considerable alterations must be made in some of the formulas.

Articles 80 and 81 are applicable in all cases, but 82, 83, and 84, must be entirely changed; the method pursued will be nearly the same as in the more simple case; but on account of the great length of some of the operations, we cannot do more than indicate a few steps, and give the results *.

To destroy the co-efficient of the term containing the product of the variables, take the formulas in (55)

$$y'=\frac{x''\,\sin.\,\theta+y''\,\sin.\,\theta'}{\sin.\,\omega}$$

$$x'=\frac{x''\,\sin.\,(\omega-\theta)+y''\,\sin.\,(\omega-\theta')}{\sin.\,\omega.}$$

Substituting in the equation $a\,y'^2+b\,x'\,y'+c\,x'^2+f'=0$, we have

$$y''^2\left\{a\,(\sin.\,\theta')^2+b\,\sin.\,\theta'\,\sin.\,\overline{\omega-\theta'}+c\,(\sin.\,\overline{\omega-\theta'})^2\right\}\frac{1}{(\sin.\,\omega)^2}$$

$$+\,x''^2\left\{a\,(\sin.\,\theta)^2+b\,\sin.\,\theta\,\sin.\,\overline{\omega-\theta}+c\,(\sin.\,\overline{\omega-\theta})^2\right\}\frac{1}{(\sin.\,\omega)^2}$$

* This article, and the following ones marked with an asterisk, had better be omitted at the first reading of the subject.

$$+ x'' y'' \Big\{ 2\, a \sin. \theta. \sin. \theta' + b \sin. \theta' \sin. \overline{\omega - \theta} + b \sin. \theta \sin. \overline{\omega - \theta'} +$$

$$2\, c \sin. \overline{\omega - \theta}. \sin. \overline{\omega - \theta'} \Big\} \frac{1}{(\sin. \omega)^2} + f' = 0.$$

Let the co-efficient of $x'' y'' = 0$; expanding $\sin. \overline{\omega - \theta}$, $\sin. \overline{\omega - \theta'}$ and dividing by $\cos. \theta. \cos \theta'$ we shall obtain the equation

$\{ a - b \cos. \omega + c (\cos. \omega)^2 \}\, 2 \tan. \theta. \tan. \theta' + \{ b - 2\, c \cos. \omega \} \sin. \omega.$
$\{ \tan. \theta + \tan. \theta' \} + 2\, c\, (\sin. \omega)^2 = 0.$

Whence for any given value of θ a value of θ' and consequently of $(\theta' - \theta)$ may be found, so that there are an infinite number of pairs of axes to which if the curve be referred, its equation may assume the form

$$a' y^2 + c' x^2 + f' = 0.$$

Let us now examine these pairs of axes, to find what systems can be rectangular:

For this purpose we must have $\theta' - \theta = \frac{\pi}{2}$ and therefore $\tan. \theta' = - \frac{1}{\tan. \theta}$.

By substituting this value of $\tan. \theta'$ in the equation containing $\tan. \theta$ and $\tan. \theta'$, we have $- 2\, \{ a - b \cos. \omega + c (\cos. \omega)^2 \} + \{ 2\, c \cos. \omega - b \}$

$$\sin. \omega. \frac{2}{\tan. 2\,\theta} + 2\, c\, (\sin. \omega)^2 = 0.$$

$$\therefore \tan. 2\,\theta = \frac{c \sin. 2\,\omega - b \sin. \omega}{a - b \cos. \omega + c \cos. 2\,\omega}.$$

There are two angles which have got the same $\tan. 2\,\theta$ separated from each other by 180°, therefore there are two angles θ, which would satisfy the above equation; however, as they are separated by an angle of 90°, the second value only applies to the new axis of y''.

Hence there is only one *system* of rectangular axes, and their position is fully determined by the last formula.

*88. To find the co-efficients a' and c' in terms of the co-efficients of the original equation, the new axes being supposed rectangular. Taking the co-efficients in the general transformed equation given above, putting $\theta' = \frac{\pi}{2} + \theta$, and multiplying by $(\sin. \omega)^2$, we have,

$$a' (\sin. \omega)^2 = a (\cos. \theta)^2 - b \cos. \theta \cos. \overline{\omega - \theta} + c\, (\cos. \overline{\omega - \theta})^2$$
$$c' (\sin. \omega)^2 = a (\sin. \theta)^2 + b \sin. \theta \sin. \overline{\omega - \theta} + c\, (\sin. \overline{\omega - \theta})^2$$

$$\therefore (a' - c') (\sin. \omega)^2 = a \{ (\cos. \theta)^2 - (\sin. \theta)^2 \} - b \{ \cos. \theta \cos. \overline{\omega - \theta} +$$
$$\sin. \theta \sin. \overline{\omega - \theta'} \} + c \{ (\cos. \overline{\omega - \theta})^2 - (\sin. \overline{\omega - \theta})^2 \}$$
$$= a \cos. 2\,\theta - b \cos. (\omega - 2\,\theta) + c \cos. (2\,\omega - 2\,\theta)$$
$$= \{ a - b \cos. \omega + c \cos. 2\,\omega \} \cos. 2\,\theta + (c \sin. 2\,\omega - b \sin. \omega) \sin. 2\,\theta.$$

Also, following the method in (83) we find from $\tan. 2\,\theta$ that

$$\cos. 2\,\theta = \frac{a - b \cos. \omega + c \cos. 2\,\omega}{\pm \text{M}}, \text{ and } \sin. 2\,\theta = \frac{c \sin. 2\,\omega - b \sin. \omega}{\pm \text{M}}$$

Where $M = \pm \sqrt{\{a^2 + b^2 + c^2 - 2b(a+c)\cos.\omega + 2ac\cos.2\omega\}}$
or $= \pm \sqrt{\{(a + c - b\cos.\omega)^2 + (b^2 - 4ac)(\sin.\omega)^2\}}$.
Hence $(a' - c')(\sin.\omega)^2 = \pm M$
and $(a' + c')(\sin.\omega)^2 = a - b\cos.\omega + c$

$$\therefore a' = \{a - b\cos.\omega + c \pm M\}\frac{1}{2(\sin.\omega)^2}$$

$$\text{and } c' = \{a - b\cos.\omega + c \mp M\}\frac{1}{2(\sin.\omega)^2}$$

Hence the final equation is

$$\{a - b\cos.\omega + c \pm M\}\frac{y''^2}{2(\sin.\omega)^2} + \{a - b\cos.\omega + c \mp M\}\frac{x''^2}{2(\sin.\omega)^2}$$
$$+ \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = 0$$

And the $\pm$ sign is to be used according as $c\sin.2\omega - b\sin.\omega$ is positive or negative, since 2θ is assumed to be positive.

These analytical transformations may be geometrically represented as in (85). In figures (1), (2), and (3) we must suppose the axes AX, AY, and also the axes A′X′ and A′Y′, to contain the angle ω.

The article (86) will equally apply when the original axes are oblique; the value of the square on the semi-axis is,

$$\frac{-2(\sin.\omega)^2}{a - b\cos.\omega + c \pm M} \cdot \left(\frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f\right)$$

89. We shall conclude the discussion of the central class by the application of the results already obtained to a few examples.

The original axes rectangular.

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \quad (1)$$

$$\left.\begin{aligned} y &= y' + n \\ x &= x' + m \end{aligned}\right\} \text{formulas to be used.}$$

$$m = \frac{2ae - bd}{b^2 - 4ac} \; (2), \quad n = \frac{2cd - be}{b^2 - 4ac} \; (3)$$

$$f' = \frac{dn + em}{2} + f = \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f \; (4)$$
$$ay'^2 + bx'y' + cx' + f' = 0 \quad (5)$$

$$\left.\begin{aligned} y' &= x''\sin.\theta + y''\cos.\theta \\ x &= x''\cos.\theta - y''\sin.\theta \end{aligned}\right\} \text{formulas to be used.}$$

$$\tan.2\theta = \frac{-b}{a-c} \quad (6)$$

$$a' = \frac{1}{2}\left\{a + c \pm M\right\} \quad (7) \qquad c' = \frac{1}{2}\left\{a + c \mp M\right\} \quad (8)$$

$$M = \pm \sqrt{(a - c)^2 + b^2}, \ \pm \text{ according as } b \text{ is } \mp$$

$$\left(-\frac{a'}{f'}\right) y''^2 + \left(-\frac{c'}{f'}\right) x''^2 = 1 \qquad (9);$$

(2) and (3) determine the situation of the centre, and together with (4) reduce the equation to the form (5); (6) determines the position of the rectangular axes passing through the centre, (7) and (8) enable us to reduce the equation to its most simple form (9): and the co-efficients of y''^2 and x''^2 inverted are respectively the squares upon the semi-axes measured along the axes of y'' and x''.

Ex. 1. $y^2 - xy + x^2 + y + x - 1 = 0$; locus an ellipse.

$$m = -1;\ n = -1;\ f' = -2;$$

$$y'^2 - x'y' + x'^2 - 2 = 0$$

tan. $2\theta = \frac{1}{0}$ $\therefore 2\theta = 90^\circ$ and $\theta = 45^\circ$; b is negative, and $\therefore$ $M = +1$

$$a' = \frac{3}{2} \text{ and } c' = \frac{1}{2}$$

$$\therefore \frac{3}{2} y''^2 + \frac{1}{2} x''^2 - 2 = 0$$

$$\text{or } \frac{3}{4} y''^2 + \frac{1}{\ } x''^2 = 1$$

The squares on the semi-axes are $\frac{4}{3}$ and 4; hence the semi-axes themselves are $\frac{2}{\sqrt{3}}$ and 2, and therefore the lengths of the axes are 4 and $\frac{4}{\sqrt{3}}$.

Ex. 2. $3y^2 - 4xy + 3x^2 + y - x - \frac{9}{10} = 0$; locus an ellipse.

The reduced equation is $5y''^2 + x''^2 = 1$. The axes are 2 and $\frac{2}{\sqrt{5}}$

Ex. 3. $2y^2 + xy + x^2 - 2y - 4x + 3 = 0$; locus an ellipse.

The reduced equation is $\frac{3 - \sqrt{2}}{2} y''^2 + \frac{3 + \sqrt{2}}{2} x''^2 = 1.$

Ex. 4. $5y^2 + 6xy + 5x^2 - 22y - 26x + 29 = 0$; locus an ellipse.

$$m = 2,\ n = 1,\ f' = \frac{dn + em}{2} + f = -8$$

$$\therefore 5y'^2 + 6x'y' + 5x'^2 - 8 = 0$$

tan. $2\theta = \infty$ $\therefore \theta = 45^\circ$; hence the formulas of transformation are

$$y' = \frac{x'' + y''}{\sqrt{2}} \text{ and } x' = \frac{x'' - y''}{\sqrt{2}};$$

$$\therefore \frac{5}{2}(x''+y'')^2 + 6\frac{x''^2-y''^2}{2} + \frac{5}{2}(x''-y'')^2 - 8 = 0$$

$$\text{or } 4y''^2 + x''^2 = 4$$

Ex. 5. $5y^2 + 2xy + 5x^2 - 12y - 12x = 0$; locus an ellipse.

$$2y''^2 + 3x''^2 = 6$$

Ex. 6. $2y^2+x^2+4y-2x-6=0$; locus an ellipse.

Let $y=y'+n$ and $x=x'+m$, hence the transformed equation is

$$2(y'+n)^2+(x'+m)^2+4(y'+n)-2(x'+m)-6=0$$

or $2y'^2+x'^2+4(n+1)y'+2(m-1)x'+2n^2+m^2+4n-2m-6=0$.

Let $n+1=0$ and $m-1=0$ $\therefore m=1$ and $n=-1$ and $f'=-9$; hence the transformed equation is

$$2y'^2+x'^2=9$$

and no further transformation is requisite. The axes are 6 and $3\sqrt{2}$.

Ex. 7. $y^2-10xy+x^2+y+x+1=0$; locus an hyperbola.

$$6y''^2-4x''^2+\frac{9}{8}=0.$$

Ex. 8. $4y^2-8xy-4x^2-4y+28x-15=0$; locus an hyperbola.

$$y''^2-x''^2=\frac{-1}{2\sqrt{2}}$$

Here the axes are each $=\sqrt[4]{2}$, that which is measured along the new axis of x'' alone meeting the curve.

Ex. 9. $y^2-2xy-x^2-2=0$; locus an hyperbola.

The origin is already at the centre, and thus only one transformation is necessary.

tan. $2\theta=1$ $\therefore 2\theta=45°$; $M=\sqrt{8}$, $a'=\sqrt{2}$, $c'=-\sqrt{2}$, $y'^2-x'^2=\sqrt{2}$.

*90. The axes oblique.

The values of m, n, and f' remain as for rectangular axes.

$$\tan. 2\theta = \frac{c \sin. 2\omega - b \sin. \omega}{a - b\cos.\omega + c\cos. 2\omega}$$

$$a' = \{a - b\cos.\omega + c \pm M\}\frac{1}{2(\sin\omega)^2}.$$

$$c' = \{a - b\cos.\omega + c \mp M\}\frac{1}{2(\sin.\omega)^2}.$$

$$M = \pm\sqrt{\{a^2+b^2+c^2-2b(a+c)\cos.\omega+2ac\cos.2\omega\}}.$$

$\pm$ as $c\sin.2\omega - b\sin\omega$ is $\pm$.

Ex. 1. $y^2+xy+x^2+y+x-\frac{1}{6}=0$; the angle between the axes being 45°.

$m=-\frac{1}{3}$, $n=-\frac{1}{3}$; tan. $2\theta=1$ $\therefore 2\theta=45°$; $M=+(1-\sqrt{2})$.

$a'=3-\frac{3}{\sqrt{2}}$, $c'=1+\frac{1}{\sqrt{2}}$ and $f'=-\frac{1}{2}$.

The reduced equation is

$$3\,(2 - \sqrt{2})\,y''^2 + (2 + \sqrt{2})\,x''^2 = 1.$$

The curve is an ellipse, and the squares upon the semi-axes are

$$\frac{1}{3\,(2 - \sqrt{2})} \text{ and } \frac{1}{2 + \sqrt{2}}.$$

Ex. 2. $7\,y^2 + 16\,x\,y + 16\,x^2 + 32\,y + 64\,x + 28 = 0$. The angle $\omega = 60°$.

$$m = -2,\ n = 0,\ f' = \frac{d\,n + e\,m}{2} + f = -36.$$

The form of the equation is now

$$7\,y'^2 + 16\,x'\,y' + 16\,x'^2 - 36 = 0\,;$$

since tan. $2\,\theta = 0$, the reduction to rectangular axes is effected by merely transferring the axis of y' through $30°$; hence, putting $\theta = 0$, and $\omega = 60$, the formulas (56) of transformation become

$$y' = \frac{2\,y''}{\sqrt{3}},\ \text{and } x' = x'' - \frac{y''}{\sqrt{3}}.$$

Substituting these values in the last equation, it becomes

$$4\,y''^2 + 16\,x''^2 - 36 = 0\,;$$

Hence the axes of the ellipse are 3 and 6.

Ex. 3. $y^2 - 3\,x\,y + x^2 + 1 = 0$; the angle $\omega = 60°$.

$$m = 0,\ n = 0,\ \text{tan. } 2\,\theta = \sqrt{3},\ \therefore \theta = 30°;\ M = +4.$$

$$a' = 5,\ c' = \frac{-1}{3},\ f' = 1, \text{ and the reduced equation is}$$

$$5\,y''^2 - \frac{1}{3}\,x''^2 = -1.$$

The curve is an hyperbola, of which the axes are $2\sqrt{3}$ and $\frac{2}{\sqrt{5}}$; the first of these, which is the greatest, is measured along the new axis of x''. The second axis never meets the curve.

91. It was observed, at the end of art. 81, that the curves corresponding to the general equation of the second order were divided into two classes, one class having a centre or point such that every chord passing through it is bisected in that point, and another class having no such peculiar point. This fact was ascertained from the inspection of the values of the two indeterminate quantities m and n introduced into the equation by means of the transformation of co-ordinates, and for the purpose of destroying certain terms in the general equation. The values of m and n were found to be infinite, that is, there was no centre when the relation among the three first terms of the co-efficients of the general equation was such that $b^2 - 4\,a\,c = 0$.

This relation $b^2 - 4\,a\,c = 0$ being characteristic of the parabola, it follows that the general equation of the second order belonging to a parabola is not capable of the reduction performed in art. (80); that is, we cannot destroy the co-efficients of both x and y, or reduce the equation to the form $a\,y^2 + b\,x\,y + c\,x^2 + f = 0$, or, finally, to the form $a\,y^2 + c\,x^2 + f = 0$.

Although, however, we cannot thus reduce the parabolic equation, we are yet able to reduce it to a very simple form, in fact to a much more simple form than that of either of the above equations. This will be effected by a process similar to that already used for the general equation, only in a different order. We shall commence by transferring the axes through an angle θ, and thus destroy two terms in the equation, so that it will be reduced to the form $a\,y^2 + d\,y + e\,x + f = 0$; we shall then transfer the axes parallel to themselves, and by that means destroy two other terms, so that the final equation will be of the form

$$a\,y^2 + e\,x = 0.$$

92. Taking the formulas in (58), let

$$y = x' \sin.\theta + y' \cos.\theta$$
$$x = x' \cos.\theta - y' \sin.\theta$$

ubstituting these values in the general equation

$$a\,y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0,$$

and arranging, we obtain the equation

$$\left.\begin{array}{l} a(\cos.\theta)^2 \\ -b\sin.\theta\cos.\theta \\ +c(\sin.\theta)^2 \end{array}\right| y'^2 + \left.\begin{array}{r} 2a\sin.\theta\cos.\theta \\ +b(\cos.\theta)^2 \\ -b(\sin.\theta)^2 \\ -2c\sin.\theta\cos.\theta \end{array}\right| x'y' + \left.\begin{array}{l} a(\sin.\theta)^2 \\ +b\sin.\theta\cos.\theta \\ +c(\cos.\theta)^2 \end{array}\right| x'^2 + \left.\begin{array}{l} d\cos.\theta \\ -e\sin.\theta \end{array}\right| y' + \left.\begin{array}{l} d\sin.\theta \\ +e\cos.\theta \end{array}\right| x' + f = 0$$

Let the co-efficient of $x'\,y' = 0$

$$\therefore\ 2\,(a - c)\sin\theta\cos.\theta + b\,\{(\cos.\theta)^2 - (\sin.\theta)^2\} = 0,$$
$$\text{or } (a - c)\sin.2\,\theta + b\cos.2\,\theta = 0,$$
$$\text{and } \tan.2\,\theta = \frac{-\,b}{a - c}, \text{ as in (82.)}$$

Hence, if the axes be transfered through an angle θ such that $\tan.2\,\theta = \dfrac{-\,b}{a - c}$ the transformed equation will have no term containing the product of the variables; that is, it will be of the form

$$a'\,y'^2 + c'\,x'^2 + d'\,y' + e'\,x' + f = 0.$$

But, since this last equation belongs to a parabola, the relation among the co-efficients of the three first terms must be such that the general condition $b^2 - 4\,a\,c = 0$ holds good. In this case, since $b' = 0$, we must have $-\,4\,a'\,c' = 0$; hence either a' or c' must $= 0$; that is, the transformation which has enabled us to destroy the co-efficient of the term containing $x'\,y'$ will of necessity destroy the co-efficient of either x'^2 or y'^2. And this will soon be observed upon examining the values of the co-efficients of x'^2 and y'^2.

93. Let the co-efficient b in the original equation be negative, that is, let $b = -\,2\sqrt{a\,c}$.

From $\tan.2\,\theta$ we have $\cos.2\,\theta = \dfrac{1}{\sqrt{1 + (\tan.2\,\theta)^2}} = \dfrac{1}{\sqrt{1 + \left(\dfrac{b}{a - c}\right)^2}}$

$$= \frac{a - c}{\pm\sqrt{b^2 + (a - c)^2}} = \frac{a - c}{\pm(a + c)} \text{ and } \sin 2\,\theta = \frac{-b}{\pm(a + c)} = \frac{-b}{(a + c)},$$

since sin. 2 θ must be positive, and b is itself negative;

$$\text{hence } \cos.\,\theta = \sqrt{\frac{1 + \cos.\,2\,\theta}{2}} = \sqrt{\tfrac{1}{2}\left(1 + \frac{a - c}{a + c}\right)} = \sqrt{\frac{a}{a + c}},$$

$$\text{and } \sin.\,\theta = \sqrt{\frac{1 - \cos.\,2\,\theta}{2}} = \sqrt{\frac{c}{a + c}}.$$

Substituting these values of sin. θ and cos. θ in the general transformed equation, we have

$$a' = \frac{a\,a}{a + c} - \frac{b\sqrt{a\,c}}{a + c} + \frac{c\,c}{a + c} = \frac{a^2 + 2\,a\,c + c^2}{a + c} = a + c.$$

$$c' = \frac{a\,c}{a + c} + \frac{b\sqrt{a\,c}}{a + c} + \frac{c\,a}{a + c} = \frac{a\,c - 2\,a\,c + a\,c}{a + c} = 0.$$

$$d' = \frac{d\sqrt{a} - e\sqrt{c}}{\sqrt{a + c}}$$

$$e' = \frac{d\sqrt{c} + e\sqrt{a}}{\sqrt{a + c}}$$

And the transformed equation is now

$$(a + c)\,y'^2 + \frac{d\sqrt{a} - e\sqrt{c}}{\sqrt{a + c}}\,y' + \frac{d\sqrt{c} + e\sqrt{a}}{\sqrt{a + c}}\,x' + f = 0.$$

And it is manifest that if b had been positive all the way through this article, the reduced equation would have been

$$(a + c)\,x'^2 + \frac{d\sqrt{c} - e\sqrt{a}}{\sqrt{a + c}}\,y' + \frac{d\sqrt{a} + e\sqrt{c}}{\sqrt{a + c}}\,x' + f = 0.$$

94. In order to reduce the equation still lower, let us transfer the axes parallel to themselves by means of the formulas $y' = y'' + n$ and $x' = x'' + m$ (54.)

then the equation $a'\,y'^2 + d'\,y' + e'\,x' + f = 0$ becomes

$$a'\,(y'' + n)^2 + d'\,(y'' + n) + e'\,(x'' + m) + f = 0,$$

$$\text{or } a'\,y''^2 + (2\,a'\,n + d')\,y'' + e'\,x'' + a'\,n^2 + d'\,n + e'\,m + f = 0.$$

And since we have two independent quantities, m and n, we can make two hypotheses respecting them; let, therefore, their values be such that the co-efficient of y' and the constant term in the equation each $= 0$ · that is, let

$$2\,a'\,n + d' = 0, \text{ and } a'\,n^2 + d'\,n + e'\,m + f = 0;$$

$$\text{whence } n = \frac{-d'}{2\,a'} \text{ and } m = \frac{d'^2 - 4\,a'\,f}{4\,a'\,e'},$$

and the reduced equation is now of the form

$$a'\,y''^2 + e'\,x'' = 0;$$

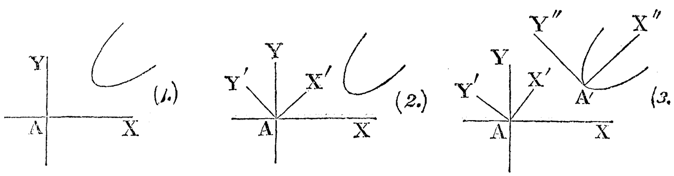
and it is manifest that if b had been positive, the equation $c' x'^2 + d' y' + e' x' + f = 0$ would have been reduced to the form

$$c' x''^2 + d' y'' = 0,$$

where the values of m and n would be found from the equations

$$m = - \frac{e'}{2 c'}, \text{ and } n = \frac{e'^2 - 4 c' f}{4 c' d'}.$$

95. The following figures will exhibit the changes which have taken place in regard to the *position* of the locus corresponding to each analytical change in the *form* of the equation :—



In fig. 1, the curve is referred to rectangular axes A X and A Y, and the equation is

$$a y^2 + b x y + c x^2 + d y + e x + f = 0.$$

In fig. 2, the axes are transferred into the position A X', A Y', the angle X A X' or θ being determined by the equation $\tan. 2 \theta = \frac{- b}{a - c}$, the corresponding equation is, for b negative,

$$a' y'^2 + d' y' + e' x' + f = 0.$$

If b is positive, the curve would originally have been situated at right angles to its present position, and the reduced equation would be

$$c' x'^2 + d' y' + e' x' + f = 0.$$

In fig. 3, the position of the origin is changed from A to A', the co-ordinates of A' being measured along A X' and A Y', and their values determined by the equations

for b negative, $n = \frac{- d'}{2 a'}$ and $m = \frac{d'^2 - 4 a' f}{4 a' e'}$

for b positive, $m = \frac{- e'}{2 c'}$ and $n = \frac{e'^2 - 4 c' f}{4 c' d'}$.

The reduced equation is

for b negative, $a' y''^2 + e' x'' = 0.$
for b positive, $c' x''^2 + d' y'' = 0.$

96 *. If the original axes are oblique, the transformation of the general equation must be effected by means of the formulas in (55). The values of a', b', and c' will be exactly the same as in (87).

We may then let $b' = 0$, and also find $\tan. 2 \theta$ when the axes are rect-

* See Note, Art. 87.

angular, whence, as in (87), we shall find that there is but one such system of axes.

The same value of θ which destroys the term in $x' y'$ will, as in (93), also destroy the term in x'^2 or y'^2; hence the reduced equation will be

for $c \sin. 2\,\omega - b \sin. \omega$ positive, $a' y'^2 + d' y' + e' x' + f = 0$.
for $c \sin. 2\,\omega - b \sin. \omega$ negative, $c' x'^2 + d' y' + e' x' + f = 0$.

97. To find the values of a', c', d', and e'.

The values of a' and c' are best deduced from those in art. (88), Since $b^2 - 4\,a\,c = 0$, we have for $c \sin. 2\,\omega - b \sin. \omega$ positive

$$M = a - b \cos. \omega + c$$

$$a' = \{a - b \cos. \omega + c\} \frac{1}{(\sin. \omega)^2}$$

$$c' = 0$$

$$\cos. 2\,\theta = \frac{a - b \cos. \omega + c \cos. 2\,\omega}{a - b \cos. \omega + c};$$

$$\sin. \theta = \frac{\sin. \omega \sqrt{c}}{\sqrt{a - b \cos. \omega + c}}, \text{ and } \cos. \theta = \sqrt{\frac{a - b \cos. \omega + c\,(\cos. \omega)^2}{a - b \cos. \omega + c}}:$$

Also $d' = \dfrac{d \cos. \theta - e \cos. (\omega - \theta)}{\sin \omega}$ from the transformed equation

$$= \frac{(d - e \cos. \omega) \sqrt{\{a - b \cos. \omega + c\,(\cos. \omega)^2\}} - e \sqrt{c}\,(\sin. \omega)^2}{\sin. \omega \sqrt{\{a - b \cos. \omega + c\}}}$$

and $e' = \dfrac{d \sin. \theta + e \sin. (\omega - \theta)}{\sin. \omega}$

$$= \frac{(d - e \cos. \omega) \sqrt{c} + e \sqrt{\{a - b \cos. \omega + c\,(\cos. \omega)^2\}}}{\sqrt{\{a - b \cos. \omega + c\}}}$$

and the reduced equation is now of the form

$$a' y'^2 + d' y' + e' x' + f = 0;$$

For $c \sin. 2\,\omega - b \sin. \omega$ negative, the corresponding values of a', c', M, d', and e' are

$$M = -(a - b \cos. \omega + c)$$

$$a' = 0$$

$$c' = (a - b \cos. \omega + c) \frac{1}{(\sin. \omega)^2}$$

$\sin. \theta$ and $\cos. \theta$ merely change values,

hence $d' = \dfrac{(d - e \cos. \omega) \sqrt{c} - e \sqrt{\{a - b \cos. \omega + c\,(\cos. \omega)^2\}}}{\sqrt{\{a - b \cos. \omega + c\}}}$

and $e' = \dfrac{(d - e \cos. \omega) \sqrt{\{a - b \cos. \omega + c\,(\cos. \omega)^2\}} + e \sqrt{c}\,(\sin. \omega)^2}{\sin. \omega \sqrt{\{a - b \cos. \omega + c\}}}$

and the reduced equation is now of the form

$$c' x'^2 + d' y' + e' x' + f = 0.$$

The transformation required to reduce the equations still lower is performed exactly as in (94); and, by making the angle between the original axes oblique, the figures in (95) will exhibit the changes in the *position* of the curve.

98. We shall conclude the discussion of this class of curves by the application of the results already obtained to a few examples.

Ex. 1. $y^2 - 6xy + 9x^2 + 10y + 1 = 0$; locus a parabola.

$$\tan. 2\theta = \frac{-b}{a-c} = -\frac{3}{4}; \text{ hence } \theta \text{ may be found by the tables.}$$

b is negative;

$$\therefore \text{by (93) } a' = a + c = 10, c' = 0, d' = \sqrt{10} \text{ and } e' = 3\sqrt{10},$$

$$\therefore 10y'^2 + \sqrt{10}\,y' + 3\sqrt{10}\,x' + 1 = 0.$$

Also by (94) $n = \frac{-d'}{2a'} = \frac{-1}{2\sqrt{10}}$, and $m = \frac{-1}{4\sqrt{10}}$;

and the final equation is

$$y''^2 + \frac{3}{\sqrt{10}}x = 0.$$

Ex. 2. $y + 2xy + x^2 + y - 3x + 1 = 0$; locus a parabola.

$$x''^2 + \sqrt{2}\,y = 0.$$

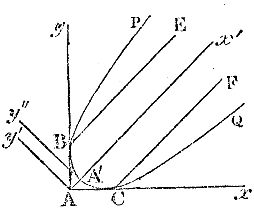
Ex. 3. $\sqrt{y} + \sqrt{x} = \sqrt{d}$. This equation may be put under the form $y + x - d = 2\sqrt{xy}$; or

$$y^2 - 2xy + x^2 - 2dy - 2dx + d^2 = 0;$$

and the locus is a parabola because it satisfies the condition

$$b^2 - 4ac = 0.$$

By tracing the curve as in (78) we shall find its position to be that of



PBCQ in the figure; and $y = x \pm d$ are the equations to the diameters BE and CF.

A x', A y', are the new axes, θ or x A x being 45°.

$$a' = 2, d' = 0, e' = -2d\sqrt{2}, n = o, m = \frac{d}{2\sqrt{2}};$$

the last two quantities are to be measured along the new axes, therefore take A A' $= \frac{d}{2\sqrt{2}}$, and A' is the new origin.

The final equation is

$$y^2 = dx\sqrt{2}.$$

Ex. 4. $y = d + ex + fx^2$. The locus is a parabola, since $b^2 - 4ac$ or $0 - 4.0.f = 0$.

Let $y = y' + n$, and $x = x' + m$;

$$\therefore y' + n = d + e(x' + m) + f(x' + m)^2;$$

$$\therefore f x'^2 + (2 m f + e) x' - y' + f m^2 + e m + d - n = 0.$$

Let $2 m f + e = 0$, and $f m^2 + e m + d - n = 0$

$$\therefore m = \frac{-e}{2 f} \text{ and } n = \frac{4 d f - e^2}{4 f},$$

and the equation is reduced at once to the form

$$f x'^2 - y' = 0.$$

99. The axes oblique.

$y^2 - 2 x y + x^2 - 6 x = 0$; the angle between the axes being 60°.

Here, $c \sin. 2\omega - b \sin. \omega$ is positive.

$$\sin. \theta = \frac{\sin. 60}{\sqrt{3}} = \frac{1}{2} \quad \therefore \theta = 30^\circ$$

$$M = 3,\ a' = 4,\ c' = 0,\ d' = 6,\ e' = -2\sqrt{3},\ m = -\frac{3\sqrt{3}}{8},$$

$$n = -\frac{3}{4};$$

$$\therefore 4 y^2 - 2\sqrt{3}\ x = 0.$$

CHAPTER VIII

THE ELLIPSE.

100. In the discussion of the general equation of the second order, we have seen that, supposing the origin of co-ordinates in the centre, there is but one system of rectangular axes to which, if the corresponding ellipse be referred, its equation is of the simple form

$$\left(\frac{-a'}{f'}\right) y^2 + \left(\frac{-c'}{f'}\right) x^2 = 1$$

$$\text{or, } P y^2 + Q x^2 = 1$$

where the coefficients P and Q are both positive. (86, 87.)

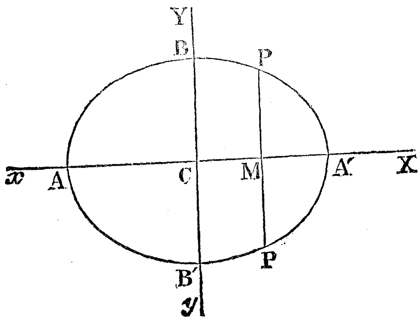
We now proceed to deduce from this equation the various properties of the ellipse.

To exhibit the coefficients in a better form; let C be the centre of the curve; $X x$, $Y y$, the rectangular axes meeting in C; $CM = x$, $MP = y$.

Then at the points where the curve cuts the axes, we have

$$y = 0, \quad Q x^2 = 1, \quad \therefore x = \pm \frac{1}{\sqrt{Q}}.$$

$$x = 0, \quad P y^2 = 1, \quad \therefore y = \pm \frac{1}{\sqrt{P}}.$$



In the axis of x take $C A' = \frac{1}{\sqrt{Q}}$ and $C A = -\frac{1}{\sqrt{Q}}$,

also in the axis of y take $C B = \frac{1}{\sqrt{P}}$ and $C B' = -\frac{1}{\sqrt{P}}$,

then the curve cuts the axes at the points A, A′, B, and B′.

Also if $C A = a$ and $C B = b$, and a be greater than b, we have $Q = \frac{1}{a^2}$ and $P = \frac{1}{b^2}$, therefore the equation to the curve becomes

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$$

$$\text{or} \quad a^2 y^2 + b^2 x^2 = a^2 b^2$$

$$\text{or} \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

101. We have already seen (76) that the curve is limited in every direction.

The points A, A′, B, and B′ determine those limits. From the last equation we have

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2} \quad (1), \quad \text{and } x = \pm \frac{a}{b}\sqrt{b^2 - y^2} \quad (2),$$

from (1) if x is greater than $\pm a$, y is impossible, and from (2) if y is greater than $\pm b$, x is also impossible; hence straight lines drawn through the points A, A′, B and B′ parallel to the axes, completely enclose the curve.

Again from (1) for every value of x less than a we have two real and equal values of y, that is, for any abscissa C M less than C A′ we have two equal ordinates M P, M P′, the $\pm$ sign determining their opposite directions.

Also as x increases from 0 to $+ a$ these values of y decrease from $\pm b$ to 0, hence we have two equal arcs B P A′, B′ P′ A′ exactly similar and opposite to one another.

If x be negative, and decrease from 0 to $- a$, x^2 is positive, and the same values of y must recur, hence there are two equal and opposite arcs B A, B′A. Therefore the whole curve is divided into two equal parts by the axis of x.

From (2) the curve appears in the same way to be divided into two equal parts by the axis of y: hence it is said to be symmetrical with respect to those axes.

Its concavity must also be turned towards the centre, otherwise it might be cut by a straight line in more points than two, which is impossible (71).

102. From the equation $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ we have

$$C P = \sqrt{x^2 + y^2} = \sqrt{x^2 + \frac{b^2}{a^2}(a^2 - x^2)} = \sqrt{b^2 + \frac{a^2 - b^2}{a^2} x^2}$$

hence C P is greatest when x is greatest, that is, when $x = a$, in which case C P becomes also equal to a, hence C A or C A′ is the greatest line that can be drawn from the centre to the curve. Again C P is least when $x = 0$, in which case C P becomes equal to b, hence C B is the least line that can be drawn from C to the curve. The axes A A′ and B B′ are thus shown to be the greatest and least lines that can be drawn through the centre. The greater A A′ is called the axis major, or greater axis, or transverse axis, and B B′ the axis minor or lesser axis.

103. The points A, B, A′ and B′ are called the vertices or summits of the curve. Any of these points may be taken for the origin, thus let A be the origin, A C the axis of x, and let the axis of y be parallel to C B, and A M $= x'$.

Then $x = \text{C M} = \text{A M} - \text{A C} = x' - a$

$$\therefore\ y^2 = \frac{b^2}{a^2}(a^2 - x^2) = \frac{b^2}{a^2}\{a^2 - (x' - a)^2\} = \frac{b^2}{a^2}(2\,a\,x' - x'^2)$$

or suppressing the accents, $y^2 = \frac{b^2}{a^2}(2\,a\,x - x^2) = \frac{b^2}{a^2}\,x\,(2\,a - x)$.

This last equation is geometrically expressed by the following proportion. The square upon M P : the rectangle A M, M A′ :: the square upon B C : the square upon A C.

Hence the square upon the ordinate varies as the rectangle contained by the segments of the axis major.

If the origin be at C, C A′ the axis of y and C B the axis of x, we have, putting x for y and y for x, the equation $y^2 = \frac{a^2}{b^2}(b^2 - x^2)$, and if the origin be at B, $y^2 = \frac{a^2}{b^2}(2\,b\,x - x^2)$.

104. If the axes major and minor were equal to one another, the equation to the ellipse would become $y^2 = a^2 - x^2$, which is that to a circle whose diameter is $2\,a$, hence we see as in (79) that the circle is a species of ellipse. As we advance we shall have frequent occasion to remark the analogy existing between these two curves.

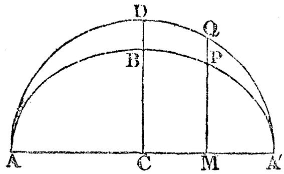
Let A D Q A′ be the circle described upon A A′ as diameter, and M Q or Y be an ordinate corresponding to the abscissa C M or x, let M P ($=y$) be the corresponding ordinate to the ellipse, then we have

$$Y^2 = a^2 - x^2$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore y^2 = \frac{b^2}{a^2} Y^2 \text{ and } y = \frac{b}{a} Y$$

$$\therefore y : Y :: b : a$$



thus the ordinate to the ellipse has to the corresponding ordinate of the circle the constant ratio of the axis minor to the axis major.

Since b is less than a the circle is wholly without the ellipse, except at A and A′ where they meet. Similarly if a circle be described on the axis minor, it is wholly within the ellipse except at B and B′. Thus the elliptic curve lies between the two circumferences.

THE FOCUS.

105. THE equation $y^2 = \frac{b^2}{a^2}(2\,a\,x - x^2)$ may be put under the form $y^2 = l\,x - \frac{l}{2\,a}x^2$, in which case the quantity $l = \frac{2\,b^2}{a}$ is called the principal Parameter or Latus Rectum.

Since $l = \frac{2\,b^2}{a} = \frac{4\,b^2}{2\,a}$ the Latus Rectum is a third proportional to the axis major and minor.

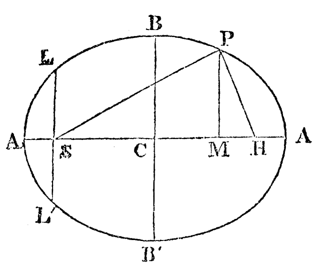
106. To find from what point in the axis major a double ordinate and be drawn equal to the Latus Rectum.

Here $4\,y^2 = l^2$ or $\frac{4\,b^2}{a^2}(a^2 - x^2) = \frac{4\,b^4}{a^2}$

$$\therefore a^2 - x^2 = b^2$$

$$\text{or } x^2 = a^2 - b^2$$

$$\text{and } x = \pm\sqrt{a^2 - b^2}.$$



With centre B and radius a describe a circle cutting the axis major in the points S and H, then we have $CH = +\sqrt{a^2 - b^2}$ and $CS = -\sqrt{a^2 - b^2}$, thus S and H are the points through either of which if an ordinate at L S L′ be drawn, it is equal to the Latus Rectum; henceforward then we shall consider this line as the Latus Rectum or principal parameter of the ellipse.

The two points S and H thus determined are called the Foci, for a reason to be hereafter explained.

107. The fraction $\frac{\sqrt{a^2 - b^2}}{a}$ which represents the ratio of C S to C A

is called the excentricity, because the deviation of this curve from the circular form, that is, its ex-centric course, depends upon the magnitude of this ratio.

If the excentricity, which is evidently less than unity, be represented by the letter e, we have $\frac{\sqrt{a^2 - b^2}}{a} = e$: whence $e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} \therefore \frac{b^2}{a^2} = 1 - e^2$ and the equation to the ellipse may be put under the form

$$y^2 = (1 - e^2)(a^2 - x^2).$$

108. The line S C is sometimes called the ellipticity; its value, as above, is $a e$; but it is also expressed by the letter c. Also since $a^2 - b^2 = a^2 e^2$ we have $b^2 = a^2 - a^2 e^2 = (a - a e)(a + a e)$; hence

The rectangle A S, S A′ = The square upon B C.

109. To find the distance from the focus to any point P in the curve.

Let $SP = r$, $HP = r'$,

$$\therefore r^2 = (y - y')^2 + (x - x')^2 \ldots (29)$$

also y', x' being the co-ordinates of S, we have $y' = 0$ and $x' = -a e$,

$$\begin{aligned} \therefore r^2 &= y^2 + (x + a e)^2 \\ &= (1 - e^2)(a^2 - x^2) + (x + a e)^2 \\ &= a^2 - x^2 - e^2 a^2 + e^2 x^2 + x^2 + 2 a e x + a^2 e^2 \\ &= a^2 + 2 a e x + e^2 x^2 \\ &= (a + e x)^2 \end{aligned}$$

$$\therefore SP = a + e x; \text{ similarly } HP = a - e x.$$

In all questions referring to the absolute magnitude of S P or H P we must give to x its proper sign; thus if P is between B and A, the absolute magnitude of S P is $a - e x$, because x is itself negative.

By the addition of S P and H P, we have $SP + HP = 2a = AA'$; that is, the sum of the distances of any point on the curve from the foci is equal to the axis major

This property is analogous to that of the circle, where the distance of any point from the centre is constant.

110. This property of the ellipse is so useful, that we shall prove the converse. To find the locus of a point P, the sum of whose distances from two fixed points S and H is constant or equal $2a$.

Let $SH = 2c$, bisect S H in C, which point assume to be the origin of rectangular axes C A′, C B; let $CM = x$, and $MP = y$,

$$\text{then } SP = \sqrt{(c + x)^2 + y^2}$$

$$HP = \sqrt{(c - x)^2 + y^2}$$

$$\text{but } SP + HP = 2a, \quad \text{or } SP = 2a - HP$$

$$\therefore \sqrt{(c + x)^2 + y^2} = 2a - \sqrt{(c - x)^2 + y^2}$$

$$\therefore (c + x)^2 + y^2 = 4a^2 - 4a\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2;$$

hence, transposing and dividing by 4, we have

$$a\sqrt{(c - x)^2 + y^2} = a^2 - c x$$

$$\therefore a^2 y^2 = a^4 - a^2 c^2 + c^2 x^2 - a^2 x^2$$
$$= (a^2 - c^2)(a^2 - x^2)$$
$$\text{and } y^2 = \frac{a^2 - c^2}{a^2}(a^2 - x^2)$$

Hence the locus is an ellipse whose axes are $2a$ and $2\sqrt{a^2 - c^2}$, and whose foci are S and H.

THE TANGENT.

111. To find the equation to the tangent to the ellipse at any point.
Let $x'\ y'$ be the point P
. . . $x''\ y''$ be any other point Q
the equation to the line P Q through these two points is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'), \quad (41)$$

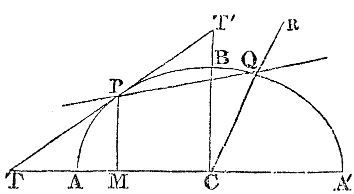
Now this cutting line or secant PQ will come to the position TPT′ or just touch the curve when Q comes to P, and the equation PQ will become the equation to the tangent PT when $x'' = x'$ and $y'' = y'$.

In this case the term $\frac{y' - y''}{x' - x''}$ becomes $\frac{0}{0}$, but its value may yet be found, for since the points $x'\ y'$, $x''\ y''$ are on the curve, we have

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2$$
$$a^2 y''^2 + b^2 x''^2 = a^2 b^2$$
$$\therefore a^2 (y'^2 - y''^2) + b^2 (x'^2 - x''^2) = 0;$$
$$\text{or } a^2 (y' - y'')(y' + y'') + b^2 (x' - x'')(x' + x'') = 0,$$
$$\therefore \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\frac{x' + x''}{y' + x''}$$
$$= -\frac{b^2}{a^2}\frac{x'}{y'} \text{ when } x'' = x' \text{ and } y'' = y.$$

∴ The equation to the tangent is

$$y - y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x - x')$$



By multiplication $a^2 y y' - a^2 y'^2 = - b^2 x x' + b^2 x'^2$

$$\therefore a^2 y y' + b^2 x x' = a^2 y'^2 + b^2 x'^2$$
$$= a^2 b^2.$$

In the figure C M is x' and M P is y', and x and y are the co-ordinates of any point in T P T′.

The equation to the tangent is easily recollected, since it may be obtained from that to the curve $a^2 y^2 + b^2 x^2 = a^2 b^2$ by putting $y y'$ for y^2 and $x x'$ for x^2.

112. That P T is a tangent is evident, since a straight line cannot cut the curve in more points than two, and here those two have gradually coalesced; it may, however, be satisfactory to show that every point in P T except P is without the curve.

Let x_1 and y_1 be the co-ordinates of any point R; then if $a^2 y_1^2 + b^2 x_1^2$ is greater than $a^2 b^2$, the point R is without the curve. For, join the point R with the centre of the ellipse by a line cutting the curve in Q, and let x and y be the co-ordinates of Q, then if $a^2 y_1^2 + b^2 x_1^2$ is greater than $a^2 b^2$, or than $a^2 y^2 + b^2 x^2$, we have $b^2 (x_1^2 - x^2)$ greater than $a^2 (y^2 - y_1^2)$; but b is less than a, therefore $x_1^2 - x^2$ must be greater than $y^2 - y_1^2$, or $x_1^2 + y_1^2$ greater than $x^2 + y^2$, and therefore C R greater than C Q (29), or R is without the curve.

In the present case we have the two equations.

$$a^2 y y' + b^2 x x' = a^2 b^2$$

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2$$

$$\therefore\ a^2 y'^2 - 2 a^2 y y' + b^2 x'^2 - 2 b^2 x x' = - a^2 b^2$$

$$\text{or } a^2 (y' - y)^2 + b^2 (x' - x)^2 = a^2 y^2 + b^2 x^2 - a^2 b^2$$

$$\therefore\ a^2 y^2 + b^2 x^2 = a^2 b^2 + a^2 (y' - y)^2 + b^2 (x' - x)^2$$

which is greater than $a^2 b^2$.

But y and x are the co-ordinates of any point in the tangent; therefore generally any point on the tangent is without the curve; in the particular case where $y = y'$, and $x = x'$, that is at P, we have the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, therefore at that point the tangent coincides with the curve.

113. If the vertex A be the origin, the equation to the curve is

$$y^2 = \frac{b^2}{a^2} (2 a x - x^2) \text{ or } a^2 y^2 + b^2 x^2 - 2 a b^2 x = 0,$$

and the equation to the tangent, found exactly as above, is

$$a^2 y y' + b^2 x x' - a b^2 (x + x') = 0;$$

If the equation to the ellipse be $y^2 = m x - n x^2$, the equation to the tangent is $y y' = \frac{m}{2} (x + x') - n x x'$.

Generally, if the equation to the curve be

$$a y^2 + b x y + c x^2 + d y + e x + f = 0,$$

the equation to the tangent is

$$y - y' = - \frac{2 c x' + b y' + e}{2 a y' + b x' + d} (x - x'),$$

$$\text{or } (2 a y' + b x' + d) y + (2 c x' + b y' + e) x + d y + e x + 2 f = 0.$$

Again let $y = \alpha x + d$ be the equation to a tangent to the ellipse; then, comparing this with the equation $a^2 y y' + b^2 x x' = a^2 b^2$, and eliminating x' and y' by means of the equation $a^2 y'^2 + b^2 x'^2 = a^2 b^2$, we have

$$a^2 \alpha^2 + b^2 = d^2,$$

and this is the necessary relation among the co-efficients of the equation $y = \alpha x + d$ when it is a tangent to the curve.

114. To find the point where the tangent cuts the axes.

In the equation $a^2 y y' + b^2 x x' = a^2 b^2$ put $y = 0 \therefore b^2 x x' = a^2 b^2$, and $x = \frac{a^2}{x'} = \text{C T}$; similarly $y = \text{C T}' = \frac{b^2}{y'}$; hence we have

The rectangle C T, C M = The square upon A C,

and The rectangle C T′, M P = The square upon B C

Since $\text{C T}\left(= \frac{a^2}{x'}\right)$ does not involve y', it is the same for all ellipses which have the same axis major, and same abscissa for the point of contact; and, as the circle on the axis major may be considered as one of these ellipses, the distance C T is the same for an ellipse and its circumscribing circle.

Again, since $\text{C T} = \frac{a^2}{x'}$ is independent of the sign of y', the tangents, at the two extremities of an ordinate, meet in the same point on the axis. The equation to the lower tangent is found by putting $-y'$ for y' in the general equation to the tangent (111).

115. The distance M T from the foot of the ordinate to the point where the tangent meets the axis of x, is called the subtangent.

In the ellipse, $\text{M T} = \text{C T} - \text{C M} = \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}$;

Hence, The rectangle C M, M T = The rectangle A M, M A′.

116. The equation to the tangent being $a^2 y y' + b^2 x x' = a^2 b^2$, let $x' = a$; and $\therefore y' = 0$, $\therefore b^2 a x = a^2 b^2$ and $x = a$; hence the tangent, at the extremity of the axis major, is perpendicular to that axis. At B, the equation to the tangent is $y = b$; hence the tangent at B is perpendicular to the axis minor.

The equation to the tangent being $a^2 y y' + b^2 x x' = a^2 b^2$, or

$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'}.$$

If P C be produced to meet the curve again in P′, the signs of the co-ordinates of P′ are both contrary to those of P; hence the co-efficient $-\frac{b^2 x'}{a^2 y'}$, remains the same for the tangent at P′, or the tangents at P and P′ are parallel (43).

117. To find the equation to the tangent at the extremity of the Latus Rectum.

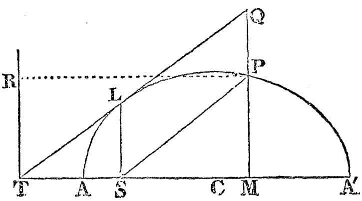
The equation to the tangent is generally

$$a^2 y y' + b^2 x x' = a^2 b^2$$

At L, $x' = -ae$ and $y' = \frac{b^2}{a}$,

$$\therefore a^2 y \frac{b^2}{a} - b^2 x\, a e = a^2 b^2$$

$$y = a + e x.$$



If the ordinate y, or M Q, cut the ellipse in P, we have S P $= a + e x$ (109)

$$\therefore \text{M Q} = \text{S P}.$$

118. To find the point where this particular tangent cuts the axis, let $y = 0$; $\therefore x = \text{CT} = -\frac{a}{e}$.

From T draw T R perpendicular to A C, and from P draw P R parallel to A C; then, taking the absolute values of C M and C T, we have

$$\text{P R} = \text{M T} = \text{C T} + \text{C M} = \frac{a}{e} + x = \frac{a + e x}{e} = \frac{1}{e}.\text{S P};$$

Consequently, the distances of any point P from S, and from the line T R, are in the constant ratio of $e : 1$.

This line T R is called the directrix; for, knowing the position of this line and of the focus, an ellipse of any excentricity may be described, as will hereafter be shown.

If $x = 0$, we have $y = a$. Thus the tangent, at the extremity of the Latus Rectum, cuts the axis of y where that axis meets the circumscribing circle.

By producing Q M to meet the ellipse again in P′, it may be proved that

The rectangle Q P, Q P′ = The square on S M.

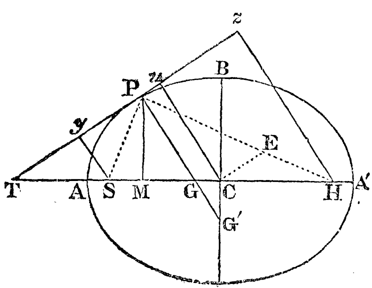
119. To find the length of the perpendicular from the focus on the tangent.

Let S y, H z, be the perpendiculars on the tangent P T.

Taking the expression in (48.) we have

$$p = -\frac{y_1 - \alpha x_1 - d}{\sqrt{1 + \alpha^2}}.$$

Where $y_1 = 0$ and $x_1 = -ae$ are co-ordinates of the point S, and $y = \alpha x + d$ is the equation to the line P T. But the equation to P T is also



$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'};$$

$$\therefore \alpha = -\frac{b^2 x'}{a^2 y'} \text{ and } d = \frac{b^2}{y'};$$

$$\therefore p = -\frac{-\frac{b^2 x'}{a^2 y'} a e - \frac{b^2}{y'}}{\sqrt{\left\{1 + \frac{b^4 x'^2}{a^4 y'^2}\right\}}}$$

$$= \frac{a b^2 . (a + e x')}{\sqrt{a^4 y'^2 + b^4 x'^2}}$$

And $a^4 y'^2 + b^4 x'^2 = a^2 (a^2 b^2 - b^2 x'^2) + b^4 x'^2 = a^2 b^2 \left\{a^2 - \frac{a^2 - b^2}{a^2} x'^2\right\}$

$$= a^2 b^2 (a^2 - e^2 x'^2);$$

$$\therefore p = \frac{a b^2 (a + e x')}{a b \sqrt{(a^2 - e^2 x'^2)}} = b \sqrt{\frac{a + e x'}{a - e x'}}.$$

Let S P or $a + e x' = r$, and H P or $a - e x' = 2a - r = r'$,

$$\therefore p^2 = \frac{b^2 r}{2a - r} = \frac{b^2 r}{r'}.$$

Similarly, if H $z = p'$, we have $p'^2 = b^2 \frac{r'}{r}$.

By multiplication we have $p p' = b^2$; Hence,

The rectangle S y, H z = The square upon B C.

120. To find the locus of y or z in the last article.

The equation to the curve at P is $a^2 y'^2 + b^2 x'^2 = a^2 b^2$ (1)

The equation to the tangent at P is $a^2 y y' + b^2 x x' = a^2 b^2$ (2).

The equation to the perpendicular S y (the co-ordinates of S being $-c, 0$) is $y = \alpha (x + c)$ and this line being perpendicular to the tangent (2), we have $\alpha = \frac{a^2 y'}{b^2 x'}$; and therefore the equation to S y is

$$y = \frac{a^2 y'}{b^2 x'} (x + c) \quad (3).$$

If we eliminate y' and x' from (1) (2) and (3), we shall have an equation involving x and y; but this elimination supposes x and y to be the same for both (2) and (3), and therefore can only refer to their intersection. Hence, the resulting equation is the locus of their intersection.

From (3) $\frac{y'}{x'} = \frac{b^2}{a^2} \frac{y}{x + c} = \frac{b^2}{x' y} - \frac{b^2 x}{a^2 y}$ from (2);

$$\therefore \frac{1}{x'} = \frac{y^2 + x (x + c)}{a^2 (x + c)}, \therefore x' = \frac{a^2 (x + c)}{y^2 + x (x + c)},$$

$$\text{and } y' = \frac{b^2 y x'}{a^2 (x + c)} = \frac{b^2 y}{y^2 + x (x + c)}.$$

Substituting these values of x' and y', in (1), we have

$$a^2 b^4 y^2 + b^2 a^4 (x + c)^2 = a^2 b^2 \{y^2 + x(x+c)\}^2$$

$$\therefore\ b^2 y^2 + a^2 (x+c)^2 = \{y^2 + x(x+c)\}^2;$$

Or, $a^2 y^2 - c^2 y^2 + a^2 (x^2 + c)^2 = y^4 + 2x(x+c)y^2 + x^2 (x+c)^2$;

$$\begin{aligned}\therefore\ a^2 \{y^2 + (x+c)^2\} &= y^4 + y^2 \{2x(x+c) + c^2\} + x^2 (x+c)^2\\ &= y^4 + y^2 x^2 + y^2 (x+c)^2 + x^2 (x+c)^2\\ &= y^2 (y^2 + x^2) + (y^2 + x^2)(x+c)^2\\ &= (y^2 + x^2)\{y^2 + (x+c)^2\};\end{aligned}$$

$$\therefore\ a^2 = y^2 + x^2.$$

This is the equation to a circle whose radius is a. Hence, the locus of y is the circle described on the axis major as diameter.

From the equation to S y, combined with that to C P $\left(y = \frac{y}{x'} x\right)$, we may prove that C P and S y meet in the directrix.

121. To find the angle which the focal distance S P makes with the tangent P T.

The equation to the tangent is $y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'}$.

The equation to S P passing through S $(-c, 0)$ and P (x', y') is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') = \frac{y'}{x' + c}(x - x');$$

And tan. S P T = tan. (P S C − P T C) $= \dfrac{\dfrac{y'}{x'+c} + \dfrac{b^2 x}{a^2 y'}}{1 - \dfrac{y'}{x'+c}\,\dfrac{b^2 x'}{a^2 y'}}$

$$= \frac{a^2 y'^2 + b^2 x'^2 + b^2 c x'}{y'\{(x'+c)a^2 - b^2 x'\}} = \frac{a^2 b^2 + b^2 c x'}{y'\{(a^2 - b^2)x' + a^2 c\}}$$

$$= \frac{b^2 (a^2 + c x')}{c y' (a^2 + c x')} = \frac{b^2}{c y'}.$$

To pass from tan. S P T to tan. H P T we must put $-c$ for c in the preceding investigation; this would evidently lead us to the equation tan. H P T $= -\frac{b^2}{c y'}$; hence, tan. H P z = tan. (180 − H P T) = − tan. H P T $= \frac{b^2}{c y'}$, or the two angles S P T, H P z are equal; thus the tangent makes equal angles with the focal distances.

It is a property of light that, if a ray proceeding from H in the direction H P be reflected by the line z P y, the angle S P y of the reflected ray will equal the angle H P z. Now, in the ellipse, these angles are equal;

hence, if a light be placed at H, all rays which are reflected by the ellipse will proceed to S. Hence, these points, S and H, are called foci.

This very important property is also thus proved from article 119.

$$S y = p = b \sqrt{\frac{r}{r'}}; \text{ and } H z = p' = b \sqrt{\frac{r'}{r}};$$

$$\therefore S y : H z :: r : r' :: SP : HP;$$

hence the triangles S P y and H P z are similar, and the angle S P y equal to the angle H P z.*

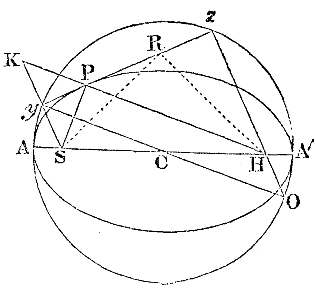
122. To find the length of the perpendicular C u, from the centre, on the tangent:

$$p = -\frac{y_1 - \alpha x_1 - d}{\sqrt{1 + \alpha^2}};$$

here $y_1 = 0$, $x_1 = 0$, $\alpha = -\frac{b^2 x'}{a^2 y'}$, and $d = \frac{b^2}{y'}$;

$$\therefore C u = \frac{\frac{b^2}{y'}}{\sqrt{\left\{1 + \frac{b^4 x'^2}{a^4 y'^2}\right\}}} = \frac{a^2 b^2}{\sqrt{\{a^4 y'^2 + b^4 x'^2\}}} = \frac{a^2 b^2}{a b \sqrt{a^2 - e^2 x'^2}} \quad (119)$$

$$= \frac{a b}{\sqrt{(a + e x')(a - e x')}} = \frac{a b}{\sqrt{r r'}}.$$



* The following geometrical method of drawing a tangent to the ellipse, and proving that the locus of the perpendicular from the focus on the tangent is the circumscribing circle, will be found useful.

Let A P A′ be the ellipse, P any point on it. Join S P and H P, and produce H P to K, making P K = P S; bisect the angle K P S by the line y P z, and join S K, cutting P y in y.

1. P y is a tangent to the ellipse; for if R be any other point in the line P y, we have S R + R H = K R + R H, greater than K H, greater than 2 a; hence, R and every other point in z P y except P is without the ellipse.

2. The locus of y is the circumscribing circle. Draw H z parallel to S y, and join C y; then, because the triangles S P y, K P y are equal, we have the angle S y P a right angle, or S y and H z are perpendicular to the tangent. Also, since S y = K y, and S C = C H, we have C y parallel to K H, and Cy = ½ K H = ½ (S P + P H) = C A.

3. The rectangle S y, H z = the square on B C. Let Z H meet the circle again in O and join C O; then, because the angle y z O is a right angle, and that the points y and O are in the circumference of the circle, the line y C O must be a straight line, and a diameter. Hence, the triangles C S y, C H O are equal; and the rectangle S y, H z = the rectangle Z H, H O = the rectangle A H, H A′ = the square on B C (108).

4. Let S P = r, H P = 2a − S P = 2a − r, S y = p, and H z = p', then $p^2 = \frac{b^2 r}{2a - r}$;

For by similar triangles, S y : S P :: H z : H P $\therefore p = \frac{r}{2a - r} p'$; and, as above,

$pp' = b^2 \therefore p^2 = \frac{b^2 r}{2a - r}$.

123. To find the locus of u:

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2 \quad (1).$$

$$a^2 y y' + b^2 x x' = a^2 b^2 \quad (2).$$

$$y = \frac{a^2 y'}{b^2 x'} x \ \ldots \ (3), \text{ the equation to C } u.$$

Proceeding, as in (120.), to eliminate x' and y', we arrive at the final equation $b^2 y^2 + a^2 x^2 = (y^2 + x^2)^2$; the locus is an oval meeting the ellipse at the extremities of the axes, and bulging out beyond the curve, something like the lowest of figures 2 in page 44. We shall have occasion to trace this curve hereafter.

124. To find the angle which the distance C P makes with the tangent, we have the equation

$$\text{to C P, } y = \frac{y'}{x'} x; \text{ and to P T, } y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'};$$

hence tan. C P T is found $= \dfrac{a^2 b^2}{c^2 x' y'}$.

125. From C u = C y sin. C y u, we have

$$\frac{a\,b}{\sqrt{r\,r'}} = a \text{ sin. C } y\,u \ \therefore \text{ sin. C } y\,u = \frac{b}{\sqrt{r\,r'}};$$

Also from H z = H P sin. H P z, we have

$$b\sqrt{\frac{r'}{r}} = r' \text{ sin. H P } z \ \therefore \text{ sin. H P } z, = \frac{b}{\sqrt{r\,r'}};$$

$\therefore$ angle C y u = angle H P z, and C y is parallel to H P.

Hence, if C E be drawn parallel to the tangent P T, and meeting H P in E, we have P E = C y = A C.

THE NORMAL.

126. The normal to any point of a curve is a straight line drawn through that point, and perpendicular to the tangent at that point.

To find the equation to the normal P G.

The equation to a straight line through the point P (x' y') is

$$y - y' = \alpha\,(x - x')$$

This line must be perpendicular to the tangent whose equation is

$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'} \quad \therefore \alpha = \frac{a^2 y'}{b^2 x'};$$

and the equation to the normal is

$$y - y' = \frac{a^2 y'}{b^2 x'}\,(x - x').$$

127. To find the points where the normal cuts the axes:

Let $y = 0 \therefore - y' = \frac{a^2 y'}{b^2 x'}(x - x') \therefore x = x' - \frac{b^2 x'}{a^2} = \frac{a^2 - b^2}{a^2} x'$ $= e^2 x' = \text{C G}.$

Let $x = 0 \therefore y = y' - \frac{a^2 y'}{b^2} = - \frac{a^2 - b^2}{b^2} y' = - \frac{a^2 e^2}{b^2} y' = \text{C G}'.$

Hence $\text{S G} = \text{S C} - \text{C G} = a e - e^2 x' = e(a - e x') = e.\ \text{S P}.$

The distance M G, from the foot of the ordinate to the foot of the normal, is called the subnormal:

Its value is $x - x' = - \frac{b^2}{a^2} x'$.

128. From the above values of M G, C G and C G′ we

$$\text{have P G} = \sqrt{\left\{ y'^2 + \frac{b^4}{a^4} x'^2 \right\}} = \sqrt{\left\{ \frac{b^2}{a^2}(a^2 - x'^2) + \frac{b^4 x'^2}{a^4} \right\}}$$

$$= \frac{b}{a}\sqrt{\left\{ a^2 - x'^2 + \frac{b^2 x'^2}{a^2} \right\}} = \frac{b}{a}\sqrt{\left\{ a^2 - \frac{a^2 - b^2}{a^2} x'^2 \right\}}$$

$$= \frac{b}{a}\sqrt{\left\{ a^2 - e^2 x'^2 \right\}} = \frac{b}{a}\sqrt{r r'};$$

and similarly $\text{P G}' = \frac{a}{b}\sqrt{r r'}$, consequently,

The rectangle, P G, P G′ $= r r' =$ the rectangle S P, H P.

The greatest value of the normal is when $x' = 0$; hence, at the extremity of the axis minor, we have the greatest value of the normal $= b$. Similarly, the least value of the normal is at the extremity of the axis major, the value being then $= \frac{b^2}{a}$, or half the Latus Rectum (105.).

Also, $\text{S G}' = \frac{a e}{b}\sqrt{r r'}$, and $\text{G G}' = \frac{a e^2}{b}\sqrt{r r'} \therefore \text{G G}' = e. \text{S G}'.$

If a perpendicular G L be drawn from G upon S P or H P, the triangles P G L, S P y, and H P z, are similar; hence

$$\text{P L} = \text{P G}.\frac{p}{r}, \text{ or } = \text{P G}\frac{p'}{r'} = \frac{b^2}{a} = \tfrac{1}{2} \text{ the Latus Rectum.}$$

129. Since the tangent makes equal angles with the focal distances, the normal, which is perpendicular to the tangent, also makes equal angles with the focal distances. This theorem may be directly proved from the above value of C G; for S G : H G :: S C − C G : H C + C G :: $a e - e^2 x' : a e + e^2 x' :: a - e x' : a + e x'$ :: S P : H P; hence, the angle S P H is bisected by the line P G.—Euclid, VI. 3, or Geometry, ii. 50 *.

THE DIAMETERS.

130. A diameter was defined in (76.) to be a line bisecting a system of parallel chords. We shall now prove that all the diameters of the ellipse

* The absolute values of S P and H P are here taken.—See 109.

are straight lines, and that they pass through the centre, which last circumstance is evident, since no line could bisect every one of a system of parallel chords without itself passing through the centre.

Let $y = \alpha x + c$ be the equation to any chord;

$$a^2 y^2 + b^2 x^2 = a^2 b^2, \text{ the equation to the curve.}$$

Transfer the origin to the bisecting point $x' y'$ of the chord, by putting $y + y'$ for y and $x + x'$ for x, then the equation to the chord becomes $y + y' = \alpha (x + x') + c$ or $y = \alpha x$, since $y' = \alpha x' + c$; also the equation to the curve becomes $a^2 (y + y')^2 + b^2 (x + x')^2 = a^2 b^2$.

To find where the chord intersects the curve, put αx for y in the second equation:

$$\therefore a^2 (\alpha x + y')^2 + b^2 (x + x')^2 = a^2 b^2;$$

$$\text{or, } (a^2 \alpha^2 + b^2) x^2 + 2 (a^2 \alpha y' + b^2 x') x + a^2 y'^2 + b^2 x'^2 = a^2 b^2.$$

But since the origin is at the bisection of the chord, the two values of x must be equal to one another, and have opposite signs, or the second term of the last equation must $= 0$.

$$\therefore a^2 \alpha y' + b^2 x' = 0.$$

This equation gives the relation between x' and y'; and, since it is independent of c, it will be the same for any chord parallel to $y = \alpha x + c$; hence, considering x' and y' as variable, it is the equation to the assemblage of all the middle points, or to their locus.

This equation is evidently that to a straight line passing through the centre. Conversely, any straight line passing through the centre is a diameter.

131. A pair of diameters are called conjugate when each bisects all the chords parallel to the other.

Hence, the axes major and minor are conjugate diameters, and the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, which we have generally employed, is that to the ellipse referred to its centre and rectangular conjugate diameters.

If the curve be referred to oblique co-ordinates, and its equation remains of the same form, that is, containing only x^2, y^2, and constant quantities, the new axes will also be conjugate diameters; for each value of one co-ordinate will give two equal and opposite values to the other. We shall, therefore, pass from the above equation to another referred to oblique conjugate diameters, by determining, through the transformation of co-ordinates, all the systems of axes, for which the equation to the ellipse preserves this same form.

Let the equation be $a^2 y^2 + b^2 x^2 = a^2 b^2$; the formulas for transformation are (57),

$$y = x' \sin. \theta + y' \sin. \theta',$$

$$x = x' \cos. \theta + y' \cos. \theta',$$

$$\therefore a^2 (x' \sin. \theta + y' \sin. \theta')^2 + b^2 (x' \cos. \theta + y' \cos. \theta')^2 = a^2 b^2,$$

$$\text{or } \{a^2 (\sin. \theta')^2 + b^2 (\cos. \theta')^2\} y'^2 + \{a^2 (\sin. \theta)^2 + b^2 (\cos. \theta)^2\} x'^2 + 2 \{a^2 \sin. \theta \sin. \theta' + b^2 \cos. \theta \cos. \theta'\} x' y' = a^2 b^2.$$

In order that this equation may be of the conjugate form, it must not contain the term $x' y'$; but since we have introduced two indeterminate quantities, θ and θ', we are enabled to put the co-efficient of $x' y' = 0$; hence we have the condition

$$a^2 \sin. \theta \sin. \theta' + b^2 \cos. \theta \cos. \theta' = 0;$$

or dividing by $a^2 \cos. \theta \cos. \theta'$,

$$\tan. \theta. \tan. \theta' = -\frac{b^2}{a^2}.$$

Now this condition will not determine both the angles θ and θ', but for any value of the one angle it gives a real value for the other; and hence there is an infinite number of pairs of axes to which, if the curve be referred, its equation is of the required conjugate form.

If, in the next figure, we draw CP making any angle θ with C A', and C D making an angle θ' (whose tangent is $-\frac{b^2}{a^2}\cot. \theta$) with C A', then C P and C D are conjugate diameters. Also since the product of the tangents is negative, if C P be drawn in the angle A' C B, C D must be drawn in the angle B C A.

132. There is no occasion to examine the above equation of condition in the case where θ or $\theta' = 0$, for then we have the original axes; but let us examine whether there are any other systems of rectangular axes.

Let $\theta' = 90^\circ + \theta$, $\therefore \sin. \theta' = \cos. \theta$, and $\cos. \theta' = -\sin. \theta$,

hence the equation of condition becomes

$$(a^2 - b^2) \sin. \theta \cos. \theta = 0,$$

and since, by the nature of the ellipse, a^2 cannot $= b^2$, we must have $\theta = 0$, or $\theta = 90^\circ$, both which values give the original axes again; hence the only system of rectangular diameters is that of the axes. This remark agrees with article 87.

We may observe in the above transformation that, although we have introduced *two* indeterminate quantities θ and θ', it does not follow that we can destroy *two* terms in the transformed equation, unless the values of these quantities are real: for example, if we attempt to destroy any other term as the second, we find $\tan. \theta = \frac{b}{a}\sqrt{-1}$, a value to which there is no corresponding angle θ; hence, in putting the co-efficient of $x' y' = 0$, we adopted the only possible hypothesis.

133. The equation to the curve is now

$$\{a^2 (\sin. \theta')^2 + b^2 (\cos. \theta')^2\} y'^2 + \{a^2 (\sin. \theta)^2 + b^2 (\cos. \theta)^2\} x'^2 = a^2 b^2.$$

If we successively make $y' = 0$, and $x' = 0$, we have the distances from the origin to the points in which the curve cuts the new axes; let these distances be represented by a_1 and b_1, the former being measured along the axis of x', and the latter along the axis of y'; then we have

$$y' = 0, \;\therefore \{a^2 (\sin. \theta)^2 + b^2 (\cos. \theta)^2\} a_1^2 = a^2 b^2,$$
$$x' = 0, \;\therefore \{a^2 (\sin. \theta')^2 + b^2 (\cos. \theta')^2\} b_1^2 = a^2 b^2.$$

And the transformed equation becomes

$$\frac{a^2 b^2}{b_1^2} y'^2 + \frac{a^2 b^2}{a_1^2} x'^2 = a^2 b^2,$$

$$\text{or,} \quad \frac{y'^2}{b_1^2} + \frac{x'^2}{a_1^2} = 1,$$

$$\text{or,} \quad a_1^2 y'^2 + b_1^2 x'^2 = a_1^2 b_1^2,$$

Where the lengths of the new conjugate diameters are $2\, a_1$ and $2\, b_1$.

134. From the transformation we obtain the three following equations:

$$a_1^2 \{a^2 (\sin.\ \theta)^2 + b^2 (\cos.\ \theta)^2\} = a^2 b^2 \quad (1),$$

$$b_1^2 \{a^2 (\sin.\ \theta')^2 + b^2 (\cos.\ \theta')^2\} = a^2 b^2 \quad (2),$$

$$\left.\begin{array}{r} a^2 \sin.\ \theta \sin.\ \theta' + b^2 \cos.\ \theta \cos.\ \theta' = 0, \\ \text{or,} \quad \tan.\ \theta \tan.\ \theta' = -\dfrac{b^2}{a^2} \end{array}\right\} \quad (3)$$

Putting $1 - (\sin.\ \theta)^2$ for $(\cos.\ \theta)^2$ in (1), we have

$$a_1^2 (a^2 - b^2)\ (\sin.\ \theta)^2 = a^2 b^2 - a_1^2 b^2,$$

$$\text{and } a_1^2 (a^2 - b^2)\ (\cos.\ \theta)^2 = a_1^2 a^2 - a^2 b^2,$$

$$\therefore (\tan.\ \theta)^2 = \frac{b^2}{a^2}\ \frac{a^2 - a_1^2}{a_1^2 - b^2}.$$

Putting b_1 for a_1 in this expression, we have the value of $(\tan.\ \theta')^2$, as found from (2)

$$(\tan.\ \theta')^2 = \frac{b^2}{a^2}\ \frac{a^2 - b_1^2}{b_1^2 - b^2};$$

hence by multiplication,

$$(\tan.\theta)^2 (\tan.\theta')^2 = \frac{b^4}{a^4}\ \frac{a^2 - a_1^2}{a_1^2 - b^2}\ \frac{a^2 - b_1^2}{b_1^2 - b^2} = \frac{b^4}{a^4} \text{ from (3}$$

$$\therefore (a^2 - a_1^2)\ (a^2 - b_1^2) = (a_1^2 - b^2)\ (b_1^2 - b^2),$$

$$\text{or, } a^4 - a^2 b_1^2 - a_1^2 a^2 + a_1^2 b_1^2 = a_1^2 b_1^2 - a_1^2 b^2 - b^2 b_1^2 + b^4;$$

$$\therefore\ a^4 - b^4 = a^2 b_1^2 + a_1^2 a^2 - a_1^2 b^2 - b^2 b_1^2,$$

$$= a^2 (a_1^2 + b_1^2) - b^2 (a_1^2 + b_1^2),$$

$$= (a^2 - b^2)\ (a_1^2 + b_1^2),$$

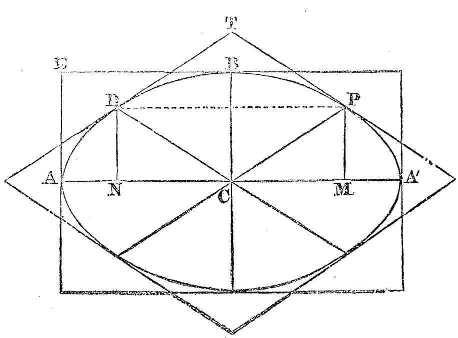
$$\therefore\ a^2 + b^2 = a_1^2 + b_1^2,$$

that is, the sum of the squares upon the conjugate diameters is equal to the sum of the squares upon the axes.

135. Again, multiplying (1) and (2) together, and (3) by itself, and then subtracting the results, we have

$$a_1^2 b_1^2 \{a^4 (\sin.\ \theta)^2 (\sin.\ \theta')^2 + b^4 (\cos.\ \theta)^2 (\cos.\ \theta')^2 + a^2 b^2 (\sin.\ \theta')^2 (\cos.\ \theta)^2 + a^2 b^2 (\sin.\ \theta)^2\ (\cos.\ \theta')^2\} = a^4 b^4,$$

$$a^4 (\sin.\ \theta)^2 (\sin.\ \theta')^2 + b^4 (\cos.\ \theta)^2 (\cos.\ \theta')^2 + 2\ a^2 b^2 \sin.\ \theta \sin.\ \theta' \cos.\ \theta \cos.\ \theta' = 0;$$



$$\therefore\ a_1^2 b_1^2 a^2 b^2 \{ (\sin.\theta')^2 (\cos.\theta)^2 - 2 \sin.\theta \sin.\theta' \cos.\theta \cos.\theta' + (\sin.\theta)^2 (\cos.\theta')^2 \} = a^4 b^4,$$

$$\text{or, } a_1^2 b_1^2 \{ \sin.\theta' \cos.\theta - \sin.\theta \cos.\theta' \}^2 = a^2 b^2$$

$$\text{or, } a_1^2 b_1^2 \{ \sin.(\theta' - \theta) \}^2 = a^2 b^2;$$

$$\therefore\ a_1 b_1 \sin.(\theta' - \theta) = a b.$$

Now $\theta' - \theta$ is the angle P C D, between the conjugate diameters C P and C D; hence drawing straight lines at the extremities of the conjugate diameters, parallel to those diameters, we have, from the above equation, the parallelogram P C D T = the rectangle A C B E, and therefore the whole parallelogram thus circumscribing the ellipse is equal to the rectangle contained by the axes *.

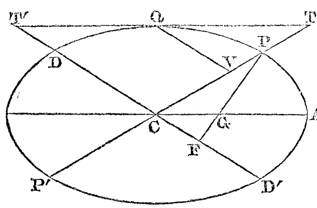
If the extremities of the conjugate diameters be joined, it is readily seen that the inscribed figure is a parallelogram, and that its area is equal to half that of the above circumscribed parallelogram.

We may remark, in passing, that the circumscribed parallelogram, having its sides parallel to a pair of conjugate diameters, is the least of all parallelograms circumscribing the ellipse; and that the inscribed parallelogram, having conjugate lines for its diameters, is the greatest of all inscribed parallelograms.

136. Returning to article (133.), the equation to the curve, suppressing the accents on x' and y', as no longer necessary, is

$$a_1^2 y^2 + b_1^2 x^2 = a_1^2 b_1^2.$$

In the figure, C P $= a_1$, C D $= b_1$ C V $= x$, and V Q $= y$.



* The theorems in articles 134 and 135 may be proved also in the following manner: Referring the curve to its rectangular axes, as in article (138.), let the co-ordinates of P be x' and y'; then the equation to C D is $a^2 y y' + b^2 x x' = 0$, and eliminating x and y between this equation and that to the curve $(a^2 y^2 + b^2 x^2 = a^2 b^2)$, we have the co-ordinates C N and D N, fig. 135, of the intersection of C D with the curve, C N $= x = \frac{a y'}{b}$ and D N $= y = \frac{b x'}{a}$; hence we have

$$a_1^2 + b_1^2 = x'^2 + y'^2 + x^2 + y^2 = x'^2 + y'^2 + \frac{a^2 y'^2}{b^2} + \frac{b^2 x'^2}{a^2} = \frac{b^2 x'^2 + a^2 y'^2}{b^2} + \frac{a^2 y'^2 + b^2 x'^2}{a^2} = \frac{a^2 b^2}{b^2} + \frac{a^2 b^2}{a^2} = a^2 + b^2.$$

Also the triangle P C D = the trapezium P M N D − the triangles P C M and D C N $= (x + x') \frac{y + y'}{2} - \frac{x y + x' y'}{2} = \frac{x' y + y' x}{2} = \frac{1}{2} \left\{ x' \frac{b x'}{a} + y' \frac{a y'}{b} \right\} = \frac{b^2 x'^2 + a^2 y'^2}{2 a b}$ $= \frac{a^2 b^2}{2 a b} = \frac{a b}{2}$, therefore the parallelogram P C D T $= a b$.

No notice has been taken of the positional value of the abscissa C N, since this is entirely a question of absolute values.

Putting the equation into the form

$$y^2 = \frac{b_1^2}{a_1^2}(a_1^2 - x^2) = \frac{b_1^2}{a_1^2}(a_1 - x)(a_1 + x),$$

we have the square upon the ordinate Q V : the rectangle P V, V P′ :: the square upon C D : the square upon C P.

137. The equation to the tangent at any point Q (x' y') found exactly as in (111.) is $a_1^2\, y\, y' + b_1^2\, x\, x' = a_1^2\, b_1^2$.

The points T and T′, where it cuts the new axes, are determined as in (114.); whence $C\,T = \frac{a_1^2}{x'}$, $C\,T' = \frac{b_1^2}{y'}$; and the tangents drawn at the two extremities of a chord meet in the diameter to that chord (114.).

138. Let the ellipse be now referred to its rectangular axes, and let the co-ordinates of P be x' y', then the equation to C P is $y = \frac{y'}{x'}x$, and the equation to C D is

$$y = x \tan.\theta' = -\frac{b^2}{a^2}\cot.\theta = -\frac{b^2\, x'}{a^2\, y'}\, x,$$

$$\text{or,} \quad a^2\, y\, y' + b^2\, x\, x' = 0.$$

But the equation to the tangent at P is

$$a^2\, y\, y' + b^2\, x\, x' = a^2\, b^2;$$

hence CD or the diameter conjugate to C P is parallel to the tangent at P.

From this circumstance the conjugate to any diameter is often defined to be the line drawn through the centre, and parallel to the tangent at the extremity of the diameter.

The equation to the conjugate diameter is readily remembered, since it is the same as that to the tangent without the last term, and therefore may be deduced from the equation to the curve, as at the end of article 111. The three equations are

$$a^2\, y^2 + b^2\, x^2 = a^2\, b^2, \text{ to the curve,}$$

$$a^2\, y\, y' + b^2\, x\, x' = a^2\, b^2, \text{ to the tangent,}$$

$$a^2\, y\, y' + b^2\, x\, x' = 0, \text{ to the conjugate.}$$

The equation to the tangent D T passing through the point D, whose co-ordinates are $\frac{b\,x'}{a}$ and $-\frac{a\,y'}{b}$ (*note* 135), and parallel to C P, is

$$y - \frac{b\,x'}{a} = \frac{y'}{x'}\left(x + \frac{a\,y'}{b}\right)$$

or reducing

$$y\,x' - x\,y' = a\,b.$$

And the equation to C P is

$$y\,x' - x\,y' = 0.$$

These equations to the tangents and conjugate diameters, combined with the equation to the curve, will be found useful in the solution of problems relating to tangents.

139. Let x' and y' be rectangular co-ordinates of P; then, from the equation $a_1^2 + b_1^2 = a^2 + b^2$, we have $b_1^2 = a^2 + b^2 - a_1^2 = a^2 + b^2 - x'^2 - y'^2 = a^2 + b^2 - x'^2 - b^2 + \frac{b^2}{a^2} x'^2 = a^2 - \frac{a^2 - b^2}{a^2} x'^2 = a^2 - e^2 x'^2$
$= (a - ex')(a + ex') = r\,r'$.
That is, the square upon the conjugate diameter C D = the rectangle under the focal distances S P and H P.

140. Draw P F perpendicular upon the conjugate diameter C D, then by (135.) the rectangle P F, C D = ab,

$$\therefore \text{PF} = \frac{ab}{b_1} = \frac{ab}{\sqrt{a^2 + b^2 - a_1^2}} = \frac{ab}{\sqrt{r\,r'}}. \;*$$

It was shown in (128.) that $\text{PG} = \frac{b}{a}\sqrt{r\,r'}$, and $\text{PG}' = \frac{a}{b}\sqrt{r\,r'}$;

hence, The rectangle P G, P F = The square on B C,
and The rectangle P G′, P F = The square on A C,
and The rectangle P G, P G′ = The square on C D.

SUPPLEMENTAL CHORDS.

141. Two straight lines drawn from a point on the curve to the extremities of a diameter are called supplemental chords. They are called principal supplemental chords if that diameter be the axis major.

Referring the ellipse to its axes, let P P′ be a diameter, Q P, Q P′ two supplemental chords; then, if x' y' be the co-ordinates of P, $-x'$, $-y'$ are those of P′; hence, the equation to Q P is $y - y' = \alpha(x - x')$,

and the equation to Q P′ is $y + y' = \alpha'(x + x')$.

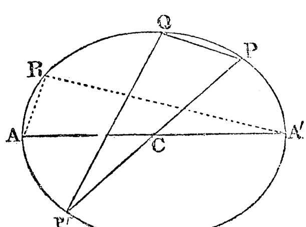
At the point of intersection, y and x are the same for both equations, being the co-ordinates of Q; hence, $y^2 - y'^2 = \alpha\alpha'(x^2 - x'^2)$;

but $a^2 y^2 + b^2 x^2 = a^2 b^2$ at Q,

and $a^2 y'^2 + b^2 x'^2 = a^2 b^2$ at P;

$$\therefore y^2 - y'^2 = -\frac{b^2}{a^2}(x^2 - x'^2);$$

$$\therefore \alpha\alpha' = -\frac{b^2}{a^2};$$



that is, The product of the tangents of the angles, which a pair of supplemental chords makes with the axis major, is constant.

* If the distance C P = u, and p = the perpendicular from the centre on the tangent at P, this equation is

$$p^2 = \frac{a^2 b^2}{a^2 + \quad - u^2}.$$

If the curve was referred to any conjugate diameters, $2\,a_1$ and $2\,b_1$, we should find exactly in the same manner that the product of the tangents of the angles, which a pair of supplemental chords makes with any axis $2\,a_1$, is constant, and equal to $-\frac{b_1^2}{a_1^2}$.

The equation to a chord Q P being $y - y' = \alpha\,(x - x')$, the equation to its supplemental chord Q P′ is $y + y' = -\frac{b^2}{a^2\,\alpha}(x + x')$.

In the circle $b = a$ $\therefore$ $\alpha\,\alpha' = -1$, which proves that in the circle the supplemental chords are at right angles to each other, a well-known property of that figure.

The converse of the proposition is thus proved.

Let A C A′ be any diameter, C the origin, and $\alpha\,\alpha' = -\frac{b_1^2}{a_1^2}$, then the equation to A R is $y = \alpha\,(x + a_1)$ (1), and the equation to A′R is $y = \alpha'\,(x - a_1) = -\frac{b^2}{a_1^2\,\alpha}(x - a_1)$ (2). To find the intersection of the lines A R and A′R, let y and x be the same for (1) and (2), and eliminate α by multiplication; hence,

$y^2 = -\frac{b_1^2}{a_1^2}(x^2 - a_1^2)$; or $a_1^2\,y^2 + b_1^2\,x^2 = a_1^2\,b_1^2$, and the locus of R is an ellipse whose axes are $2\,a_1$ and $2\,b_1$.

142. The equation $\alpha\,\alpha' = -\frac{b^2}{a^2}$ is remarkable, as showing that $\alpha\,\alpha'$ is the same, not only for different pairs of chords drawn to the extremities of the same diameter, but also for pairs of chords drawn to the extremities of any diameter; hence, if from the extremity of the axis major we can draw one chord A R parallel to Q P′, the supplemental chord R A′ will be parallel to Q P: this is possible in all cases, except when one chord is parallel or perpendicular to the axis.

143. To find the angle between two supplemental chords.

Let x, y be the co-ordinates of Q, and $x'\,y'$ those of P,

$$\text{Then tan. P Q P}' = \frac{\alpha - \alpha'}{1 + \alpha\,\alpha'} = \frac{\dfrac{y - y'}{x - x'} - \dfrac{y + y'}{x + x'}}{1 - \dfrac{b^2}{a^2}} = \frac{2.a^2}{a^2 - b^2}\;\frac{x'\,y - y'\,x}{x^2 - x'^2};$$

$$\text{or,} = -\frac{2\,b^2}{a^2 - b^2}\;\frac{x'\,y - y'\,x}{y^2 - y'^2}.$$

For the principal supplemental chords, we have $x' = a$, $y' = 0$;

$$\therefore \text{tan. A R A}' = -\frac{2\,b^2}{a^2 - b^2}\;\frac{a}{y}.$$

This value of the tangent being negative, the angle A R A′ is always obtuse, which is also evident, since all the points on the ellipse are within the circumscribing circle.

As y increases, the numerical value of the tangent decreases, or the angle increases (since the greater the obtuse angle, the less is its tangent); hence, the angle is a maximum when y is, that is, when $y = b$. This shows that the angle A B A′ is the greatest angle contained by the principal supplemental chords, and therefore by any supplemental chords. Also, its supplement B A B′ is the least angle contained by any supplemental chords. The angle between the chords being thus limited by the angles A′B A, B A B′, of which the former is greater, and the latter less, than a right angle, chords may be drawn containing any angle between these limits. This is done by describing a segment of a circle, containing the given angle, upon any diameter, except the axis, and joining the extremities of the diameter with the points of intersection of the ellipse and circle. Also, from the value of tan. P Q P′, it appears that, if the angle be a right angle, the two chords are perpendicular to the axes.

144. It was shown in (131.) that if θ and θ' were the angles which conjugate diameters make with the axis major, $\tan.\,\theta.\,\tan.\,\theta' = -\frac{b^2}{a^2}$, but α, α' being tangents of the angles which two supplemental chords make with the same axis, we have $\alpha\,\alpha' = -\frac{b^2}{a^2}$; $\therefore \tan.\,\theta.\,\tan.\,\theta' = \alpha\,\alpha'$; hence, if $\tan.\,\theta = \alpha$, we have $\tan.\,\theta' = \alpha'$; or if one diameter be parallel to any chord, the conjugate diameter is parallel to the supplemental chord.

145. Since supplemental chords can be drawn containing any angle within certain limits, conjugate diameters parallel to these chords may be drawn containing any given angle within the same limits.

Also, since the angle between the principal supplemental chords is always obtuse, the angle P C D between the conjugate diameters is also obtuse, and is the greatest when they are parallel to A B and A B′. In this case, being symmetrically situated with respect to the axes, they are equal to one another.

The magnitude of the equal conjugate diameters is found from the equation $a_1^2 + b_1^2 = a^2 + b^2$, $\therefore a_1^2 = \frac{a^2 + b^2}{2}$.

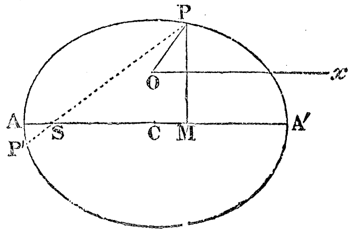
The equation to the ellipse referred to its equal conjugate diameters is $y^2 + x^2 = a_1^2$; however, this must not be confounded with the equation to the circle, which only assumes this form when referred to *rectangular* axes.

THE POLAR EQUATION.

146. Instead of an equation between rectangular co-ordinates x and y, we may obtain one between polar co-ordinates u and θ.

Let the curve be referred to the centre C, and to rectangular axes, and let the co-ordinates of the pole O be x' and y', θ the angle which the radius vector O P, or u, makes with a line O x parallel to the axis of x; then, by (61.), or by inspection of the figure, we have

$$y = y' + u \sin.\,\theta$$
$$x = x' + u \cos.\,\theta.$$



also $a^2 y^2 + b^2 x^2 = a^2 b^2$;

$\therefore$ by substitution, $a^2 (y' + u \sin. \theta)^2 + b^2 (x' + u \cos. \theta)^2 = a^2 b^2$;

Whence u may be found in terms of θ and constant quantities.

147. Let the centre be the pole:

$$\therefore x' = 0 \text{ and } y' = 0,$$

$$\therefore a^2 u^2 (\sin. \theta)^2 + b^2 u^2 (\cos. \theta)^2 = a^2 b^2;$$

$$u^2 = \frac{a^2 b^2}{a^2 (\sin. \theta)^2 + b^2 (\cos. \theta)^2} = \frac{a^2 b^2}{a^2 (\sin. \theta)^2 + (a^2 - a^2 e^2)(\cos. \theta)^2}$$

$$= \frac{a^2 b^2}{a^2 - a^2 e^2 (\cos. \theta)^2} = \frac{a^2 (1 - e^2)}{1 - e^2 (\cos. \theta)^2}.$$

148. Let the focus S be the pole:

$$\therefore y' = 0,\ x' = -ae = -c, \text{ and } u \text{ becomes } r;$$

hence the transformed equation (146.) becomes

$$a^2 (r \sin. \theta)^2 + b^2 (-c + r \cos. \theta)^2 = a^2 b^2;$$

$$\therefore a^2 r^2 (\sin. \theta)^2 + b^2 r^2 (\cos. \theta)^2 - 2 b^2 r c \cos. \theta + b^2 c^2 = a^2 b^2;$$

or, $a^2 r^2 (\sin. \theta)^2 + a^2 r^2 (\cos. \theta)^2 - c^2 r^2 (\cos. \theta)^2 - 2 b^2 r c \cos. \theta = a^2 b^2 - b^2 c^2 = b^4$ since $a^2 - b^2 = c^2$.

$$\text{or, } a^2 r^2 = c^2 r^2 (\cos. \theta)^2 + 2 b^2 r c \cos. \theta + b^4$$

$$= (c r \cos. \theta + b^2)^2;$$

$$\therefore a r = c r \cos. \theta + b^2$$

$$r = \frac{b^2}{a - c \cos. \theta} = \frac{a^2 (1 - e^2)}{a - a e \cos. \theta} = \frac{a (1 - e^2)}{1 - e \cos. \theta}.$$

149. Let any point on the curve be the pole:
Expanding the terms of the polar equation in (146.), and reducing by means of the equation $a^2 y'^2 + b^2 x'^2 = a^2 b^2$, we have

$$u = -2 \frac{a^2 y' \sin. \theta + b^2 x' \cos. \theta}{a^2 (\sin. \theta)^2 + b^2 (\cos. \theta)^2}.$$

If the pole is at A, we have $y' = 0$, and $x' = -a$,

$$\therefore u = \frac{2 b^2 a \cos. \theta}{a^2 (\sin. \theta)^2 + b^2 (\cos. \theta)^2} = \frac{2 a (1 - e^2) \cos. \theta}{1 - e^2 (\cos. \theta)^2}.$$

150. When the focus is the pole, the equation is often obtained directly from some known property of the curve.

Let $SP = r$, $CM = x$, and $ASP = \theta$,

$$\text{then } SP = a + e x \ (109.)$$

$$= a + e (SM - SC)$$

$$= a + e (-r \cos. \theta - a e)$$

$$\therefore r + e r \cos. \theta = a - a e^2 \text{ and } r = \frac{a (1 - e^2)}{1 + e \cos. \theta}.$$

This is the equation generally used in astronomy, the focus S being the place of the sun, and the ellipse the approximate path of the planet.

Let $a\,(1 - e^2) = \frac{b^2}{a} = p$, where p is the parameter. (105.)

Then the last equation may be written under the following forms:

$$r = \frac{p}{2} \cdot \frac{1}{1 + e\cos.\theta} = \frac{p}{2} \cdot \frac{1}{1 - e + 2e\left(\cos.\frac{\theta}{2}\right)^2}$$

$$= \frac{p}{2} \cdot \frac{1}{(1+e)\left(\cos.\frac{\theta}{2}\right)^2 + (1-e)\left(\sin.\frac{\theta}{2}\right)^2}$$

If θ be measured, not from S A, but from a line passing through S, and making an angle α with S A, the polar equation is

$$r = \frac{p}{2} \cdot \frac{1}{1 + e\cos.(\theta - \alpha)}.$$

151. If P S meet the curve again P′, let S P′ $= r'$,

$$\text{then } r = \frac{p}{2} \cdot \frac{1}{1 + e\cos.\theta}$$

$$\text{and } r' = \frac{p}{2} \cdot \frac{1}{1 + e\cos.(\pi - \theta)} = \frac{p}{2} \cdot \frac{1}{1 - e\cos.\theta};$$

$$\therefore r + r' = \mathrm{PP'} = \frac{p}{1 - e^2(\cos.\theta)^2}$$

$$\text{and } r\,r' = \frac{p^2}{4} \cdot \frac{1}{1 - (e^2\cos.\theta)^2} = \frac{p}{4}(r + r')$$

or the rectangle S P, S P′ $= \frac{1}{4}$ of the rectangle under the principal parameter and focal chord.

152. Let CD, or b_1, be the semi-diameter parallel to S P, then (147.)

$$b_1^2 = \frac{a^2(1 - e^2)}{1 - e^2(\cos.\theta)^2} = \frac{a}{2}\,\frac{p}{1 - e^2(\cos.\theta)^2} = \frac{a}{2}(r + r'),$$

$$\therefore r + r = \frac{2\,b_1^2}{a};$$

that is, a focal chord at any point P, is a third proportional to the axis major and diameter to that chord.

CHAPTER IX.

THE HYPERBOLA.

153. In the discussion of the general equation of the second order, we observed that, referring the curve to the centre and rectangular axes, the equation to the hyperbola assumed the form

$$\left(\frac{a'}{-f'}\right)y^2 + \left(\frac{c'}{-f'}\right)x^2 = 1$$

where the co-efficients have different signs, 85. 86.

Let $\left(\frac{a'}{-f'}\right)$ be negative, then the equation becomes

$$-\left(\frac{a'}{-f'}\right)y^2 + \left(\frac{c'}{-f'}\right)x^2 = 1;$$

$$\text{or } P\,y^2 - Q\,x^2 = -1.$$

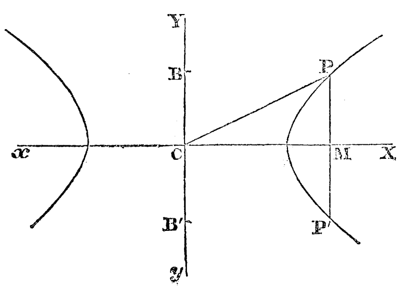
We now proceed to investigate this equation, and to deduce from it all the properties of the hyperbola.

154. Let the curve be referred to its centre C, and rectangular axes X x, Y y, meeting in C; C M $= x$, and M P $= y$; then, at the point where the curve cuts the axes, we have

$$y = 0,\ Q\,x^2 = 1,\ \therefore\ x = \pm\frac{1}{\sqrt{Q}},$$

$$x = 0,\ P\,y^2 = -1,\ \therefore\ y = \pm\sqrt{\frac{-1}{P}}$$

In the axis of x take C A $= \frac{1}{\sqrt{Q}}$, and CA′ $= -\frac{1}{\sqrt{Q}}$, and the curve cuts the axis X x in A and A: Since the value of y is impossible, the other axis never meets the curve; nevertheless we mark off two points, B and B′, in that axis, whose distances from C are C B $= +\frac{1}{\sqrt{P}}$ and C B′ $= -\frac{1}{\sqrt{P}}$.



Also if C A $= a$, and C B $= b$, we have $Q = \frac{1}{a^2}$, $P = \frac{1}{b^2}$; therefore the equation to the curve becomes

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1;$$

$$\text{or } a^2 y^2 - b^2 x^2 = -a^2 b^2;$$

$$\text{or } y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

155. From the last equation we have

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \ (1) \text{ and } x = \pm \frac{a}{b}\sqrt{y^2 + b^2} \ (2).$$

From (1) if x be less than $\pm a$, y is impossible; if, therefore, lines be drawn through A and A′, parallel to C Y, no part of the curve is found between these lines.

Again, for every value of x, greater than a, we have two real and equal values of y; that is, for any abscissa C M, greater than C A, we have two equal and opposite ordinates, M P, M P′.

Also as x increases from a to ∞, these values of y increase from 0 to $\pm\infty$; hence, we have two arcs A P, A P′, exactly equal and opposite to each other, and extending themselves indefinitely.

If x be negative, x^2 being positive, the same values of y must recur; hence, there are again two equal and opposite arcs which form another branch extending from A′ to ∞; thus the whole curve is divided into two equal parts by the axis of x.

From (2) it appears to be divided into two equal parts by the axis of y; hence it is symmetrical with respect to the axes; and its concavity is turned towards the axis of x, otherwise it might be cut by a straight line in more points than two, (71.)

156. If P be any point on the curve, we have

$$C P = \sqrt{x^2 + y^2} = \sqrt{x^2 + \frac{b^2}{a^2}(x^2 - a^2)} = \sqrt{\frac{a^2 + b^2}{a^2}x^2 - b^2};$$

hence C P is least when x is least, that is, when $x = a$, in which case C P becomes also equal to a; hence C A, or C A′, is the least line that can be drawn from the centre to the curve: thus, the axis A A′ is the least line that can be drawn through the centre to meet the curve. The other axis, B B′, never meets the curve.

In the equation $P y^2 - Q x^2 = -1$, the imaginary axis may be greater or less than the real one, according as Q is greater or less than P; hence the appellation of axis major cannot be generally applied to the real axis of the curve. In this treatise we shall call A A′ the transverse axis, and B B′ the conjugate axis.

157. The points A, A′ are called the vertices, or summits of the curve: either of these points may be taken for the origin by making proper substitutions.

Let A be the origin, A M $= x'$;

$$\text{Then } x = C M = C A + A M = a + x';$$

$$\therefore y^2 = \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2}{a^2}\{(a + x')^2 - a^2\} = \frac{b^2}{a^2}\{2\,a\,x' + x'^2\},$$

or, suppressing accents, $y^2 = \frac{b^2}{a^2}(2\,a\,x + x^2) = \frac{b^2}{a^2}\,x\,(2\,a + x)$.

This last equation is geometrically expressed by the following proportion:

The square upon M P : rectangle A M, M A′ :: the square upon B C : the square upon A C.

If the origin be at A′, the equation is $y^2 = \frac{b^2}{a^2}(x^2 - 2\,a\,x)$.

158. If $a = b$, the equation to the hyperbola becomes $y^2 - x^2 = -a^2$; this curve is called the equilateral hyperbola, and has, to the common hyperbola, the same relation that the circle has to the ellipse.

159. The analogy between the ellipse and hyperbola will be found to be very remarkable; the equations to the two curves differ only in the sign of b^2; for if, in the equation to the ellipse $a^2\,y^2 + b^2\,x = a^2\,b^2$, we put $-\,b^2$ for b^2, we have the equation to the hyperbola: hence we might conclude that many of the algebraical results found in the one curve will be true for the other, upon changing b^2 into $-\,b^2$ in those results; and in fact this is the case, the same theorems are generally true for both, and may be proved in the same manner: for this reason we shall not enter at length into the demonstration of all the properties of the hyperbola, but merely put down the enunciations and results, with a reference at the end of each article to the corresponding one in the ellipse, except in those cases where there may be any modification required in the working. To prevent any doubt about the form of the figure, we shall insert figures in those places where they may be wanted; and, with this assistance, we trust that the present plan will offer no difficulty.

THE FOCUS.

160. The equation $y^2 = \frac{b^2}{a^2}(2\,a\,x + x^2)$ may be put under the form $y^2 = l\,x + \frac{l}{2\,a}\,x^2$, in which case the quantity $l = \frac{2\,b^2}{a}$ is called the principal parameter, or the Latus Rectum.

Since $l = \frac{2\,b^2}{a} = \frac{4\,b^2}{2\,a}$, the Latus Rectum is a third proportional to the transverse and conjugate axes.

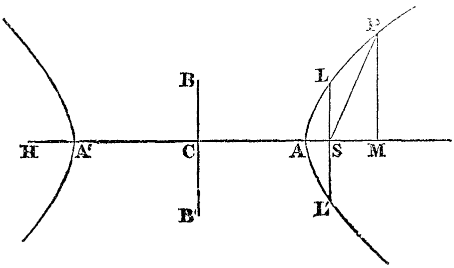
161. To find from what point in the transverse axis a double ordinate can be drawn equal to the Latus Rectum,

$$\text{Here } 4\,y^2 = l^2, \text{ or } \frac{4\,b^2}{a^2}(x^2 - a^2) = \frac{4\,b^4}{a^2};$$

$$\therefore x^2 - a^2 = b^2.$$

$$\text{or, } x^2 = a^2 + b^2;$$

$$\therefore x = \pm\sqrt{a^2 + b^2}.$$



Join A B, then $AB = \sqrt{a^2 + b^2}$; with centre C and radius A B describe a circle cutting the transverse axis in the points S and H, we have then $CS = \sqrt{a^2 + b^2}$, and $CH = -\sqrt{a^2 + b^2}$; thus S and H are the points through either of which, if an ordinate as L S L′ be drawn, it is equal to the Latus Rectum.

The two points S and H, thus determined, are called the foci.

162. The fraction $\frac{\sqrt{a^2 + b^2}}{a}$, which represents the ratio of C S to C A, is called the eccentricity: if this quantity, which is evidently greater than unity, be represented by the letter e, we have $\sqrt{a^2 + b^2} = a\,e$, whence $e^2 = \frac{a^2 + b^2}{a^2} = 1 + \frac{b^2}{a^2}$; $\therefore \frac{b^2}{a^2} = e^2 - 1$, and the equation to the hyperbola may be put under the form

$$y^2 = (e^2 - 1)(x^2 - a^2).$$

163. Since $a^2 + b^2 = a^2 e^2$, we have $b^2 = a^2 e^2 - a^2 = (a e - a)(a e + a)$;

Or the rectangle A S, S A′ = the square upon B C.

164. To find the distance from the focus to any point P in the curve, proceeding exactly as in (109.) we find

$$SP = e x - a, \quad HP = e x + a;$$

Hence $HP - SP = 2a = AA'$, that is the difference of the distances of any point in the curve from the foci is equal to the transverse axis.

165. Conversely, To find the locus of a point, the difference of whose distances from two fixed points S and H is constant or equal $2a$.

If $SH = 2c$, the locus is an hyperbola, whose axes are $2a$ and $2\sqrt{a^2 + c^2}$, and whose foci are S and H. (110.)

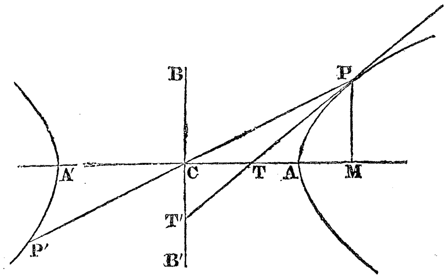
THE TANGENT.

166. To find the equation to the tangent at any point P $(x'\,y')$,

The required equation obtained as in (111.) is

$$a^2 y y' - b^2 x x' = -a^2 b^2.$$

This form is easily recollected, since it may be obtained from the equation to the curve $a^2 y^2 - b^2 x^2 = - a^2 b^2$, by putting $y\,y'$ for y^2, and $x\,x'$ for x^2.



167. To find the points where the tangent cuts the axes;

Let $y = 0$, $\therefore x = \dfrac{a^2}{x'} = \text{C T}$; similarly $y = \text{C T}' = - \dfrac{b^2}{y'}$; hence we have

The rectangle C T, C M = the square upon A C;
and The rectangle C T′, M P = the square upon B C.

Since C T $\left(= \dfrac{a^2}{x'}\right)$ is always less than C A, the tangent to any point of the branch P A cuts the transverse axis between C and A.

The subtangent M T $= x' - \dfrac{a^2}{x'} = \dfrac{x'^2 - a^2}{x'}$. (115.)

The tangent at the extremity A of the transverse axis is perpendicular to that axis (116.).

If P C be produced to meet the curve again in P′, the tangents at P and P′ will be found to be parallel (116.).

168. To find the equation to the tangent at the extremity of the Latus Rectum,

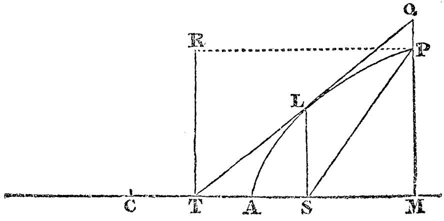
Generally the equation to the tangent is

$$a^2 y\,y' - b^2 x\,x' = - a^2 b^2;$$

at L, $x' = a\,e$, $y' = \dfrac{b^2}{a}$;

$$\therefore a^2 y \frac{b^2}{a} - b^2 x\,a\,e = - a^2 b^2,$$

$$y = e\,x - a.$$



Let the ordinate y, or M Q, cut the curve in P, then we have S P $= e\,x - a$ (164.).

$$\therefore \text{M Q} = \text{S P}:$$

Also C T $= \dfrac{a}{e}$, hence from T draw T R perpendicular to A C, and from P draw P R parallel to A C, then we have

$$\text{P R} = \text{M T} = \text{M C} - \text{C T} = x - \frac{a}{e} = \frac{e\,x - a}{e} = \frac{1}{e}.\text{S P}.$$

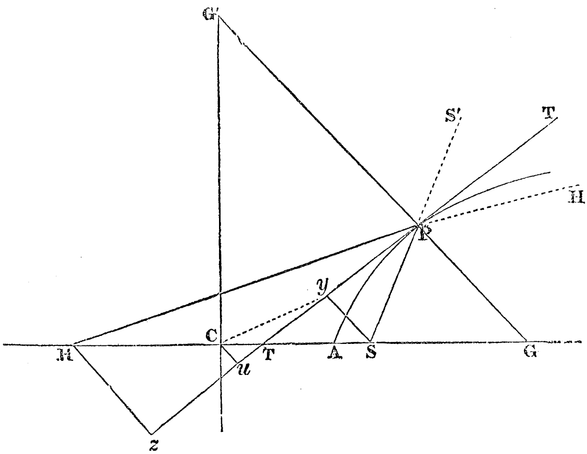
Consequently, the distances of any point P from S, and from the line T R, are in the constant ratio of $e : 1$.

The line T R is called the directrix.

If $x = 0$, we have $y = -a$; hence the tangent at the extremity of the Latus Rectum cuts the axis of y at the point where the circle on the transverse axis cuts the axis of y.

169. To find the length of the perpendicular from the focus on the tangent.

Let S y, H z be the perpendiculars on the tangent P T.



Taking the expression in (48.) we have

$$p = -\frac{y_1 - \alpha\,x_1 - d}{\sqrt{1 + \alpha^2}}$$

here $y_1 = 0$ and $x_1 = a\,e$ are co-ordinates of the point S, and $y = \alpha\,x + d$ is the equation to P y; but the equation to P y (166.) is also

$$y = \frac{b^2\,x'}{a^2\,y'}\,x - \frac{b^2}{y'}\ \therefore\ \alpha = \frac{b^2\,x'}{a^2\,y'} \text{ and } d = -\frac{b^2}{y'};$$

$$p = -\frac{-\dfrac{b^2\,x'}{a^2\,y'}\,a\,e + \dfrac{b^2}{y'}}{\sqrt{\left\{1 + \dfrac{b^4\,x'^2}{a^4\,y'^2}\right\}}} = +\frac{a\,b^2\,(e\,x' - a)}{\sqrt{\{a^4\,y'^2 + b^4\,x'^2\}}}$$

$$= \frac{a\,b^2\,(e\,x' - a)}{a\,b\,\sqrt{\{e^2\,x'^2 - a^2\}}} = b\sqrt{\frac{e\,x' - a}{e\,x' + a}}.$$

Let $SP = r$, and $HP = 2a + r = r'$ $\therefore p = b\sqrt{\frac{r}{r'}}$, or $p^2 = b^2 \frac{r}{2a+r}$

Similarly if $Hz = p'$, we have $p'^2 = b^2 \frac{r'}{r}$.

By multiplication we have $p\,p' = b^2$; : hence

The rectangle Sy, Hz = the square upon BC.

170. To find the locus of y or z in the last article.

The equation to the curve at P is $a^2 y'^2 - b^2 x'^2 = - a^2 b^2$

The equation to the tangent at P is $a^2 y\,y' - b^2 x\,x' = - a^2 b^2$.

$$\text{The equation to } S\,y \text{ is } y = \frac{-a^2 y'}{b^2 x'}(x - c)$$

By eliminating x' and y', exactly as in (120.), we arrive at the equation

$$a^2 = y^2 + x^2;$$

Hence the locus of y is a circle described on the transverse axis as diameter.

171. To find the angle which the focal distance S P makes with the tangent P T.

The equation to the tangent is $y = \frac{b^2 x'}{a^2 y'} x - \frac{b^2}{y'}$, and the equation to S P is, $y - y' = \frac{y'}{x' - c}(x - x')$,

$$\text{hence } \tan.\ SPT = \tan.\ (PSX - PTX)$$

$$= \frac{\frac{y'}{x' - c} - \frac{b^2 x'}{a^2 y'}}{1 + \frac{b^2 x'}{a^2 y'} \frac{y'}{x' - c}} = \frac{a^2 y'^2 - b^2 x'^2 + b^2 c x'}{a^2 y' x' - a^2 c y' + b^2 x' y'}$$

$$= \frac{b^2 (c x' - a^2)}{y' c (c x' - a^2)} = \frac{b^2}{c y'}.$$

Similarly $\tan.\ HPT = \frac{b^2}{c y'}$, $\therefore$ the angles SPT, HPT are equal; thus the tangent makes equal angles with the focal distances.

Produce SP to S′, then it is a property of light, that if a ray proceeding from H be reflected by the line TPT′, the angle S′PT′ of the reflected ray will equal the angle HPT. Now, in the hyperbola, these angles are equal; hence if a light be placed at H, all rays which are incident on the curve will be reflected as if diverging from S; or if a body of rays proceeding to S be incident on the curve, they will converge to H. Hence these points S and H are called foci.

This important property of the curve is also thus proved from article (169.),

$$S\,y = p = b\sqrt{\frac{r}{r'}}, \text{ and } H\,z = p' = b\sqrt{\frac{r'}{r}};$$

$$\therefore S\,y : H\,z :: r \,..\, r' :: S\,P : H\,P;$$

$\therefore$ angle $S\,P\,y = H\,P\,z$, and the tangent makes equal angles with the focal distances *.

172. To find the length of the perpendicular C u from the centre on the tangent.

$$p = -\frac{y_1 - \alpha\,x_1 - d}{\sqrt{1+\alpha^2}},$$

here $y_1 = 0$, $x_1 = 0$, $\alpha = \frac{b^2\,x'}{a^2\,y'}$, and $d = -\frac{b^2}{y'}$, $\therefore$ C $u = \frac{a\,b}{\sqrt{r\,r}}$

173. To find the locus of u.

The equation to C u is $y = -\frac{a^2\,y'}{b^2\,x'}\,x$, eliminating x' y' from this equation, and the equation to the tangent, we find, as in (123.), the resulting equation to be $a^2\,x^2 - b^2\,y^2 = (x^2+y^2)^2$, which cannot be discussed at present.

174. From the equation to the tangent, and that to C P, we find, as in (124.),

$$\tan.\ C\,P\,T = \frac{a^2\,b^2}{c^2\,x'\,y'}.$$

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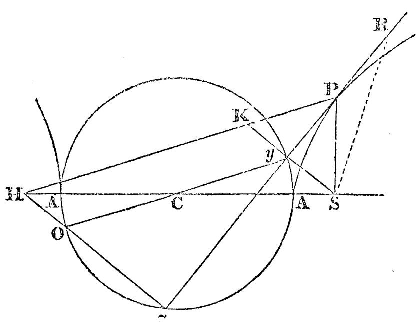
* The following geometrical method of drawing a tangent to the hyperbola, and proving that the locus of the perpendicular from the focus on the tangent is the circle on the transverse axis, will be found useful.



Let A P be the hyperbola, P any point on it; join S P and H P, and in H P take P K = P S; bisect the angle S P K by the line P y z, and join S K, cutting P y in y.

1. P y is a tangent to the hyperbola; for if R be any other point in the line P y, we have H R − S R = H R − K R is less than H K (Geom. i. 10) less than $2\,a$, hence R, and every other point in P y, is without the curve.

2. The locus of y is the circle on the transverse axis: draw H z parallel to S y, and join C y; then, because the triangles S P y, K P y are equal, we have the angle S y P a right angle, or S y and H z are perpendicular to the tangent. Also since S y = K y, and S C = C H, we have C y parallel to H K, and C $y = \frac{1}{2}$ H K $= \frac{1}{2}$ (H P − S P) = C A.

3. The rectangle S y, H z = the square on B C. Let z H meet the circle again in O, and join C O; then the line O C y is a straight line and a diameter, hence the triangles C S y, C H O are equal, and the rectangle S y, H z = the rectangle H O, H z = the rectangle H A′, H A = the square upon B C.

4. Let S P = r, H P = $2\,a + r$, S $y = p$ and H $z = p'$, then $p^2 = \frac{b^2\,r}{2\,a + r}$; for by similar triangles, S y : S P :: H z : H P, $\therefore p = \frac{r}{2\,a+r}\,p'$, and, as above, $p\,p' = b^2$, $\therefore p^2 = \frac{b^2\,r}{2\,a+r}$.

From $C u = C y \sin. C y u$, we have

$$\frac{a b}{\sqrt{r r'}} = a \sin. C y u \therefore \sin. C y u = \frac{b}{\sqrt{r r'}}.$$

Also from $H z = H P \sin. H P z$, we have

$$b \sqrt{\frac{r'}{r}} = r' \sin. H P z, \therefore \sin. H P z = \frac{b}{\sqrt{r r'}},$$

$\therefore$ angle $C y u =$ angle $H P z$ and $C y$ is parallel to H P.

And if C E be drawn parallel to the tangent P T, and meeting H P in E, we have $P E = C y = A C$.

THE NORMAL.

175. The equation to the line passing through the point P $(x' y')$, and perpendicular to the tangent $\left(y = \frac{b^2 x'}{a^2 y'} x - \frac{b^2}{y'}\right)$ is

$$y - y' = - \frac{a^2 y'}{b^2 x'} (x - x').$$

To find where the normal P G cuts the axes.

$$\text{Let } y = 0 \therefore - y' = - \frac{a^2 y'}{b^2 x'} (x - x') \therefore x = x' + \frac{b^2 x'}{a^2}$$

$$= \frac{a^2 + b^2}{a^2} x' = e^2 x' = C G.$$

$$\text{Let } x = 0 \therefore y = y' + \frac{a^2 y'}{b^2} = \frac{a^2 + b^2}{b^2} y' = \frac{a^2 e^2}{b^2} y' = C G'.$$

Also the subnormal $M G = x - x' = \frac{b^2 x'}{a^2}$; and $S G = e. S P$.

176. From the above values of C G, C G′, and M G′, we may demonstrate that $P G = \frac{b}{a} \sqrt{r r'}$, $P G' = \frac{a}{b} \sqrt{r r'}$, and consequently that

The rectangle P G, P G′ $= r r' =$ the rectangle S P, H P.

Also $S G' = \frac{a e}{b} \sqrt{r r'}$, $G G' = \frac{a e^2}{b} \sqrt{r r'}$, and $\therefore G G' = e . S G'$.

177. Since the tangent makes equal angles with the focal distances, the normal, which is perpendicular to the tangent, also makes equal angles with the focal distances, one of them being first produced as to H′. This theorem may be directly proved from the above value of C G; for $S G : H G :: e^2 x' - a e : e^2 x' + a e :: e x' - a : e x' + a :: S P : H P$, hence the angle S P H′ is bisected by the line P G.

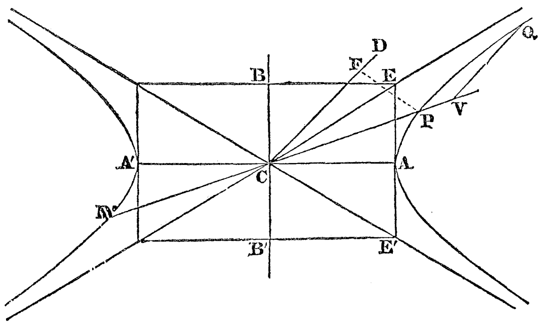
THE DIAMETERS.

178. It may be proved as for the ellipse (130.), that all the diameters of the hyperbola pass through the centre, and that any line through the

centre is a diameter. If $y = \alpha x + c$ be the equation to any chord, $a^2 x y - b^2 x = 0$ is the equation to the diameter bisecting all chords parallel to $y = \alpha x + c$.

179. In the ellipse all the diameters must necessarily meet the curve; but this is not the case in the hyperbola, as will appear by finding the co-ordinates of intersection of the diameter and the curve.

Let $y = \beta x$ be the equation to a diameter C P, and substitute this value of y in the equation to the curve.



$$a^2 y^2 - b^2 x^2 = - a^2 b^2,$$
$$\therefore a^2 \beta^2 x^2 - b^2 x^2 = - a^2 b^2,$$
$$x^2 = \frac{a^2 b^2}{b^2 - a^2 \beta^2},$$
$$\therefore x = \pm \frac{a\ b}{\sqrt{b^2 - a^2 \beta^2}}.$$

These values are impossible, if $a^2 \beta^2$ is greater than b^2, that is, if β is greater than $\frac{b}{a}$; and if $\beta = \pm \frac{b}{a}$, the diameter meets the curve only at an infinite distance. The limits of the intersecting diameters are thus determined; through A, B and B′ draw lines parallel to the axes meeting in E and E′, then tan. E C A $= \frac{b}{a}$, and tan. E′ C A $= - \frac{b}{a}$, hence C E and C E′ produced are the lines required. Hence, in order that a diameter meet the curve, it must be drawn within the angle E C E′; thus the line C D never meets the curve.

The curve is symmetrical with respect to these lines C E, C E′, since the axis bisects the angle E C E′.

180. The hyperbola has an infinite number of pairs of conjugate diameters. This is proved by referring the equation to other axes by means of the formulas of transformation (57.)

$$y = x' \sin. \theta + y' \sin. \theta',$$
$$x = x' \cos. \theta + y' \cos. \theta';$$

hence the equation $a^2 y^2 - b^2 x^2 = - a^2 b^2$ becomes

$$\{a^2 (\sin. \theta')^2 - b^2 (\cos. \theta')^2\} y'^2 + \{a^2 (\sin. \theta)^2 - b^2 (\cos. \theta)^2\} x'^2$$
$$+ 2 \{a^2 \sin. \theta \sin. \theta' - b^2 \cos. \theta \cos. \theta'\} x' y' = - a^2 b^2.$$

In order that this equation be of the conjugate form, let the co-efficient of $y' x' = 0$,

$$\therefore\ a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta' = 0,$$

$$\text{or,}\quad \tan.\theta \tan.\theta' = \frac{b^2}{a^2}.$$

Hence for any value of θ, we have a real value of θ', that is, there is an infinite number of pairs of axes to which, if the curve be referred, its equation is of the required conjugate form.

If $\tan.\theta$ be less than $\frac{b}{a}$, $\tan.\theta'$ must be greater than $\frac{b}{a}$, that is, if one diameter C P, in the last figure, meets the curve, the conjugate diameter C D does not; therefore in each system of conjugate diameters one is imaginary. Also, since the product of the tangents is positive, both angles are acute, or both obtuse; in the figure they are both acute, but for the opposite branch they must be both obtuse.

181. As in article (132.), it appears that there can be only one system of rectangular conjugate diameters.

182. The equation to the curve is now

$$\{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\} y'^2 + \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\} x'^2 = - a^2 b^2.$$

If we successively make $y' = 0$, and $x' = 0$, we have the distances from the origin to the points in which the curve cuts the new axes; but as we already know (180.) that one of these new axes never meets the curve, we must represent one of these distances by an imaginary quantity.

Let the axis of x' meet the curve at a distance a_1 from the centre, and let the length of the other semi-axis be b_1 connected with the symbol $\sqrt{-1}$, that is, let the new conjugate diameters be $2\,a_1$ and $2\,b_1 \sqrt{-1}$, then we have

$$y = 0\ \therefore\ \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\} a_1^2 = - a^2 b^2,$$

$$x = 0\ \therefore\ \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\} (- b_1^2) = - a^2 b^2,$$

And the transformed equation becomes

$$\frac{a^2 b^2}{b_1^2} y'^2 - \frac{a^2 b^2}{a_1^2} x'^2 = - a^2 b^2,$$

$$\text{or,}\quad \frac{y'^2}{b_1^2} - \frac{x'^2}{a_1^2} = - 1,$$

$$\text{or,}\quad a_1^2 y'^2 - b_1^2 x'^2 = - a_1^2 b_1^2.$$

183. From the transformation we obtain the three following equations:

$$a_1^2 \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\} = - a^2 b^2 \qquad (1),$$

$$b_1^2 \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\} = + a^2 b^2 \qquad (2),$$

$$\left.\begin{array}{l} a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta' = 0, \\ \text{or,}\quad \tan.\theta \tan.\theta' = \dfrac{b^2}{a^2} \end{array}\right\} \qquad (3).$$

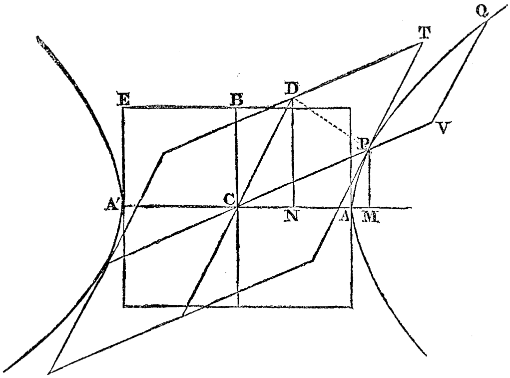
Following the steps exactly as in article (134.), or, which amounts to the same thing, putting $- b^2$ for b^2, and $- b_1^2$ for b_1^2 all through that article, we arrive at the result

$$a_1^2 - b_1^2 = a^2 - b^2,$$

or, the difference of the squares upon the conjugate diameters is equal to the difference of the squares upon the axes.

184. Again, multiplying (1) and (2) together, and (3) by itself, then subtracting the results, and reducing, as in the article (135.), we have

$$a_1 \, b_1 \sin. (\theta' - \theta) = a \, b.$$



Now $\theta' - \theta$ is the angle P C D between the conjugate diameters C P and C D; hence, drawing straight lines at the extremities of the conjugate diameters, parallel to those diameters, we have, from the above equation, the parallelogram P C D T = the rectangle A′ C B E, and hence the whole parallelogram thus inscribed in the figure is equal to the rectangle contained by the axes *.

185. Returning to article (182.), the equation to the curve, suppressing the accents on x' and y', as no longer necessary, is

$$a_1^2 \, y_1^2 - b_1^2 \, x^2 = - a_1^2 \, b_1^2;$$

* The theorems in articles 183 and 184 may be proved also in the following manner:—

Referring the curve to its rectangular axes, as in art. (187.), let the co-ordinates of P be x' and y'; then the equation to C D is $a^2 \, y \, y' - b^2 \, x \, x' = 0$, and eliminating x and y between this equation and that to the curve ($a^2 \, y^2 - b^2 \, x^2 = - a^2 \, b^2$) we have the co-ordinates C N and D N, independent of the sign $\sqrt{-1}$, with which they are both affected,

$$\text{C N} = x = \frac{a \, y'}{b}, \text{ and D N} = y = \frac{b \, x'}{a};$$

Hence we have

$$a_1^2 - b_1^2 = x'^2 + y'^2 - x^2 - y^2 = x'^2 + y'^2 - \frac{a^2 \, y'^2}{b^2} - \frac{b^2 \, x'^2}{a^2} = \frac{b^2 \, x'^2 - a^2 \, y'^2}{b^2}$$

$$+ \frac{a^2 \, y'^2 - b^2 \, x'^2}{a^2} = \frac{a^2 \, b^2}{b^2} + \frac{- a^2 \, b^2}{a^2} = a^2 - b^2.$$

Also the triangle P C D = the trapezium P M N D + the triangle D C N − the triangle P C M

$$= (x' - x) \frac{y + y'}{2} + \frac{x \, y - x' \, y'}{2} = \frac{x' \, y - y' \, x}{2} = \frac{1}{2} \left\{ x' \frac{b \, x'}{a} - y' \frac{a \, y'}{b} \right\}$$

$$= \frac{b^2 \, x'^2 - a^2 \, y'^2}{2 \, a \, b} = \frac{a^2 \, b^2}{2 \, a \, b} = \frac{a \, b}{2},$$

therefore the parallelogram P C D T = $a \, b$.

In the last figure, C P $= a_1$, C D $= b_1$, C V $= x$ and Q V $= y$: Putting the equation into the form

$$y^2 = \frac{b_1^2}{a_1^2}(x^2 - a_1^2) = \frac{b_1^2}{a_1^2}(x - a_1)(x + a_1),$$

we have the square upon Q V : the rectangle P V, V P′ :: the square upon C D : the square upon C P.

186. The equation to the tangent at any point Q $(x' \, y')$ is

$$a_1^2 \, y \, y' - b_1^2 \, x \, x' = - a_1^2 b_1^2.$$

187. Let the curve be referred to its axes C A, C B, and let the co-ordinates of P be $x' \, y'$, then the equation to C P being $y = \frac{y'}{x'} x$, the equation to C D is $y = x \tan. \theta = x \frac{b^2}{a^2} \cot. \theta = \frac{b^2 \, x'}{a^2 \, y'} x$, or,

$$a^2 \, y \, y' - b^2 \, x \, x' = 0.$$

But the equation to the tangent at P is

$$a^2 \, y \, y' - b^2 \, x \, x' = - a^2 \, b^2;$$

hence C D, or the diameter conjugate to C P, is parallel to the tangent at P.

The equation to the conjugate diameter is the same as that to the tangent, omitting the last term $- a^2 \, b^2$.

188. Let x' and y' be the rectangular co-ordinates of P; then from the equation $a_1^2 - b_1^2 = a^2 - b^2$, we have

$$b_1^2 = a_1^2 - a^2 + b^2 = x'^2 + y'^2 - a^2 + b^2 = x'^2 + \left(\frac{b^2}{a^2} x'^2 - b^2\right) - a^2 + b^2$$

$$= \frac{a^2 + b^2}{a^2} x'^2 - a^2 = e^2 \, x'^2 - a^2 = (e \, x' - a)(e \, x' + a) = r \, r';$$

That is, the square upon the conjugate diameter C D = the rectangle under the focal distances S P and H P.

189. If P F be drawn perpendicular from P upon the conjugate C D, (see the last figure but one,) we have the rectangle P F, C D $= a \, b$, (184.).

$$\therefore \text{P F} = \frac{a \, b}{b_1} = \frac{a \, b}{\sqrt{a_1^2 - a^2 + b^2}} = \frac{a \, b}{\sqrt{r \, r'}};*$$

Also P G $= \frac{b}{a} \sqrt{r \, r'}$, and P G′ $= \frac{a}{b} \sqrt{r \, r'}$,

Hence the rectangle P G, P F = the square on B C;

And the rectangle P G′, P F = the square on A C;

And the rectangle P G, P G′ = the square on C D.

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* If the distance C P $= u$, and $p =$ the perpendicular from the centre on the tangent, this equation is

$$p^2 = \frac{a^2 \, b^2}{u^2 - a^2 + b^2}.$$

SUPPLEMENTAL CHORDS.

190. Two straight lines drawn from a point on the curve to the extremities of a diameter are called supplemental chords; they are called principal supplemental chords if that diameter be the transverse axis.

The equations to a pair of chords are

$$y - y' = \alpha\,(x - x')$$
$$y + y' = \alpha'\,(x + x');$$

Whence $\alpha\,\alpha' = \dfrac{b^2}{a^2}$ as in (141.); hence the product of the tangents of the angles which a pair of supplemental chords makes with the transverse axis is constant; the converse is proved as in 141.

191. The angle between two supplemental chords is found from the expression

$$\tan.\ \mathrm{P\,Q\,P'} = \frac{2\,b^2}{a^2 + b^2}\,\frac{x'\,y - y'\,x}{y^2 - y'^2}.$$

And, if A R, A′ R be principal supplemental chords drawn to any point R on the curve,

$$\tan.\ \mathrm{A\ R\ A'} = \frac{2\,a\,b^2}{(a^2 + b^2)\,y}.$$

The angle A R A′ is always acute, and diminishes from a right angle to 0; the supplemental angle A A′ R′ increases at the same time from a right angle to 180°; hence, the angle between the supplemental chords may be any angle between 0 and 180°.

Chords may be drawn containing any angle between these limits, by describing on any diameter, except the axes, a segment of a circle containing the given angle, and then joining the extremities of the diameter with the point where the circle intersects the hyperbola. And therefore principal supplemental chords parallel to these may be drawn.

192. Conjugate diameters are parallel to supplemental chords (144.); and therefore they may be drawn containing any angle between 0 and 90°.

193. There are no equal conjugate diameters in the hyperbola, but in that particular curve where $b = a$, we have the equation

$$a_1^2 - b_1^2 = a^2 - b^2 = 0;$$

hence the conjugate diameters a_1 and b_1 are always equal to each other.

The equation to this curve, called the equilateral hyperbola, is

$$y^2 - x^2 = -a^2.$$

THE ASYMPTOTES.

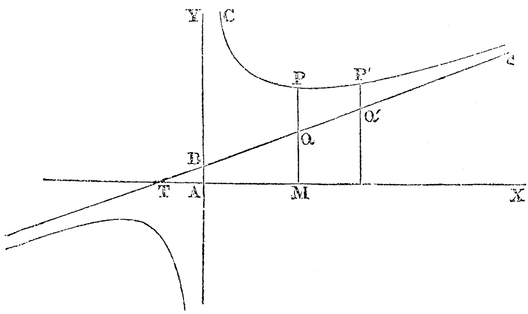
194. We have now shown that most of the properties of the ellipse apply to the hyperbola with a very slight variation: there is, however, a whole class of theorems quite peculiar to the latter curve, and these arise from the curious form of the branches extending to an infinite distance; it appears from the equation $\tan.\ \theta\ .\ \tan\ \theta' = \dfrac{b^2}{a^2}$ in (180.), that as $\tan.\ \theta$

approaches to $\frac{b}{a}$, tan. θ' approaches also to $\frac{b}{a}$, and thus, as a point P recedes along the curve from the origin, the conjugate diameters for that point approach towards a certain line C E, fig. (179.), and finally at an infinite distance come indefinitely near to that line.

We now proceed to show that the curve itself continually approaches to the same line C E, without ever actually coinciding with it. But as this species of line is not confined to the hyperbola, we shall state the theory generally.

195. Let C P P′ be a curve whose equation has been reduced to the form

$$y = a\,x + b + \frac{c}{x};$$



And let T B S be the line whose equation is

$$y = a\,x + b.$$

For any value of x we can find from this last equation a corresponding ordinate M Q, and by adding $\frac{c}{x}$ to M Q, we determine a point P in the curve: similarly we can determine any number of corresponding points (P′, Q′, &c.) in the curve and straight line.

Since $\frac{c}{x}$ decreases as x increases, the line P′ Q′ will be less than P Q, and the greater x becomes, the smaller does the corresponding P′ Q′ become; so that when x is infinitely great, P′ Q′ is infinitely small, or the curve approaches indefinitely near to the line T B S, but yet never actually meets it: hence T B S is called an asymptote to the curve, from three Greek words signifying " never coinciding."

The equation to the asymptote T B S is $y = a\,x + b$, or is the equation to the curve, with the exception of the term involving the inverse power of x.

196. The reasoning would have been as conclusive if there had been more inverse powers of x; and in general if the equation to a curve can be put into the form

$$y = \&c. + m\,x^3 + n\,x^2 + a\,x + b + \frac{c}{x} + \frac{d}{x^2} + \&c.$$

Then the equation to the curvilinear asymptote is

$$y = \&c. + m\,x^3 + n\,x^2 + a\,x + b$$

Also the equation $y = \&c. + m\,x^3 + n\,x^2 + a\,x + b + \frac{c}{x}$ gives a curve much more asymptotic than the preceding equation, and hence arises a series of curves, each " more nearly coinciding" with the original curve.

197. Let us apply this method to lines of the second order, whose general equation is (75.)

$$y = -\frac{b\,x + d}{2\,a} \pm \frac{1}{2\,a}\sqrt{\{(b^2 - 4\,a\,c)\,x^2 + 2\,(b\,d - 2\,a\,e)\,x + d^2 - 4\,a\,f\}}$$

$$= -\frac{b\,x + d}{2\,a} \pm \sqrt{\{m\,x^2 + n\,x + p\}}\text{, by substitution,}$$

$$= -\frac{b\,x + d}{2\,a} \pm x\sqrt{m}\left\{1 + \frac{n\,x + p}{m\,x^2}\right\}^{\frac{1}{2}},$$

$$= -\frac{b\,x + d}{2\,a} \pm x\sqrt{m}\left\{1 + \tfrac{1}{2}\left(\frac{n\,x + p}{m\,x^2}\right) - \tfrac{1}{8}\left(\frac{n\,x + p}{m\,x^2}\right)^2 + \&c.\right\}$$

$$= -\frac{b\,x + d}{2\,a} \pm \sqrt{m}\left\{x + \tfrac{1}{2}\frac{n}{m}\right\} \pm \frac{\text{constant terms}}{\text{powers of } x}.$$

Hence the equation to the asymptote is

$$y = -\frac{b\,x + d}{2\,a} \pm \sqrt{m}\left\{x + \tfrac{1}{2}\frac{n}{m}\right\}$$

$$= -\frac{b\,x + d}{2\,a} \pm \frac{\sqrt{b^2 - 4\,a\,c}}{2\,a}\left\{x + \frac{b\,d - 2\,a\,c}{b^2 - 4\,a\,c}\right\}$$

Now $b^2 - 4\,a\,c$ is negative in the ellipse, and therefore there is no locus to the above equation in this case; also if $b^2 - 4\,a\,c = 0$, the equation to the asymptote, found as above, will contain the term $\sqrt{x}$, and therefore will belong to a curvilinear asymptote; hence the hyperbola is the only one of the three curves which admits of a rectilinear asymptote.

It appears from the $\pm$ sign, that there are two asymptotes, and that the diameter $y = -\frac{b\,x + d}{2\,a}$ bisects them. Also these asymptotes pass through the centre; for giving to x the value $\frac{2\,a\,e - b\,d}{b^2 - 4\,a\,c}$, we have

$$y = -\frac{b\,x + d}{2\,a} = \frac{2\,c\,d - b\,e}{b^2 - 4\,a\,c},$$

and these values of x and y are the co-ordinates of the centre (80.).

198. If the equation want either of the terms x^2 or y^2, a slight operation will enable us to express the equation in a series of inverse powers of y or x; thus if the equation be

$$b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0,$$

we have $y = -\frac{c\,x^2 + e\,x + f}{b\,x + d} = -\frac{c\,x^2 + e\,x + f}{b\,x\left(1 + \frac{d}{b\,x}\right)}$

$$= -\frac{c\,x^2 + e\,x + f}{b\,x}\left(1 + \frac{d}{b\,x}\right)^{-1}$$

$$= -\left(\frac{c\,x}{b} + \frac{e}{b} + \frac{f}{b\,x}\right)\left(1 - \frac{d}{b\,x} + \left(\frac{d}{b\,x}\right)^2 -, \&c.\right)$$

Hence the equation to the asymptote, found by multiplying and neglecting inverse powers of x, is

$$y = -\frac{c\,x}{b} - \frac{e}{b} + \frac{c\,d}{b^2},$$

$$\text{or,}\quad b\,y + c\,x = \frac{c\,d - b\,e}{b}.$$

The other asymptote is determined by the consideration that if, for any finite value of x, we obtain a real infinite value of y, that value of x determines the position of an asymptote.

Here when $b\,x + d = 0$, we have $y = \infty$; hence a line drawn parallel to the axis of y, and through the point $x = -\frac{d}{b}$, is the required asymptote.

If the equation be

$$a\,y^2 + b\,x\,y + d\,y + e\,x + f = 0,$$

the equations to the asymptotes are

$$a\,y + b\,x = \frac{a\,e - b\,d}{b}, \text{ and } b\,y + e = 0;$$

and the second asymptote is parallel to the axis of y.

If the equation be

$$b\,x\,y + d\,y + e\,x + f = 0,$$

the equations to the asymptotes are

$$b\,x + d = 0, \text{ and } b\,y + e = 0;$$

the former asymptote being parallel to the axis of y, and the latter parallel to that of x.

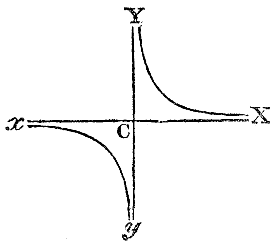
199. Lastly, if the equation be

$$b\,x\,y + f = 0,$$

the asymptotes are then the axes themselves, and the curve is referred to its centre and asymptotes as axes.

The position of the curve in this case is directly obtained from the equation $y = -\frac{f}{b\,x} = \frac{m}{x}$, by substitution.

Let C X and C Y be the axes, then for $x = 0$, $y = \infty$; as x increases y decreases, and when $x = \infty$, $y = 0$; hence we have the branch Y X.



For x negative, y is negative; and as x increases from 0 to ∞, y decreases from ∞ to 0; hence another branch $y\,x$, equal and similar to the former.

200. To find the equation to the asymptotes from the equation to the hyperbola referred to its centre and axes,

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}} = \pm \frac{b}{a} x \left\{1 - \tfrac{1}{2}\frac{a^2}{x^2} +, \&c.\right\}$$

Hence the equation to the asymptotes is

$$y = \pm \frac{b}{a} x.$$

To draw these lines, complete the parallelogram on the principal axes (see the figure, art. 179.); the diagonals of this parallelogram are the loci of the last equation, and therefore are the asymptotes required: thus C E and C E′, when produced, are the asymptotes.

The equation to the asymptotes, referred to the centre and rectangular axes, is readily remembered, since it is the same as the equation to the curve without the last term; the two equations are

$$a^2 y^2 - b^2 x^2 = -a^2 b^2, \text{ to the curve,}$$
$$a^2 y^2 - b^2 x^2 = 0 \quad , \text{ to the asymptotes.}$$

If the curve be referred to conjugate axes, the equations are

$$a_1^2 y^2 - b_1^2 x^2 = -a_1^2 b_1^2, \text{ to the curve,}$$
$$a_1^2 y^3 - b_1^2 x^2 = 0 \quad , \text{ to the asymptotes.}$$

201. If $b = a$, the equation to the hyperbola referred to its centre and rectangular axes is $y^2 - x^2 = -a^2$, therefore the equation to the asymptotes is $y^2 - x^2 = 0$, or $y = \pm x$; hence these asymptotes cut the axes at an angle of 45°, or the angle between them is 90°; hence the equilateral hyperbola is also called the rectangular hyperbola.

202. If the curve be referred to its vertex A and rectangular axes, the equation to the curve is

$$y = \pm \frac{b}{a}(x^2 - 2\,a\,x)^{\frac{1}{2}} = \pm \frac{b}{a} x \left(1 + \frac{2\,a}{x}\right)^{\frac{1}{2}},$$

and, expanding and neglecting inverse powers of x, the equation to the asymptotes is

$$y = \pm\left(\frac{b}{a} x + b\right).$$

203. If we take the equation to any line $\left(y = \frac{b}{a} x + c\right)$ parallel to the asymptote, and eliminate y between this equation and the equation to the curve, we find only one value of x; and thus a straight line parallel to the asymptote cuts the hyperbola only in one point.

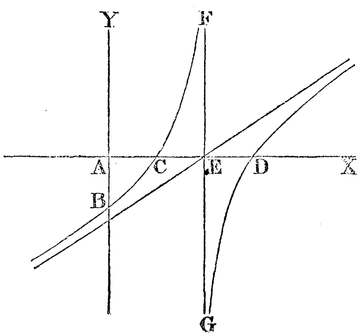
204. In article (77.) it was stated that, in some cases, the form of the curve could not be readily ascertained: thus, when the curve cuts neither diameter, there might be some difficulty in ascertaining its correct position: the asymptotes will, however, be found very useful in this respect: for example, if the equation is $x y = x^2 + b x + c^2$, or $y = x + b + \frac{c^2}{x}$, we have for $x = 0$, $y = \infty$; and when x becomes very great, y approximates to $x + b$; hence the lines A Y and T B S, in figure (194), will represent the asymptotes of the curve; and since the curve never cuts the axes, its course is entirely confined within the angle Y B S and the opposite angle T B A; hence the position of the curve is at once determined, as in figure (194).

Ex. 2. $y\,(x - 2) = (x - 1)\,(x - 3)$, or $y = \frac{(x-1)\,(x-3)}{x-2}$

In the first place we ascertain that the curve is an hyperbola by the test $b^2 - 4\,a\,c$ being positive; then draw the rectangular axes A X, A Y: to find the points where the curve cuts the axes,

$$\text{Let } x = 0, \quad \therefore\ y = -\tfrac{3}{2} = \text{A B},$$
$$\text{Let } y = 0, \quad \therefore\ x = 1 = \text{A C},$$
$$\text{also } x = 3 = \text{A D},$$

thus the curve passes through the points B, C, and D.



Again, to find the asymptotes, we have $y = \infty$ for $x = 2$; hence, if A E $= 2$, the line F E G, drawn perpendicular to A X, is one asymptote.

To find the other, we have

$$y = \frac{(x-1)\,(x-3)}{x-2} = \frac{(x-1)\,(x-3)}{x\left(1 - \frac{2}{x}\right)} = \frac{(x-1)\,(x-3)}{x}\left(1 - \frac{2}{x}\right)^{-1}$$

$$= \frac{x^2 - 4x + 3}{x} \left\{ 1 + \frac{2}{x} +, \&c. \right\} = \left(x - 4 + \frac{3}{x} \right)$$

$$\left\{ 1 + \frac{2}{x} +, \ \&c. \right\} = x - 4 + 2 + \frac{a}{x} +, \&c.;$$

hence the equation to the asymptote is $y = x - 2$, and therefore this line must be drawn through the point E, making an angle of 45° with A X.

We can now trace the course of the curve completely; for all values of x less than 1, y is negative, hence the branch B C; for x greater than 1, but less than 2, y is positive and increases from 0 to ∞, hence the branch C F; for x greater than 2, but less than 3, y is negative, hence the branch G D; and for x greater than 3, y is positive and approximating to $x - 2$, hence the branch from D extending to the second asymptote.

For negative values of x, y is negative, and increases from $-\frac{3}{2}$ to ∞, approximating also to the value $-x-2$; hence the curve extends downwards from B towards the asymptote.

Ex. 3. $y(x-a) = x(x - 2a)$. Here $x = a$ and $y = x - a$, are the equations to the asymptotes. The figure is like the last, supposing that A and C coincide.

Ex. 4. $y^{\frac{1}{2}} = a\, x^{-\frac{1}{2}} + x^{\frac{1}{2}}$. The axis of y is one asymptote, since $x = 0$ gives $y = \infty$: Also

$$y = x\left(1 + \frac{a}{x}\right)^2 = x + 2a + \frac{a^2}{x};$$

hence $y = x + 2a$ gives the other asymptote.

205. In order to discuss an equation of the second order completely, we have given, in Chapter VII., a general method of reducing that equation to its more simple forms.

In that chapter we showed that the equation, when belonging to an hyperbola, could be reduced to the form $a y^2 + c x^2 + f = 0$. (84.)

Now the same equation can be reduced also to the form $x y = k^2$; and as this form is of use in all discussions about asymptotes, we shall proceed to its investigation.

206. Let the general equation be referred to rectangular axes, and let it be

$$a y^2 + b x y + c x^2 + d y + e x + f = 0.$$

Let $x = x' + m$, and $y = y' + n$, and then, as in article (80.), put the co-efficients of x' and y' each $= 0$; by this means the curve is referred to its centre, and its equation is reduced to the form

$$a y'^2 + b x' y' + c x'^2 + f' = 0.$$

Again, to destroy the co-efficients of x'^2 and y'^2, take the formulas of transformation from rectangular to oblique co-ordinates (57.).

$$y' = \ \ x'' \sin.\theta + y'' \sin.\theta',$$

$$x' = \ \ x'' \cos.\theta + y'' \cos.\theta';$$

then, by substituting and arranging, the central equation becomes

$$y''^2 \{a(\sin.\theta')^2 + b \sin.\theta' \cos.\theta' + c(\cos.\theta')^2\}$$

$$+ x''^2 \{a(\sin.\theta)^2 + b \sin.\theta \cos.\theta + c(\cos.\theta)^2\}$$

$$+ x'' y'' \{2 a \sin.\theta' \sin.\theta + b (\sin.\theta \cos.\theta' + \sin.\theta' \cos.\theta) + 2 c \cos.\theta' \cos.\theta\} + f' = 0.$$

There are two new indeterminate quantities θ and θ' introduced; therefore we may make two suppositions respecting the co-efficients in the transformed equation; hence, letting the co-efficients of x''^2 and $y''^2 = 0$, we have

$$a (\sin.\theta)^2 + b \sin.\theta \cos.\theta + c (\cos.\theta)^2 = 0 \qquad (1),$$

$$a (\sin.\theta')^2 + b (\sin.\theta') (\cos.\theta') + c (\cos.\theta')^2 = 0 \quad (2).$$

Dividing the first of these two equations by $(\cos.\theta)^2$, we have

$$a (\tan.\theta)^2 + b \tan.\theta + c = 0;$$

$$\text{hence} \quad \tan.\theta = \frac{b \pm \sqrt{b^2 - 4 a c}}{2 a}.$$

From the similarity of the equation (1) and (2), it is evident that we shall arrive at the same value for $\tan.\theta'$; hence, letting one of the above values refer to θ, the other will refer to θ; or both the new axes are de termined in position from the above values of $\tan.\theta$.

The equation is now reduced to the form

$$b' x'' y'' + f' = 0.$$

207. To find the value of b', we have

$$b' = 2 a \sin.\theta' \sin.\theta + b (\sin.\theta \cos.\theta' + \sin.\theta' \cos.\theta) + 2 c \cos.\theta' \cos.\theta,$$

$$= \cos.\theta' \cos.\theta \{2 a \tan.\theta' \tan.\theta + b (\tan.\theta' + \tan.\theta) + 2 c\}.$$

From the eauation involving $\tan.\theta$, we have

$$\tan.\theta . \tan.\theta' = \frac{c}{a}, \quad \tan.\theta + \tan.\theta' = -\frac{b}{a},$$

$$\text{and therefore} \quad \cos.\theta \cos.\theta' = \frac{a}{\sqrt{(a - c)^2 + b^2}};$$

$$\therefore b' = \frac{a}{\sqrt{(a - c)^2 + b^2}} \left\{2 c - \frac{b^2}{a} + 2 c\right\} = - \frac{b^2 - 4 a c}{\sqrt{(a - c)^2 + b^2}}.$$

$$\text{Also} \quad f' = \frac{a e^2 - c d^2 - b d e}{b^2 - 4 a c} + f \quad (80.).$$

Hence the final equation is

$$- \frac{b^2 - 4 a c}{\sqrt{(a - c)^2 + b^2}} x'' y'' + \frac{a e^2 + c d^2 - b d e}{b^2 - 4 a c} + f = 0.$$

208. If the original axes are oblique we must take the formulas in (56.), and then, following the above process, we find

$$\tan.\theta = \frac{\pm \sqrt{b^2 - 4 a c} - b + 2 c \cos.\omega}{2 (a + c (\cos.\omega)^2 - b \cos.\omega)},$$

$$b' = \frac{-(b^2 - 4 a c)}{\sqrt{\{(a + c - b \cos.\omega)^2 + (b^2 - 4 a c) (\sin.\omega)^2\}}}.$$

209. The following examples relate to the reduction of the general equation referred to rectangular axes, to another equation referred to the asymptotes.

Ex 1. $y^2 - 10\,x\,y + x^2 + y + x + 1 = 0,$

$$m = \frac{1}{8}, n = \frac{1}{8}, f' = \frac{9}{8}; \text{ tan. } \theta = 5 \pm 2\sqrt{6}, b' = -\frac{48}{5};$$

$$\therefore\ -\frac{48}{5}x''\,y'' + \frac{9}{8} = 0,$$

$$\text{or,}\quad x''\,y'' = \frac{15}{128}.$$

Ex. 2. $4\,y^2 - 8\,x\,y - 4\,x^2 - 4\,y + 28\,x - 15 = 0,$

$$b' = -8\sqrt{2},\ f' = 2,$$

$$\therefore\ -8\sqrt{2}\,x''\,y'' + 2 = 0,$$

$$\text{or,}\quad x''\,y'' = \frac{1}{4\sqrt{2}}.$$

Ex. 3. $\frac{a}{x} + \frac{b}{y} = 1,$ or $x\,y = a\,y + b\,x.$

The axes are here parallel to the asymptotes (198.): in order to transfer the origin to the centre, let $y = y' + n$ and $x = x' + m$, hence we have $m = a$, $n = b$, and the reduced equation is

$$x'\,y' = a\,b.$$

210. If θ and θ' be the angles which the asymptotes make with the original rectangular axes, we have from the equation (206.),

$$a\,(\text{tan. }\theta)^2 + b\text{ tan. }\theta + c = 0,$$

$$\therefore\ \text{tan. }\theta.\ \text{tan. }\theta' = \frac{c}{a}$$

Now when $c = -a$, this equation becomes tan. θ. tan. $\theta' = -1$, or, tan. θ. tan. $\theta' + 1 = 0$; hence by (47.), the angle between the asymptotes is in this case $= 90°$; and thus whenever, in the general hyperbolic equation, we have $c = -a$, the cnrve is a rectangular hyperbola.

Ex. 4. $y^2 - x^2 = \sqrt{2}$.

The curve is a rectangular hyperbola, and is referred to its centre and rectangular axes; also taking the two values of tan. θ in (206.), we have tan. $\theta = 1$, and tan. $\theta' = -1$; hence $\theta = 45°$ and $\theta' = -45°$, and the formulas of transformation become

$$y = \frac{x' - y'}{\sqrt{2}},\ x = \frac{x' + y'}{\sqrt{2}};$$

$$\therefore\ \left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' + y'}{\sqrt{2}}\right)^2 = \sqrt{2},$$

$$\text{or}\quad -2\,x'\,y' = \sqrt{2},$$

$$\text{and } x'\,y' = -\frac{1}{\sqrt{2}}.$$

In this example the curve is placed as in the next figure, and at first was referred to the axes C X and C Y, but now is referred to the asymptotes C x and C y, supposing C y and C x to change places, and the angle x C $y = 90°$.

211. Conversely given the equation $x y = k^2$, to find the equation referred to the rectangular axes, and thence to deduce the lengths of the axes

For this purpose we use the formulas of transformation from oblique to rectangular axes (56.).

$$y = \frac{x' \sin. \theta + y' \cos. \theta}{\sin. \omega},$$

$$x = \frac{x' \sin. (\omega - \theta) - y' \cos. (\omega - \theta)}{\sin. \omega};$$

substituting these values in the equation $x y = k^2$, we have

$$x'^2 \sin. \theta \sin (\omega - \theta) - y'^2 \cos. \theta \cos. (\omega - \theta)$$
$$+ x' y' \{\cos. \theta \sin. (\omega - \theta) - \sin. \theta \cos. (\omega - \theta)\} = k^2 (\sin. \omega)^2.$$

Let the co-efficient of $x' y' = 0$,

$\therefore \cos. \theta \sin. (\omega - \theta) - \sin. \theta \cos. (\omega - \theta)$, or $\sin. (\omega - 2\theta) = 0$;

$$\therefore \omega = 2\theta, \text{ and } \theta = \frac{\omega}{2};$$

hence the new rectangular axis of x, determined by the angle θ, bisects the angle ω between the asymptotes; this agrees with the remark at the end of (179.).

The transformed equation, putting $\theta = \frac{\omega}{2}$, is

$$x'^2 \left(\sin. \frac{\omega}{2}\right)^2 - y'^2 \left(\cos. \frac{\omega}{2}\right)^2 = k^2 (\sin. \omega)^2,$$

or, putting $2 \sin. \frac{\omega}{2} \cos. \frac{\omega}{2}$ for $\sin. \omega$, and dividing

$$\frac{y'^2}{4 k^2 \left(\sin. \frac{\omega}{2}\right)^2} - \frac{x'^2}{4 k^2 \left(\cos. \frac{\omega}{2}\right)^2} = -1;$$

Comparing this with the equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1$, we have

$$a = 2 k \cos. \frac{\omega}{2}, \text{ and } b = 2 k \sin. \frac{\omega}{2};$$

hence the lengths of the semi-axes are determined.

If the equation had been $x y + a x + b y + c = 0$, first refer the curve to its centre, and then proceed as above.

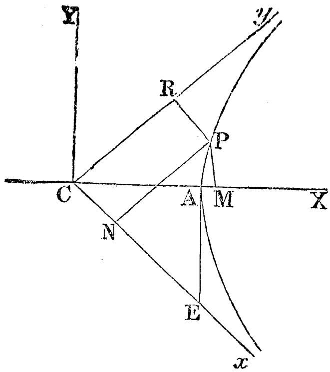
212. To deduce the equation $x y = k^2$ from the equation to the curve referred to the centre and rectangular axes.

Let C X, C Y be the rectangular axes,

C x, C y the asymptotes, or the new axes,

$\left.\begin{array}{l} CM = x \\ MP = y \end{array}\right\}$ the original co-ordinates of P,

$\left.\begin{array}{l} CN = x' \\ NP = y' \end{array}\right\}$ the new co-ordinates of P.



Then taking the formulas of transformation from rectangular to oblique axes (57),

$$y = x' \sin.\theta + y' \sin.\theta',$$

$$x = x' \cos.\theta + y' \cos.\theta',$$

and substituting in the equation $a^2 y^2 - b^2 x^2 = - a^2 b^2$, we have

$$a^2 (x' \sin.\theta + y' \sin.\theta')^2 - b^2 (x' \cos.\theta + y' \cos.\theta')^2 = - a^2 b^2,$$

$$\text{or,}\ \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\} y'^2 + \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\} x'^2,$$

$$+ 2\{a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta'\} x' y' = - a^2 b^2.$$

In order that this equation may be of the required form, it must not contain the terms in x'^2 and y'^2; but since we have introduced two indeterminate quantities, we can make the two suppositions that the co-efficients of these terms shall $= 0$;

$$\therefore\ a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2 = 0,$$

$$a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2 = 0,$$

From the last of these equations we have $\tan.\theta = \pm \frac{b}{a}$, and as we obtain from the other equation the same value of $\tan.\theta'$, it follows that the values of θ and θ' are both contained in the equation $\tan.\theta = \pm \frac{b}{a}$, that is, if $\tan.\theta \left(= - \frac{b}{a}\right)$ refers to the axis of x, then $\tan.\theta \left(= + \frac{b}{a}\right)$ refers to the axis of y, (we have chosen $\tan.\theta = - \frac{b}{a}$ for the axis of x, in order to agree with the figure).

The equation to the curve referred to its asymptotes is now

$$2\{a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta'\} x' y' = - a^2 b^2,$$

$$\text{or,}\quad 2 \cos.\theta \cos.\theta' \{a^2 \tan.\theta \tan.\theta' - b^2\} x' y' = - a^2 b^2;$$

but since $\tan.\theta = \pm \frac{b}{a}$, we have

$$\cos.\theta = \frac{1}{\sqrt{1+(\tan.\theta)^2}} = \frac{a}{\sqrt{a^2+b^2}} = \cos.\theta';$$

$$\therefore 2\frac{a^2}{a^2+b^2}\left\{-a^2\frac{b^2}{a^2}-b^2\right\}x\,y' = -a^2b^2,$$

$$\text{or,}\quad -\frac{4a^2b^2}{a^2+b^2}x'y' = -a^2b^2;$$

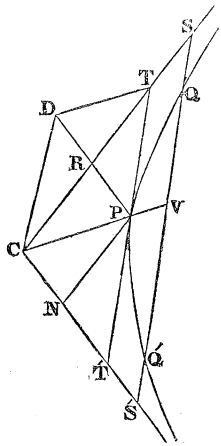
$$\therefore x'y' = \frac{a^2+b^2}{4}.$$

If $b = a$, or the curve be the rectangular hyperbola, the equation referred to the asymptotes is $x\,y = \frac{a^2}{2}$.

213. The angle between the asymptotes is $2\,\theta$; if therefore P R be drawn parallel to C N, the area P N C R $= x$ sin. $2\,\theta = x\,y$. 2 sin. θ cos. $\theta = \frac{a^2+b^2}{4} \cdot 2 \cdot \frac{b}{\sqrt{a^2+b^2}} \cdot \frac{a}{\sqrt{a^2+b^2}} = \frac{a\,b}{2}$.

Thus all the parallelograms constructed upon co-ordinates parallel to the asymptotes are equal to each other, and to half the rectangle in the semi-axes.

214. Let C S, C S′ be the asymptotes to the curve referred to conjugate diameters C P, C D $(a_1\,b_1)$, then if P T be parallel to C D, it is a tangent at P (187.); T P T′ is also a double ordinate to the asymptote, for the equation to C S is $y = \pm\frac{b_1}{a_1}x$, and when $x = a_1, y = \pm\,b_1$. Hence P T = P T′, or the parts of the tangent contained between the point of contact and the asymptotes are equal to each other, and to the semi-conjugate diameter.



215. Join D T, then D P is a parallelogram; also because C D is equal and parallel to P T′, we have the line D P parallel to the asymptote

C S′. Hence, if the conjugate diameters be given, the asymptotes may always be found by completing the parallelogram upon the conjugate diameters, and then drawing the diagonals. Also, if the asymptotes be given, a conjugate diameter to C P may be found by drawing P R parallel to C S′ and taking P D double of P R.

If the asymptotes be given, a tangent may be drawn by taking C T double of C R, and joining P T.

If the position of the focus is known, the length of the conjugate axis is equal to the perpendicular, from the focus on the asymptote.

216. To find the equation to the tangent P T, when referred to the asymptotes as axes,

Let x', y' be the co-ordinates of P, and x'' y'' co-ordinates of another point on the curve.

$$\therefore \; y' = \frac{k^2}{x'}, \text{ and } y'' = \frac{k^2}{x''}$$

$$\therefore \; \frac{y' - y''}{x' - x''} = -\frac{k^2}{x'\,x''} = -\frac{y'}{x''}$$

$$\therefore \; y - y' = -\frac{y'}{x''}(x - x') \text{ is the equation to a secant.}$$

When $x'' = x'$ we have the equation to the tangent

$$y - y' = -\frac{y'}{x'}(x - x') = -\frac{y'}{x'}\,x + y'$$

$$\therefore \; x'y + y'x = 2\,x'y' = 2\,k^2.$$

This equation to the tangent is readily obtained from the equation to the curve ($xy = k^2$ or $xy + xy = 2k^2$) by putting $x'y$ and xy' successively for xy, and then adding the results.

Let $y = 0$ ∴ $C T' = 2\,x' = 2\,C N$; and $C T = 2\,y' = 2\,N P$;

The triangle $C T T' = \frac{1}{2}.\,2\,x'.\,2\,y' \sin.\, T C T' = 2\,x'\,y' \sin.\,2\,\theta = a\,b$, (213.)

217. The two parts S Q, S′ Q′ of any secant S Q Q′ S comprised between the curve and its asymptote are equal; for if the diameter C P V and its conjugate C D be drawn, we have V Q = V Q′ from the equation to the curve $\left(y = \pm \frac{b_1}{a_1}\sqrt{x^2 - a_1^2}\right)$, and from the equation to the asymptotes $\left(y = \pm \frac{b_1}{a_1}\,x,\right)$ we have V S = V S′ ∴ S Q = S′ Q′.

218. If Y and y are the ordinates V S, V Q respectively, we have

$$Y^2 - y^2 = \frac{b_1^2}{a_1^2}\,x^2 - \frac{b_1^2}{a_1^2}\,(x^2 - a_1^2)$$

$$\text{or } (Y - y)\,(Y + y) = b_1^2.$$

Thus the rectangle S Q, Q S′ = the square upon C D.

THE POLAR EQUATION.

219. Let the curve be referred to the centre C, and to rectangular axes C A, C B, and let the co-ordinates of the pole O be x' and y', O being situated anywhere in the plane of the curve and P any point on the curve, as in (146.), θ the angle which the radius vector O P or u makes with a line parallel to the axis of x. Then we have by (61.)

$$y = y' + u \sin.\theta$$
$$x = x' + u \cos.\theta$$
$$\text{also,} \quad a^2 y^2 - b^2 x^2 = -a^2 b^2$$
$$\therefore \; a^2 (y' + u \sin.\theta)^2 - b^2 (x' + u \cos.\theta)^2 = -a^2 b^2$$

220. Let the centre be the pole, $\therefore$ $x' = 0$, and $y' = 0$,

$$\therefore u^2 = \frac{-a^2 b^2}{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2} = \frac{a^2 (e^2 - 1)}{e^2 (\cos.\theta)^2 - 1.}$$

221. Let the focus S be the pole,

$$\therefore \; y' = 0, x' = a e \text{ and } u \text{ becomes } r,$$

Substituting these values, and following the steps in (148.), we find

$$r = \frac{b^2}{a - c \cos.\theta} = \frac{a (e^2 - 1)}{1 - e \cos \theta}$$

If the angle A S P $= \theta$, we have

$$r = \frac{a (e^2 - 1)}{1 + e \cos.\theta}$$

This is the equation generally used. It may easily be obtained from the equation $r = e x - a$, fig. (161) $= e (a e - r \cos.\theta) - a$,

$$\therefore \; r = \frac{a (e^2 - 1)}{1 + e \cos.\theta}.$$

222. If $\frac{p}{2} = a (e^2 - 1)$ we have $r = \frac{p}{2} . \frac{1}{1 + e \cos.\theta} =$ S P, and if P S meet the curve again in P′, we have the rectangle S P, S P′ $= \frac{p}{4}$ (P S + S P′) $= \frac{p}{4}$ P P′.

The length of the chord through the focus $= 2 \frac{b_{\prime}^{2}}{a}$ where $b_{\prime}$ is the diameter to that chord.

THE CONJUGATE HYPERBOLA.

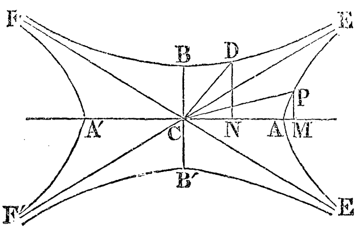
223. There is another equation to the hyperbola, not yet investigated. If $\left(\frac{c'}{-f'}\right)$ be negative in article 153, the equation is $P y^2 - Q x^2 = 1$.

or $a^2 y^2 - b^2 x^2 = a^2 b^2$ if $P = \frac{1}{b^2}$, and $Q = \frac{1}{a^2}$. If we examine the course of this curve, we shall find that $B B' = 2 b$ is the real or transverse axis, and $A A'$, or $2 a$, is the conjugate axis, and that the curve extends indefinitely from B to B′, so that it is, in form, like the hyperbola already investigated, but only placed in a different manner.

Both curves are represented in the next figure; the real axis of the one being the conjugate or imaginary axis of the other.

It is evident from the form of the equations that both curves have got common asymptotes E C E′, F C F′.

224. Let C P and C D be two conjugate diameters to the original hyperbola A P E, it is required to find the locus of D.



Let $C M = x'$, $M P = y'$, $C N = x$, $N D = y$,

$$\text{then } a_1^2 - b_1^2 = a^2 - b^2;$$

$$\therefore x'^2 + y'^2 = x^2 + y^2 + a^2 - b^2$$

but the equation to C D is

$$a^2 y y' - b^2 x x' = 0 \qquad \therefore x' = \frac{a^2 y y}{b^2 x},$$

$$\therefore x'^2 + y'^2 = \frac{a^4 y^2 + b^4 x^2}{b^4 x^2} y'^2 = x^2 + y^2 + a^2 - b^2,$$

$$\therefore y'^2 = \frac{b^4 x^2}{a^4 y^2 + b^4 x^2} (x^2 + y^2 + a^2 - b^2),$$

$$\text{and } x'^2 = \frac{a^4 y^2}{a^4 y^2 + b^4 x^2} (x^2 + y^2 + a^2 - b^2).$$

Substituting these values in the equation $a^2 y'^2 - b^2 x'^2 = - a^2 b^2$, and reducing, we have $a^2 y^2 \quad b^2 x^2 = a^2 b^2$, hence the locus of D is the conjugate hyperbola, and hence arises its name.

By changing the sign of the constant term in the equation to any hyperbola, referred to its centre, we directly obtain the equation to its conjugate, referred to the same axes of x and y. Both curves are comprised in the form

$$(a^2 y^2 - b^2 x^2)^2 = a^4 b^4, \text{ or } x^2 y^2 = k^4.$$

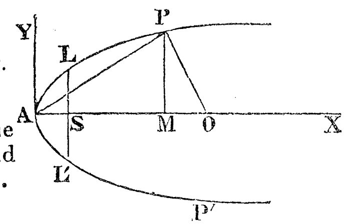
CHAPTER X.

THE PARABOLA.

225. THE equation to the parabola, referred to rectangular axes, has been reduced to the form $a' y^2 + e' x = 0$ (94.).

From this equation we proceed now to deduce all the important properties of the parabola.

Let $\frac{e'}{-a'} = p$, $\therefore y^2 = p\ x$.



Let A be the origin; A X, A Y the axes; then for $x = 0$ we have $y = 0$, and the curve passes through the origin A.

For each positive value of x there are two equal and opposite values of y, which increase from 0 to ∞, according as x increases from 0 to ∞; hence there are two equal arcs, A P and A P', proceeding from A, without any limit. This curve is symmetrical with respect to its axis A X, and its concavity is turned towards that axis, otherwise it could be cut by a straight line in more points than one.

For every negative value of x, y is imaginary.

226. The point A is called the vertex of the parabola; A X, A Y the principal axes; but, generally speaking, A X alone is called the Axis of the parabola. Thus the equation to the curve referred to its axis and vertex is $y^2 = p\,x$.

From this equation we have The square upon the ordinate = The rectangle under the abscissa and a constant quantity; or the square upon the ordinate varies as the abscissa.

227. The last property of this curve points out the difference between the figures of the hyperbola and parabola; both have branches extending to infinity, but of a very different nature; for the equation to the hyperbola is $y^2 = \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2}{a^2} x^2 \left(1 - \frac{a^2}{x^2}\right)$, and therefore, for large values of x, the values of y^2 increase nearly as the corresponding values of x^2 or y varies nearly as x; hence the hyperbolic branch rises much more rapidly than that of the parabola, whose ordinate varies only as $\sqrt{x}$. When x is very great, the former takes nearly the course of the line $y = \frac{b}{a} x$, but in the parabola, y is not much increased by an increase of x, and therefore the curve tends rather towards parallelism with the axis of x.

228. The equation to the parabola may be derived from that of the ellipse by considering the axis major of the ellipse to be infinite.

Let C be the centre, and S the focus of an ellipse whose equation is

$$y^2 = \frac{2\,b^2}{a}\,x - \frac{b^2}{a^2}\,x^2. \quad (105.)$$

$$\text{Let } m = \text{A S} = \text{A C} - \text{S C} = a - \sqrt{a^2 - b^2}, \text{ (fig. 106.)}$$

$$\therefore b^2 = 2\,a\,m - m^2;$$

$$\therefore y^2 = \left(4\,m - \frac{2\,m^2}{a}\right)x - \left(\frac{2\,m}{a} - \frac{m^2}{a^2}\right)x^2.$$

Now if a be considered to vary, this will be the equation to a series of ellipses, in which the distance A S, or m, is the same for all, but the axis major different for each; thus giving to a any particular value, we have a corresponding ellipse. Let now a be infinite, then, since all the other terms vanish, the equation becomes $y^2 = 4\,m\,x$; hence the ellipse has gradually approached to the parabolic form, as its axes enlarged, and finally coincided with it when the axis major was infinite *.

In the same manner the equation to the parabola may be derived from that to the hyperbola.

THE FOCUS.

229. The quantity p, which is the co-efficient of x in the equation to the parabola, is called the principal parameter, or Latus Rectum of the parabola.

Since $p = \frac{y^2}{x}$, the principal parameter is a third proportional to any abscissa and its corresponding ordinate.

In article (228.) we have used the equation $y^2 = 4\,m\,x$ for the parabola, merely to avoid fractions with numerical denominators; it appears that many of the operations in this chapter are similarly shortened, without losing any generality, by merely putting $4\,m$ for p; hence we shall use the equation $y^2 = 4\,m\,x$ in most of the following articles, recollecting that all the results can be expressed in terms of the principal parameter, by putting $\frac{p}{4}$ for m wherever m occurs.

230. To find the position of the double ordinate which is equal to the Latus Rectum.

Let $2\,y = 4\,m$, $\quad \therefore 4\,y^2 = 16\,m^2$, or $16\,m\,x = 16\,m^2$, and $x = m$.

In A X take A S $= m$, then the ordinate L S L′ drawn through S, is the Latus Rectum.

The point S is called the focus.

The situation of the focus S may be also thus determined:

Let A M $= x$, M P $= y$, join A P, and draw P O perpendicular to A P,

Then A M : M,P :: M P : M O $= \frac{y^2}{x} = 4\,m$, $\therefore$ A S $= m = \frac{1}{4}$ M O

* If x is very small when compared with a, the equation to the ellipse is very nearly that to a parabola; and this is the reason that the path of a comet near its perihelion appears to be a portion of a parabola.

231. To find the distance of any point P in the curve from the focus:

Let $S P = r$, $A M = x$, $M P = y$; also at s, $y' = 0$, and $x' = m$,

$$\therefore r^2 = (y - y')^2 + (x - x')^2 = y^2 + (x - m)^2 = 4 m x + (x - m)^2 = (x + m)^2;$$

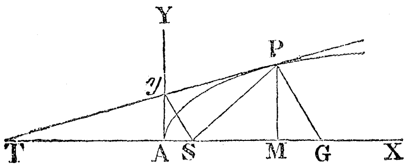
$$\therefore r = S P = x + m.$$

THE TANGENT.

232. To find the equation to the tangent at any point P (x', y') of the parabola.

The equation to a secant through two points on the curve (x', y') $(x'' y'')$ is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$



Also $y'^2 = 4 m x'$, and $y''^2 = 4 m x''$;

$$\therefore y'^2 - y''^2 = 4 m (x' - x''),$$

and $$\frac{y' - y''}{x' - x''} = \frac{4 m}{y' + y''}.$$

Thus the equation to the secant becomes

$$y - y' = \frac{4 m}{y' + y''}(x - x');$$

but when the two points coincide $y'' = y'$, and the secant becomes a tangent,

$$\therefore y - y' = \frac{4 m}{2 y'}(x - x'),$$

or $$y y' - y'^2 = 2 m (x - x'),$$

$$\therefore y y' = y'^2 + 2 m (x - x') = 4 m x' + 2 m (x - x'$$

$$\therefore y y' = 2 m (x + x').$$

This equation is immediately deduced from that to the curve

$$(y^2 = 4 m x = 2 m (x + x))$$

by writing $y y'$ for y^2, and $x + x'$ for $x + x$.

233. To find the points where the tangent cuts the axes.

Let $y = 0$, $\therefore x + x' = 0$, $\therefore x = - x'$, or $A T = - A M$;

Hence the absolute value of the sub-tangent $M T$ is $2 A M$.

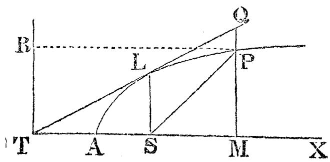
Let $x = 0$, $\therefore y = 2m\frac{x'}{y'} = \frac{y'^2}{2y'} = \frac{y'}{2}$, $\therefore Ay = \frac{1}{2}MP$.

234. The equation to the tangent being $yy' = 2m(x + x')$, we have at the vertex A, x' and y' each $= 0$, therefore the equation to the tangent becomes $2mx = 0$, or $x = 0$;

But $x = 0$ is the equation to the axis A Y;

Hence the tangent at the vertex of the parabola coincides with the axis of y.

235. To find the equation to the tangent at the extremity of the principal parameter.



$$yy' = 2m(x + x')$$

At L we have $x' = m$, and $y' = 2m$,

$$\therefore 2my = 2m(x + m),$$

$$\therefore y = x + m,$$

If the ordinate y or M Q cut the parabola in P, we have $SP = x + m$ (231.), $\therefore MQ = SP$.

236. To find the point where this particular tangent cuts the axis of x.

$$\text{Let } y = 0, \therefore x = AT = -m = -AS.$$

From T draw T R perpendicular to A X, and from P draw P R parallel to A X, then taking the absolute value of A T, we have

$$PR = AT + AM = m + x = SP.$$

Consequently the distances of any point P from S, and from the line T R, are equal to one another.

This line, T R, is called the directrix; for knowing the position of this line and of the focus, a parabola may be described.

This tangent cuts the axis at an angle of 45°. (35. Ex. 3.)

237. To find the length of the perpendicular S y from the focus on the tangent.

Taking the expression in (48.) we have

$$Sy = -\frac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}}.$$

But from the figure 232, we have $y_1 = 0$, and $x_1 = m$ for the coordinates of the point S, and $y = \alpha x + b$ is the equation to the line P T; also the equation to P T is

$$y = \frac{2m}{y'}(x + x'),$$

$$\therefore \alpha = \frac{2m}{y'}, \text{ and } b = \frac{2mx'}{y'},$$

$$\therefore Sy = -\frac{\frac{2m}{y'}m + \frac{2mx'}{y'}}{\sqrt{\left\{1 + \frac{4m^2}{y'^2}\right\}}} = \frac{2m(m + x')}{\sqrt{\{y'^2 + 4m^2\}}} = \frac{2m(m + x')}{\sqrt{\{4mx' + 4m^2\}}}$$

$$= \sqrt{m(m + x')} = \sqrt{mr}, \text{ if } SP = r;$$

Hence the square on S y = the rectangle S P, S A;

$$\text{or, } \text{S P} : \text{S}\,y :: \text{S}\,y : \text{S A}.$$

238. To find the locus of y in the last article.

The equation to the tangent P T, fig. 232, is $y = \frac{2\,m}{y'}(x + x')$;

Hence the equation to S y passing through the point $(m, 0)$, and perpendicular to P T, is

$$y = -\frac{y'}{2\,m}(x - m).$$

To find where this line cuts the axis of y, put $x = 0$, $\therefore\ y = \frac{y'}{2}$, but this is the point where the tangent at P cuts the same axis (233.); hence the tangent and the perpendicular on it from the focus meet in the axis A Y, or the locus of y is the axis A Y.

239. Again, to find where the perpendicular S y cuts the directrix, put

$$x = -m,\ \therefore\ y = -\frac{y'}{2\,m}(x - m) = -\frac{y'}{2\,m}(-m - m) = y',$$

but this is the ordinate M P; hence a tangent being drawn at any point P, the perpendicular on it from the focus cuts the directrix in the point where the perpendicular from P on the directrix meets that directrix.

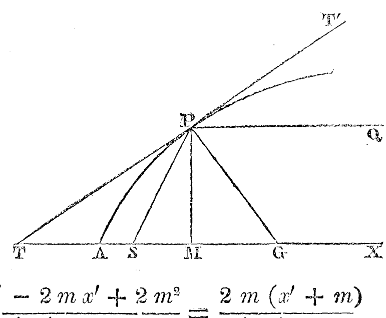
240. To find the angle which the tangent makes with the focal distance

The equation to the tangent P T is $y = \frac{2\,m}{y'}(x + x')$.

The equation to the focal distance S P through the points S $(= 0, m)$ and P $(= x', y')$ is

$$y = \frac{y'}{x' - m}(x \quad m),$$

And tan. S P T = tan. (P S X − P T X)



$$= \frac{\dfrac{y'}{x' - m} - \dfrac{2\,m}{y'}}{1 + \dfrac{y'}{x' - m}\,\dfrac{2\,m}{y'}}$$

$$= \frac{y'^2 - 2\,m\,(x' - m)}{y'\,(x' - m) + 2\,m\,y'} = \frac{4\,m\,x' - 2\,m\,x' + 2\,m^2}{y'\,(x' + m)} = \frac{2\,m\,(x' + m)}{y'\,(x' + m)}$$

$$= \frac{2\,m}{y'} = \frac{y'}{2\,x'},\quad \text{since } 2\,m = \frac{y'^2}{2\,x'}:$$

But M P = M T tan. P T M, $\therefore$ tan. P T M $= \frac{y'}{2\,x'}$, $\therefore$ tan. S P T

= tan. S T P = tan. T′ P Q, if P Q be drawn parallel to the axis of x. Thus the tangent at P makes equal angles with the focal distance, and with a parallel to the axis through P.

This important theorem may also be deduced from the property in article 233. It is there proved that the absolute value of A T is A M, hence we have $ST = SA + AT = m + x = SP$, and therefore the angle S P T = angle S T P = angle Q P T′.

If a ray of light, proceeding in the direction Q P, be incident on the parabola at P, it will be reflected to S, on account of the equal angles Q P T′, S P T: similarly all rays coming in a direction parallel to the axis, and incident on the curve, would converge to S; and if a portion of the curve revolve round its axis, so as to form a hollow concave mirror, all rays from a distant luminous point in the direction of the axis would be concentrated in S. Thus, if a parabolic mirror be held with its axis pointing to the sun, a very powerful heat will be found at the focus.

Again, if a brilliant light be placed in the focus of such a mirror, all the rays, instead of being lost in every direction, will proceed in a mass parallel to the axis, and thus illuminate a very distant point in the direction of that axis. This property of the curve has led to the adoption of parabolic mirrors in many light-houses.

THE NORMAL.

241. To find the equation to the normal P G, at a point P $(x' y')$.

The equation to a straight line, through P, is $y - y' = \alpha\,(x - x')$, and as this line must be perpendicular to the tangent whose equation is $y = \frac{2\,m}{y'}\,(x + x')$, we have $\alpha = -\frac{y'}{2\,m}$, hence the equation to the normal is $y - y' = -\frac{y'}{2\,m}\,(x - x')$.

242. To find the point where the normal cuts the axis of x.

Let $y = 0 \quad \therefore\ x - x' = m$, or the subnormal M G is constant and equal to half the principal parameter.

Hence $SG = SM + MG = x' - m + 2\,m = x' + m = SP$

And $PG = \sqrt{y'^2 + 4\,m^2} = \sqrt{4\,m\,x' + 4\,m^2} = \sqrt{4\,m\,(x' + m)} = \sqrt{4\,m\,r}$.

Hence the normal P G is a mean proportional between the principal parameter and the distance S P.

THE DIAMETERS.

243. It was shown in article 81, that the parabola has no centre.

Since for every positive value of x there are two equal and opposite values of y, the axis of x is a diameter, but that of y is not; hence the axes cannot be called conjugate axes. The parabola has an infinite number of diameters, all parallel to the axis; to prove this,

Let $y = \alpha\,x + b$ be the equation to any chord,

$y^2 = p\,x$ the equation to the curve.

Transfer the origin to the bisecting point $x'\,y'$ of the chord, then the equations become $y = \alpha\,x$, and $(y + y')^2 = p\,(x + x')$.

To find where the chord intersects the curve, put $a\,x$ for y in the second equation.

$$\therefore\ (a\,x + y')^2 = p\,(x + x')$$
$$\text{or } a^2 x^2 + (2\,a\,y' - p)\,x + y'^2 - p\,x' = 0$$

But since the origin is at the bisection of the chord, the two values of x must be equal to one another, and have opposite signs; hence the second term of the last equation must $= 0$, $\therefore\ 2\,a\,y' - p = 0$.

This equation gives the value of y', and since it is independent of b, it will be the same for any chord parallel to $y = a\,x + b$; hence $y = \dfrac{p}{2\,a}$ is the equation to the locus of all the middle points of a system of parallel chords, and this equation is evidently that to a straight line parallel to the axis; and conversely.

244. To transform the equation into another referred to a new origin and to new axes, and so that it shall preserve the same form,

$$\text{Let } x = a + x' \cos.\,\theta + y' \cos.\,\theta',$$
$$\text{and}\quad y = b + x' \sin.\,\theta + y' \sin.\,\theta', \quad (57.)$$

Substituting these values in the equation $y^2 = p\,x$ and arranging, we have

$$y'^2 (\sin.\,\theta')^2 + x'^2 (\sin.\,\theta)^2 + 2\,x'\,y' \sin.\,\theta \sin.\theta' + y'\,(2\,b \sin.\theta' - p \cos.\,\theta')$$
$$+ x'\,(2\,b \sin.\,\theta - p \cos.\,\theta) + b^2 - a\,p = 0.$$

And as this equation must be of the form $y^2 = p\,x$, we must have

$$(\sin.\,\theta)^2 = 0 \quad . \quad . \quad . \quad (1),$$
$$2 \sin.\,\theta.\ \sin.\,\theta' = 0 \quad . \quad . \quad (2),$$
$$2\,b \sin.\,\theta' - p \cos.\,\theta' = 0 \quad (3),$$
$$b^2 - a\,p = 0 \quad . \quad . \quad . \quad (4).$$

Hence the equation becomes

$$y'^2 (\sin.\,\theta')^2 + (2\,b \sin.\,\theta - p \cos.\,\theta)\,x' = 0;$$
$$\text{or since } \theta = 0,\quad y'^2 (\sin.\,\theta')^2 - p\,x' = 0.$$

245. On the examination of the equations (1) (2) (3) and (4), it appears from (1) that the new axis of x' is parallel to the original axis of x; and θ being 0 from (1), of course (2) is destroyed, and thus the equations of condition are reduced to three: but there are four unknown quantities, hence there are an infinite number of points to which, if the origin be transferred, the equation may be reduced to the same simple form.

We may take the remaining three quantities a, b and θ', in any order, and arrive at the same results. Suppose a is known, then from (4), $b^2 = p\,a$, this equation shows that a must be taken in a positive direction from A, and also that the new origin must be taken on the curve itself, or the new origin is at some point P on the curve, as in the next figure.

From (3) we have $\tan.\,\theta = \dfrac{p}{2\,b} = \dfrac{b}{2\,a}$;

but this is exactly the value of the tangent of the angle which a tangent P T to the curve makes with the Axis (240.): hence the new axis of y is a tangent to the curve at the new origin P.

The results are therefore these,—the new origin is at any point P on the curve (see the next figure). The axes are one (P X′) parallel to the axis A X, and the other (P Y′) is a tangent at the new origin P. Lastly, from the form of the equation, the new axis of x is a diameter.

246. The equation is $y'^2 = \frac{p}{(\sin.\ \theta')^2}\, x' = p' x$ where $p' = \frac{p}{(\sin\ \theta')^2}$

$= p\,(\text{cosec.}\ \theta')^2 = p\,(1 + \cot.\theta')^2 = p\left(1 + \frac{4\,a^2}{b^2}\right) = p + 4\,a$

$= 4\left(\frac{p}{4} + a\right) = 4$ S P (231.)

Hence the new parameter at P is four times the focal distance S P.

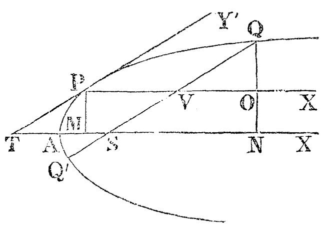
247. The equation to the parabola, when we know the position and direction of the new axes, is readily obtained from the original equation referred to rectangular co-ordinates.

Let the point P be the new origin, P X′, P Y′ the new axes, angle Y′ P X′ $= \theta$.

Also, let A N $= x$, N Q $= y$ be the rectangular co-ordinates of Q.

And A M $= a$, M P $= b$ . . . . . P.

P V $= x'$, V Q $= y'$ be the new co-ordinates of Q.



Then $y =$ Q N $=$ M P + O Q $= b + y' \sin.\ \theta$,

$x =$ A N $=$ A M + P V + V O $= a + x' + y' \cos.\ \theta$.

Substituting these values in the equation $y^2 = p\,x$,

we have $(b + y' \sin.\ \theta)^2 = p\,(a + x' + y' \cos\ \theta)$;

$\therefore\ y'^2 (\sin.\ \theta)^2 + (2\,b \sin.\ \theta - p \cos.\ \theta)\, y' + b^2 = p\,a + p\,x$;

but $b^2 = p\,a$, and $\tan.\ \theta = \tan.$ P T M $= \frac{b}{2\,a} = \frac{b}{2\frac{b^2}{p}} = \frac{p}{2\,b}$,

$$\therefore\ 2\,b \sin.\theta - p \cos.\theta = 0,$$

and the equation is reduced to the form

$$y'^2 (\sin.\ \theta)^2 = p\,x'.$$

Also from $(\tan.\ \theta)^2 = \frac{b^2}{4\,a^2}$ we have $(\cos.\ \theta)^2 = \frac{4\,a^2}{4\,a^2 + b^2}$

and $(\sin.\ \theta)^2 = \frac{b^2}{4\,a^2 + b^2} = \frac{p}{4\,a + p}$;

$$\therefore y'^2 \frac{p}{4a+p} = p x'$$

$$y'^2 = (4a+p)\,x' = 4\left(a + \frac{p}{4}\right) x' = p'\,x', \text{ where } p' = 4\,SP.$$

Hence the square upon the ordinate = the rectangle under the abscissa and parameter.

248. To find the length of the ordinate which passes through the focus:

Here, $x = PV = ST = SP = r \;\therefore\; y^2 = px = 4r\,.\,r = 4r^2$

$$\therefore y = 2r$$

Hence, $QQ' = 4\,SP$.

Thus the ordinate through the focus is equal to four times the focal distance S P, is equal to the parameter at the point P.

Hence, generally, if the origin of co-ordinates be at any point P on the parabola, and if the axes be a diameter and a tangent at P, the parameter to the point P is that chord which passes through the focus.

249. The equation to a tangent at any point Q $(x'\ y')$, referred to the new axes PX', PY', is

$$y y' = \frac{p'}{2}\left(x + x'\right)$$

Let $y = 0 \;\therefore\; x = -x'$, hence the sub-tangent = twice the abscissa.

Let $x = 0 \;\therefore\; y = \dfrac{p'x'}{2y} = \dfrac{y'}{2} = \frac{1}{2}$ the ordinate.

For y put $-y$, then we have the equation to the tangent at the other extremity Q' of the ordinate QVQ'; hence it may be proved that tangents at the two extremities of a chord meet in a diameter to that chord.

250. If the chord QVQ' pass through the focus, as in the figure, the co-ordinates of Q are $y' = 2\,SP = 2\,r$, and $x' = PV = SP = r$, also

$p' = 4r$; hence the equation to the tangent at Q, or $y y' = \dfrac{p'}{2}(x + x')$

becomes $y = x + r$, and similarly the equation to the tangent at Q' is $-y = x + r$, and these lines meet the axis PX' at a distance $-r$ from P, that is, tangents at the extremity of any parameter meet in the directrix.

Also, the angle between these tangents is determined from the equation

$$\tan.\,\theta = \frac{(\alpha - \alpha')\sin.\,\omega}{1 + \alpha\alpha' + (\alpha + \alpha')\cos.\,\omega} \qquad (51.)$$

$$= \frac{(1+1)\sin.\,\omega}{1 - 1 + 0} \text{ (since } \alpha = 1 \text{ and } \alpha = -1)$$

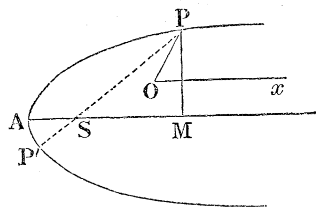
$$= \frac{1}{0} = \tan.\,90^\circ.$$

Hence, pairs of tangents drawn at the extremities of any parameter meet in the directrix at right-angles.

THE POLAR EQUATION.

251. To find the polar equation to the curve.

Let the co-ordinates of any point O be x' and y', and let θ be measured from a line O x, which is parallel to the axis of the curve:



Then by (61.), or by inspection of the figure, we have

$$y = y' + u \sin.\theta$$

$$x = x' + u \cos.\theta$$

Substituting these values of x and y in the equation $y^2 = p\,x$, we have

$$(y' + u \sin.\theta)^2 = p\,(x' + u \cos.\theta)$$

252. Let the pole be at any point on the curve,

$$\therefore\ y'^2 + 2\,u\,y' \sin.\theta + u^2 (\sin.\theta)^2 = p\,x' + p\,u \cos.\theta\,;$$

$$\text{or, } u\,(\sin.\theta)^2 = p \cos.\theta - 2\,y' \sin.\theta, \quad \text{since } y'^2 = p\,x'\,;$$

$$\therefore\ u = \frac{p \cos.\theta - 2\,y' \sin.\theta}{(\sin.\theta)^2},$$

And if the vertex be the pole, we have $y' = 0$;

$$\therefore\ u = \frac{p \cos.\theta}{(\sin.\theta)^2}.$$

253. Let the focus S be the pole, $\therefore\ y' = 0$, $x' = \frac{p}{4}$ and u becomes r;

hence the general equation $(y' + u \sin.\theta)^2 = p\,(x' + u \cos.\theta)$ becomes

$$r^2 (\sin.\theta)^2 = \frac{p^2}{4} + p\,r \cos.\theta,$$

$$\text{or } r^2 (\sin.\theta)^2 + r^2 (\cos.\theta)^2 = \frac{p^2}{4} + p\,r \cos.\theta + r^2 (\cos.\theta)^2$$

$$\therefore\ r^2 = \left(\frac{p}{2} + r \cos.\theta\right)^2$$

$$r = \frac{p}{2} + r \cos.\theta; \quad \text{or } r = \frac{p}{2} \cdot \frac{1}{1 - \cos.\theta}.$$

The polar equation in this case is also easily deduced from article (231). Let angle A S P $= \theta$,

$$\text{then } r = \text{SP} = \text{AM} + \text{AS} = 2\,\text{AS} + \text{SM} = \frac{p}{2} - r \cos.\theta;$$

$$\therefore\ r = \frac{p}{2}\,\frac{1}{1 + \cos.\theta} = \frac{p}{\left(\cos.\frac{\theta}{2}\right)^2}$$

254. If P S meet the curve again in P′, we have S P′

$$\frac{p}{2} \cdot \frac{1}{1 - \cos.(\pi - \theta)} = \frac{p}{2} \frac{1}{1 + \cos.\theta};$$

hence the rectangle

$$PS, SP' = \frac{p}{4} \frac{p}{1 - (\cos.\theta)^2} = \frac{p}{4} (SP + SP') = \frac{p}{4}, PP'.$$

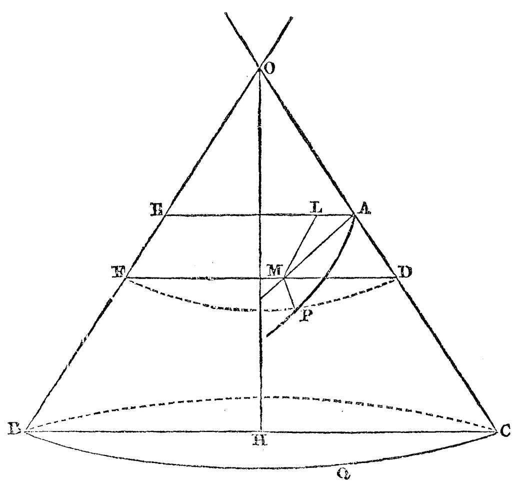
CHAPTER XI

THE SECTIONS OF A CONE.

255. It is well known that the three curves, the ellipse, the hyperbola, and parabola, were originally obtained from the section of a cone, and that hence they were called the conic sections. We shall now show the manner in which a cone must be cut by a plane, in order that the section may be one of these curves.

A right cone is the solid generated by the revolution of a right-angled triangle about one of its perpendicular sides.

The fixed side, O H, about which the triangle revolves, is called the axis; and the point O, where the hypothenuse of the triangle meets the axis, is called the vertex of the cone. If the revolving hypothenuse be produced above the vertex, it will describe another cone, having the same axis and vertex. Any point in the hypothenuse of the triangle describes a circle; hence, the base of the triangle describes a circular area called the base of the cone.



Section made by planes which pass through the vertex and along the axis are called vertical sections; these are, evidently, triangles.

If a plane pass through the cone in any direction, the intersection of it with the surface of the cone is called a conic section. The nature of the ine thus traced will be found to be different, according to the various positions of the cutting plane. It is our purpose to show, generally, to what class of curves a section must necessarily belong; and, afterwards, to point out the particular species of curve due to a given position of the cutting plane.

256. Let O B Q C be a right cone, O the vertex, O H the axis, B C Q the circular base, P A the line in which the cutting plane meets the surface of the cone; A being the point in the curve nearest to the vertex O. Let O B H C A be a vertical plane passing through the axis O H and perpendicular to the cutting plane P A M.

A M, the intersection of these planes, is a straight line, and is called the axis of the conic section, the curve being symmetrically placed with regard to it.

Let F P D be a section parallel to the base, it is therefore a circle, and F M D, its intersection with the vertical plane O B H C A, is a diameter.

Since both this last plane F P D and the cutting plane P A M are perpendicular to the vertical plane O B H C, M P the intersection of the two former is perpendicular to the vertical plane, (Euc. xi. 19, or Geometry iv. 18,) and, therefore, to all lines meeting it in that plane. Hence M P is perpendicular to F D and to A M.

Let the angle O A M, which is the inclination of the cutting plane to the side of the cone, $= \alpha$, and let the $\angle$ A O B $= \beta$, draw A E parallel to B H and M L parallel to O B.

Let A M $= x$, M P $= y$, and A O $= a$.

Then by the property of the circle

The square on M P = the rectangle F M, M D;

$$\text{and } MD = \frac{MA \sin. MAD}{\sin. MDA} = x \frac{\sin. \alpha}{\cos. \frac{\beta}{2}};$$

$$\text{Also, } FM = EA - AL = \frac{AO \sin. AOE}{\sin. OEA} - \frac{AM \sin. AML}{\sin. ALM};$$

But angle O E A $= 90^\circ - \frac{\beta}{2}$, angle A L M $= 90^\circ + \frac{\beta}{2}$, and if we produce M L to meet O A, we shall find that the angle A M L $= 180^\circ - (\alpha + \beta)$;

$$\text{hence } FM = a \frac{\sin. \beta}{\cos. \frac{\beta}{2}} - x \frac{\sin. (\alpha + \beta)}{\cos. \frac{\beta}{2}};$$

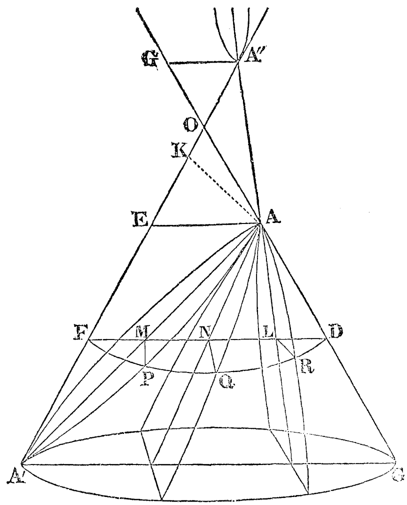
$$\therefore\ y^2 = x \frac{\sin. \alpha}{\cos. \frac{\beta}{2}} \left\{ a \frac{\sin. \beta}{\cos. \frac{\beta}{2}} - x \frac{\sin. (\alpha + \beta)}{\cos. \frac{\beta}{2}} \right\}$$

$$\text{or } y^2 = \frac{\sin. \alpha}{\left(\cos. \frac{\beta}{2}\right)^2} \{ a \sin. \beta \,.\, x - \sin. (\alpha + \beta)\, x^2 \};$$

which equation being of the second degree, it follows that the sections of the cone are curves of the second degree.

Comparing this with the equation $y^2 = p\,x + q\,x^2$, which represents an ellipse, a parabola, or an hyperbola, according as q is negative, nothing, or positive; we observe that the section is an ellipse, a parabola, or an hyperbola, according as sin. $(\alpha + \beta)$ is positive, nothing, or negative. To investigate these various cases, we shall suppose the cutting plane to move about A, so that α may take all values from 0 to 180°.

257. Let $\alpha = 0$, $\therefore$ $y^2 = 0$, and $y = 0$; this is the equation to the straight line which is the axis of x.



And this appears, also, from the figure; for when $\alpha = 0$, the cutting plane just touches the cone, and hence the line of intersection A M is in the position A O.

258. Let $\alpha + \beta$ be less than 180°. The curve is an ellipse. In the figure the angles A O E and O A M being together less than 180°, the lines O E and A M meet in A′, or the sectional plane cuts both sides of the cone.

259. Let M be the centre of the ellipse, then F M $= \frac{1}{2}$ A E and M D $= \frac{1}{2}$ A′ G;

$\therefore$ The square on the axis minor = The rectangle A E, A′ G.

Also by drawing perpendiculars from A and E upon A′ G, it may be proved that

The square on the axis major = The square on A G + The rectangle A E, A′ G.

And $\therefore$ The distance between the foci = A G.

If the straight line A K be drawn making the angle E A K = the angle E A A′, then A K is the latus rectum of the section.

And if a circle be inscribed in the triangle A′ A O, it will touch the line A A′ in the focus of the section. (Geometry, Appendix, prop. 21.)

260. Let $\alpha = 90° - \frac{\beta}{2}$, then $\frac{\sin.\ \alpha \sin.\ (\alpha + \beta)}{\left(\cos.\ \frac{\beta}{2}\right)^2} = 1$,

and the equation is that to a circle, the cutting plane being parallel to the base.

261. Let $\alpha + \beta = 180°$, $\therefore \sin.\ (\alpha + \beta) = 0$, and the curve is a parabola. The plane, continuing to turn, has now come into the position A N Q, the axis A N being parallel to O F, or the cutting plane parallel to a side of the cone.

The equation to the parabola is $y^2 = 4\ a \left(\sin.\ \frac{\beta}{2}\right)^2 x$.

If A K be drawn making the angle E A K = the angle A O K, then A K is the latus rectum of the section, and the circle which touches A O, A N and O F, will touch A N in the focus of the parabola.

262. Let $\alpha + \beta$ be greater than 180° $\therefore$ sin. $(\alpha + \beta)$ is negative, and the curve is an hyperbola; The cutting plane is now in the position A L R; in this case the lines A L, E O must meet if produced backwards, or the plane cuts both cones, and the curve consists of two branches, one on the surface of each cone.

As in the ellipse, it may be proved that the square on the conjugate axis = the rectangle A E, A″ G′; that A G′ is the distance between the foci, that A K is the latus rectum, and that the circle touching A′ O, O A and A L touches A L at the focus.

263. We may also suppose a to have different values, or the cutting plane to meet the cone in some other point than A, for example:

$$\text{Let } a = 0 \quad \therefore\ y^2 = -\frac{\sin.\ \alpha \sin.\ (\alpha + \beta)}{\left(\cos.\ \frac{\beta}{2}\right)^2} x^2;$$

Since sin. α and $\left(\cos.\ \frac{\beta}{2}\right)^2$ are positive, the rationality of this equation will depend upon sin. $(\alpha + \beta)$.

If $\alpha + \beta$ is less than 180° the radical quantity is impossible, and the only solution of the equation is $x = 0$ and $y = 0$, or the section is a point; this is the case when the cutting plane passes through the vertex O, and is parallel to any elliptic section A P A′.

If $\alpha + \beta$ is greater than 180° we have two straight lines which cut each other at the origin. In this case the cutting plane is drawn through O, parallel to A L R, and the intersection with the cone is two straight lines meeting in O.

264. We may conclude from this discussion, that the conic sections are seven: a point, a straight line, two straight lines which intersect, a circle, an ellipse, an hyperbola, and a parabola or all the curves of the second degree and their varieties, with the exception of two parallel lines, which is a variety of the parabola.

The three latter sections, the ellipse, hyperbola, and parabola, are those which are usually termed "conic sections," and which have been the study and delight of mathematicians since the time of Plato. In his school they were first discovered; and his disciples, excited, no doubt, by the many beautiful properties of these curves, examined them with such industry, that in a very short time several complete treatises on the conic sections were published. Of these, the best still extant is that of Apollonius of Perga. It is in eight books, four of which are elementary; and four on the abstruser properties of these curves. The whole work is well worth attention, as showing how much could be done by the ancient analysis, and as giving a very high opinion of the geometrical genius of the age.

Apollonius gave the names of ellipse and hyperbola to those curves—Hyperbola, because the square on the ordinate is equal to a figure "exceeding" ("ὑπερβαλλον") the rectangle under the abscissa and latus rectum by another rectangle.—B. i. p. 13.

Ellipse, because the square on the ordinate is "defective" ("ελλειπον") with regard to the same rectangle.—p. 14.

It is not known who gave the name of parabola to that curve—probably Archimedes, because the square of the ordinate is equal ("παραβαλλον") to the rectangle of the abscissa and latus rectum.

Thus, the ancients viewed these curves geometrically, in the same manner as we are accustomed to express them by the equations:

$$y^2 = p\,x + \frac{p}{2\,a}\,x^2,$$

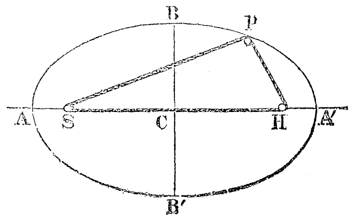
$$y^2 = p\,x - \frac{p}{2\,a}\,x^2,$$

$$y^2 = p\,x.$$

DESCRIPTION OF THE CONIC SECTIONS BY CONTINUED MOTION.

265. The conic sections being curves of great importance, not only from their mathematical properties, but also from their usefulness in the arts and sciences, it becomes necessary that we should be able to describe these curves with accuracy. Now, a curve may be drawn in two ways, either by "mechanical description" or by "points." As an instance of the first method we may mention the circle, described by the compasses, or by means of a string fastened at one end to the centre, and the other carried round by the hand, the hand tracing the curve. This mechanical method, or, as it is sometimes called, "that by continued motion," is not always practicable: no curve is so simple, in this respect, as the circle; hence we are often obliged to have recourse to the second method, or that by points: this is done by taking the equation to the curve and from some property expressed geometrically, finding a number of points, all of which belong to the curve, and then neatly joining these points with a pen or other instrument. We shall commence with the mechanical description of these curves.

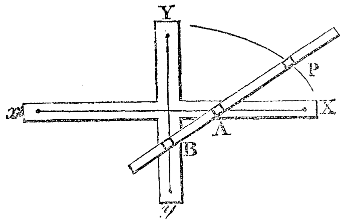
266. To trace an ellipse of which the axes are given:



Let A A′, B B′ be the axes: with centre B and radius A C describe a circle cutting A A′ in S and H, these points are the foci. Place pegs at S and H. Let one extremity of a string be held at A, and pass the string round H back again to A, and there join its two ends by a knot, so that its length shall be just double of A H; place a pen or other pointed instrument within this string, and move it round the points S and H, so that the string be always stretched; the pen will trace out the required ellipse. For if P be one of its positions, we have

$$S P + P H + H S = 2 A H = A A' + H S;$$
$$\therefore S P + P H = A A'.$$

267. Another method is by means of an instrument called the elliptic compasses, or the trammel.



Let $X x$ and $Y y$ be two rulers with grooves in them, and fastened at right angles to each other. Let B P be a third ruler, on which take B P equal to the semi-axis major, and P A the semi-axis minor. At B a peg is so fixed that the point B with the peg can move along $Y y$; a similar peg is fixed at A. By turning the ruler B P round, a pen placed at P will trace out the curve. Suppose C to be the point where the axes meet, C M $= x$ and M P $= y$, the rectangular co-ordinates of P, and suppose that B N is drawn parallel to C M and meeting P M in N, then A M $= \frac{b}{a}$ B N, and

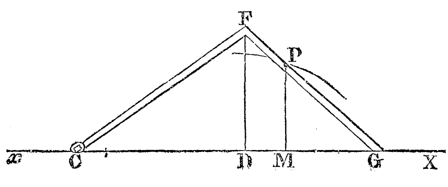
The square on A P = the square on P M + the square on A M,

$$\text{or } b^2 = y^2 + \frac{b^2}{a^2} x^2,$$
$$\therefore a^2 y^2 + b^2 x^2 = a^2 b^2.$$

268. The following is also a very simple method of describing the ellipse. $X x$ is a ruler of any length, C F, F G are two rulers, each equal to half the sum of the semi-axes. These rulers are fastened together by a moveable joint at F, and F C turns round a pivot at C; F P is taken equal to half the difference of the semi-axes. Let the point G slide along the line $X x$, then the point P will trace out the curve. Draw F D and P M perpendicular to C X, and let C M $= x$, and M P $=$ y, then

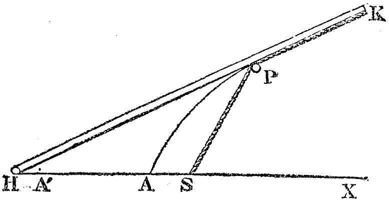
The square on F G = the square on F D + the square on D G;

$$\text{or } \left(\frac{a+b}{2}\right)^2 = \left(\frac{a+b}{2}\frac{y}{b}\right)^2 + \left(\frac{a+b}{2}\frac{x}{a}\right)^2$$
$$\therefore \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1.$$



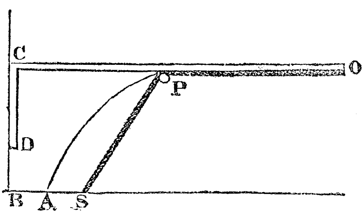
For a description of the Elliptograph, and other instruments for describing ellipses, we must refer our readers to the treatise on Practical Geometry, where an extremely good account is given of all the instruments, and also the advantages and disadvantages of each are well exhibited

269. To trace the hyperbola by continued motion, let A A′ be the transverse axis, S H the distance between the foci, H P K a ruler movable about H. A string, whose length is less than H K by A A′ is fastened to K and S; when the ruler is moved round H, keep the string stretched, and in part attached to the ruler by a pencil as at P; then, since the difference of H P and P S is constantly the same, the point P will trace out the curve.



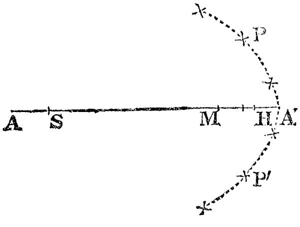
If the length of the string be H K, a straight line perpendicular to H S will be traced out; and if the string be greater than H K, the opposite branch, or that round H, will be described.

270. To trace the parabola by continued motion. Let S be the focus, and B C the directrix. Apply a carpenter's square O C D to the ruler B C, fasten one end of a thread whose length is C O to O, and the other end to the focus S; slide the square D C O along B C, keeping the thread tight by means of a pencil P, and in part attached to the square. Then since S P = P C, the point P will describe a parabola.

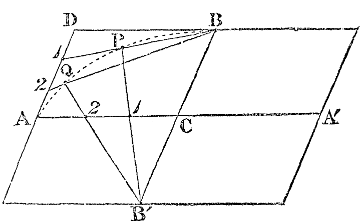


Description of the Conic Sections by Points.

271. Given the axes of an ellipse to describe the curve. Let A A′ be the axis major, S and H the foci. With centre S, and any radius A M less than A A′, describe a circle, and with centre H and radius A′ M describe a second circle, cutting the former in two points P and P′; then since S P + P H = A M + M A′ = A A′, P is a point in the required curve; and thus any number of points may be found, and the curve described.



272. Given a pair of conjugate diameters to describe the curve. Let A A′, B B′, be the conjugate diameters. Through B draw B D parallel to A C, and through A draw A D parallel to B C. Divide A D and A C into the same number of equal parts as three. From B draw lines to the dividing points in A D, and from B′ draw lines to the dividing points in A C; the intersections P, Q, of these lines are points in the ellipse.



For let C be the origin; $CA = a_1$, $CB = b_1$,

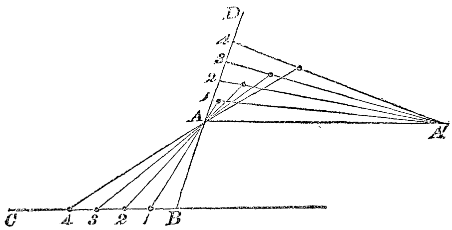
Then the equation to B P is $y - b_1 = \frac{b_1}{3a_1}x$;

and the equation to B′P is $y + b_1 = -\frac{3b_1}{a_1}x$.

Hence the product of the tangents of the angles which these lines B P, B′P make with the axis of $x = -\frac{b_1}{3a_1}\frac{3b_1}{a_1} = -\frac{b_1^2}{a_1^2}$ and is constant; therefore P is a point in the curve (141).

Innumerable points may be thus found in the four quadrants of the figure.

273. The following is perhaps the best method of tracing the ellipse by points:



Let A A′ be a diameter and A B equal and parallel to the conjugate diameter. Through B draw B C parallel to A A′ and equal to any multiple of A A′. In B A produced, take A D the same multiple of A B. Divide B C into any number of equal parts, and A D into the same number of equal parts. Through A draw lines to the points of division in B C, and through A′ draw lines to the points of division in A D; the intersections of corresponding lines will give points in an ellipse whose conjugate diameters are A A′ and A B. The proof is the same as in the last case.

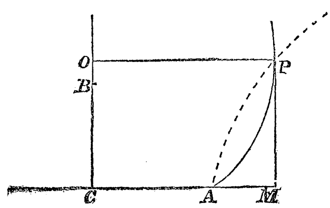
274. Given the axes of an hyperbola to trace the curve.

Let A A′ be the transverse axis, S and H the foci, which are given points; with centre S and any radius A M greater than A A′, describe a circle, and with centre H and radius A′ M describe a second circle, cutting the former in two points P and P, these are points in the required curve.

The proof is much the same as that for the ellipse (271.)

Again, if, in article 273, B C was taken to the right of B instead of the left, as in the figure, the intersections of the corresponding lines will give an hyperbola.

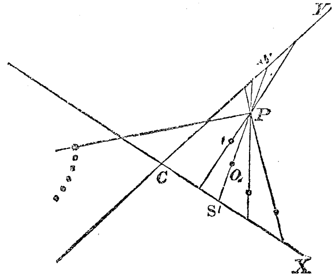
275. To describe the rectangular hyperbola by points.



Let C A, C B be the equal semi-axes with any centre O in C B produced and with radius O A, describe a circle; draw O P perpendicular to C O meeting the circle in P, then P is a point in the curve; Let $CM = x$, $MP = y$; then the square on C O = the square on O A − the square on C A;

$$\text{or } y^2 = x^2 - a^2.$$

276. Given the asymptotes C X, C Y of an hyperbola, and one point P in the curve, to describe the curve by points.

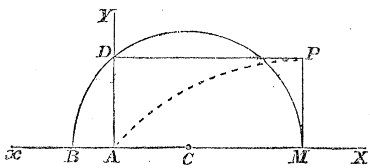


Through P draw any line S P S′ terminated by the asymptotes; in it take S′Q = S P; then Q is a point in the curve (217), and similarly any number of points may be found.

Together with the asymptotes, another condition must always be given to enable us to trace the curve, for the position of the asymptotes only gives us the ratio of the axes, and not the axes themselves.

277. To describe the parabola by points, when the principal parameter p is given.

Let A X, A Y be the rectangular axes; in A x take A B = p; with any centre C in A X and radius C B describe a circle B D M, cutting A Y in D and A X in M, draw D P and M P perpendicular to A Y and A X respectively; then P is a point in the curve.

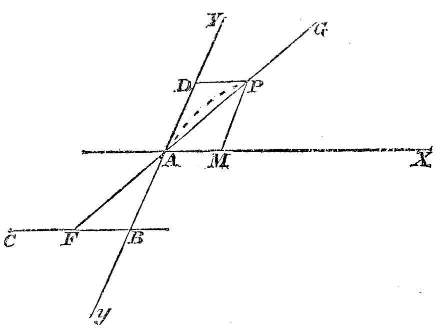


Let A M = x, M P = y; then the square on A D = the rectangle B A, A M,

$$\text{or } y^2 = p\,x.$$

278. Given the angle between the axes and any parameter p' to describe the curve.

Let A X, Y A y be the axes, A B the parameter. Through B draw C B parallel to A X. Through A draw any line F A G, meeting B C in F; in A Y take A D = B F, and draw D P parallel to A X, cutting A G in P, then P is a point in the curve.



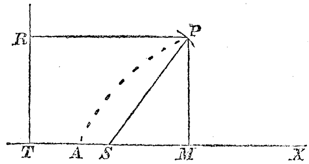
Draw M P parallel to A Y, and let

$$\text{A M} = x, \text{ and M P} = y,$$

$$\text{then M P : M A :: A B : F B},$$

$$\text{or } y : x :: p' : y; \quad \therefore y^2 = p'\,x.$$

279. Given the position of the directrix T R and the focus S, to trace any of the conic sections by points.



Draw S T perpendicular to T R, then T S produced will be the axis of the curve.

Let $e : 1$ be the ratio of the distance of any point P in the curve from the focus and from the directrix; hence if A S : A T :: e : 1; A is a point in the curve. Take any point M in A X, and with centre S and radius equal to e times T M, describe a circle; draw M P perpendicular to A X, and meeting the circle in P, then P is a point in the curve.

Let A be the origin of rectangular co-ordinates, $\text{AM} = x$, $\text{MP} = y$, $\text{AS} = m$, and $\therefore \text{AT} = \frac{m}{e}$;

$$\text{then SP} = e \,.\, \text{TM} = e \,.\, \text{PR}$$

$$\therefore \quad y^2 + (x - m)^2 = e^2\left(x + \frac{m}{e}\right)^2$$

$$\text{or } y^2 + x^2 - 2\,m\,x + m^2 = e^2\,x^2 + 2\,e\,m\,x + m^2;$$

$$\therefore \quad y^2 + (1 - e^2)\,x^2 - 2\,m\,x\,(1 + e) = 0;$$

which is the equation to the curves of the second order.

Let e be less than unity, $\therefore \; y^2 = (1 - e^2)\left\{\frac{2\,m}{1 - e}\,x - x^2\right\}$.

Comparing this equation with that to the ellipse $y^2 = \frac{b^2}{a^2}\,(2\,a\,x - x^2)$, we have

$$2\,a = \frac{2\,m}{1 - e} \text{ and } \frac{b^2}{a^2} = 1 - e^2,$$

$$\therefore \; b^2 = \frac{m^2}{(1 - e)^2}\,(1 - e^2) = m^2\frac{1 + e}{1 - e};$$

hence the curve is an ellipse whose axes are $\frac{2\,m}{1 - e}$ and $2\,m\sqrt{\frac{1 + e}{1 - e}}$.

Let e be greater than unity, $\therefore \; y^2 = (e^2 - 1)\left\{\frac{2\,m}{e - 1}\,x + x^2\right\}$;

and the curve is an hyperbola, whose axes are $\frac{2\,m}{e - 1}$ and $2\,m\sqrt{\frac{e + 1}{e - 1}}$.

Let e be equal to unity, $\therefore \; y^2 = 4\,m\,x$;

the curve is a parabola, whose principal parameter is $4\,m$.

280. The general equation to all the conic sections being

$$y^2 + (1 - e^2)\,x^2 - 2\,m\,x\,(1 + e) = 0,$$

it follows that if we find any property of the ellipse from this equation, it will be true for the hyperbola and parabola, making the necessary changes in the value of e:

Thus the equation to the tangent is

$$y\,y' + (1 - e^2)\,x\,x' - m\,(1 + e)\,(x + x') = 0, \text{ for the ellipse,}$$

$$y\,y' - (e^2 - 1)\,x\,x' - m\,(1 + e)\,(x + x') = 0, \text{ for the hyperbola,}$$

$$\text{and } y\,y' - 2\,m\,(x + x') = 0, \text{ for the parabola.}$$

Also most of the results found in Chapter VIII. for the ellipse will be true for the hyperbola, by putting $-b^2$ for b^2; and will be true for the parabola by transferring the origin to the vertex of the ellipse, by then putting $\frac{m}{1-e}$ for a, and $m^2 \frac{1+e}{1-e}$ for b^2; and then making $e = 1$. Thus the equation to the tangent at the extremity of the Latus Rectum in the ellipse, when the origin is at the vertex, is

$$y = a + e\,(x - a)\ (117),$$

$$\text{or,}\quad y = a\,(1 - e) + e\,x,$$

$$\text{for } a \text{ put } \frac{m}{1-e}, \text{ and then let } e = 1;$$

$$\therefore\ y = m + x, \text{ as in (235).}$$

281. If $S\,P = r$, and $A\,S\,P = \theta$, the polar equation to the curve thus traced is easily found:

$$S\,P = e.\,P\,R = e\,(T\,S + S\,M),$$

$$\text{or}\quad r = e\,(m\,\frac{1+e}{e} - r\cos.\theta);$$

$$\therefore\ r = \frac{m\,(1+e)}{1 + e\cos.\theta}.$$

Or since $m\,(1+e) = \frac{b^2}{a}$ for the ellipse and hyperbola, and $= 2\,m$ for the parabola, we have (putting p for the principal parameter) the general polar equation to the three curves,

$$r = \frac{p}{2}\frac{1}{1 + e\cos.\theta}\ (150).$$

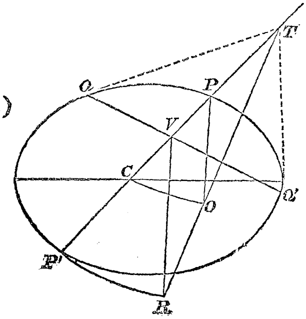
282. To draw a tangent at a given point P on the ellipse.

Draw the ordinate M P, and produce it to meet the circumscribing circle in Q, from Q draw a tangent to the circle meeting the axis major produced in T, join P T; this line is a tangent to the ellipse (114).

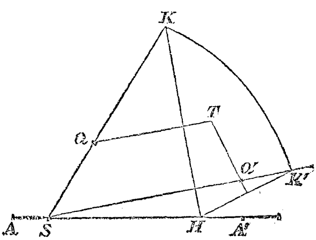
Again, taking the figure in the note appended to Art. 121, join S P, H P, and produce H P to K, so that $P\,K = P\,H$; join S K; the line $P\,y$ bisecting S K is a tangent.

283. To draw a tangent to the ellipse from a point T without the curve.

(1)



(2)



Draw the line T P C P′ through the centre, fig. 1.; draw a conjugate diameter to C P: then the question is reduced to finding a point V in C P, through which a chord Q V Q′ is to be drawn, so that T Q and T Q′ may be tangents.

Take C V a third proportional to C T and C P, then V is the required point (136).

Again, with centre T and radii T P′, T C describe circles C O, P′ R, draw any line T O R, cutting these circles in O and R; join P O, and draw R V parallel to P O: then it may be proved by similar triangles that C V is a third proportional to C T and C P, and therefore V is the required point.

284. If the axes, and not the ellipse, are given, take S and H the foci, fig. 2, with centre S and radius A A′ describe a circle, and with centre T and radius T H describe another circle, cutting the former in the points K and K′; join S K and SK′, H K and H K′; from T draw the lines T Q and TQ perpendicular to H K and H K′, these lines meet S K and SK′ in the required points Q and Q′. The proof will readily appear upon joining H Q and H Q′, and referring to the note, page 77.

285. To draw a tangent to the hyperbola at a given point P on the curve.

Join S P and H P, note, page 97; in H P take PK = S P, and join S K; the line P Y bisecting S K is the required tangent.

286. To draw a tangent from a given point T without the curve.

The two methods given (283) for the ellipse will apply, with the necessary alteration of figure, to the hyperbola.

287. To draw a tangent to the parabola at a given point P on the curve.

Draw an ordinate P M to the axis, fig. 232, and in the axis produced take A T = A M, join P T; this line is a tangent (233); or take S T = S P, and join P T.

288. To draw a tangent to a parabola from a given point T without the curve.

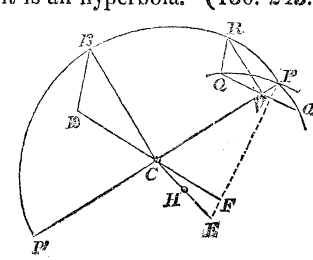
Draw a diameter T P V parallel to the axis, and cutting the curve in P, take P V = P T, and draw an ordinate Q V Q′ to the abscissa P V, then T Q and T Q′ are the required tangents (249).

If the directrix and focus be given, but not the curve; with centre T and radius T S describe a circle, cutting the directrix in the points R and R′, join R S and R′ S; draw R Q and R′ Q′ parallel to the axis, and then TQ and T Q′ perpendicular to R S and R′ S (239).

289. An arc Q P Q′ of a conic section, being traced on a plane to find to which of the curves it belongs; and also the axes and focus of the section.

Draw a line L through the middle of two parallel chords, and another line L′ through the middle of other two parallel chords, if the lines L, L′ are parallel, the curve is a parabola, if they meet on the concave side of the curve it is an ellipse, if on the convex side it is an hyperbola. (130. 243.)

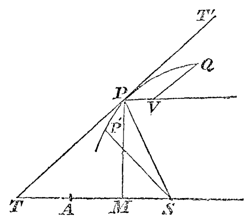
290. Let the curve be an ellipse, the point where the lines L L′ meet is the centre C; let P P′ be a diameter, its conjugate C D is thus found; describe a circle on P P′ as diameter, and draw V R, C B perpendicular to P P′; join R Q, and draw B D parallel to R Q, meeting a line parallel to Q V, passing through C; then C D is the conjugate diameter (136).



To find the length and position of the axes; draw P F perpendicular on C D, and produce it to E, making P E = C D, join C E, and bisect C E in H; join P H; then from the triangle C P E we have the side C E in terms of C P and C D $= \sqrt{\{a_1^2 + b_1^2 - 2\, a_1 b_1 \sin. (\theta' - \theta)\}} = \sqrt{\{a^2 + b^2 - 2\, a\, b\}} = a - b$ ∴; C H $= \frac{a-b}{2}$; also from the same triangle we have P H $= \frac{a+b}{2}$; hence P H + H E is the small-axis major, and P H − H E is the semi-axis minor.

In H P take H K = H E, then C K is the direction of the axis-major.

291. If the arc $Q P Q'$ be an hyperbola, the conjugate diameter may be found by a process somewhat similar to that for the ellipse; the asymptotes may then be drawn by Art. 215. The direction of the axes bisects the angle of the asymptote, and their length is determined by drawing a tangent P T, and perpendicular P M, to the axis, and taking C A a mean proportional between C M and C T (167).



292. If the arc be a portion of a parabola, draw $T P T'$ parallel to Q V, and then draw P S, making the angle S P T = the angle $T' P V$; repeat this construction for another point P', then the junction of PS and $P'S$ determines the focus (240); the axis is parallel to P V, and the vertex is found by drawing a perpendicular on the axis, and then bisecting T M (233).

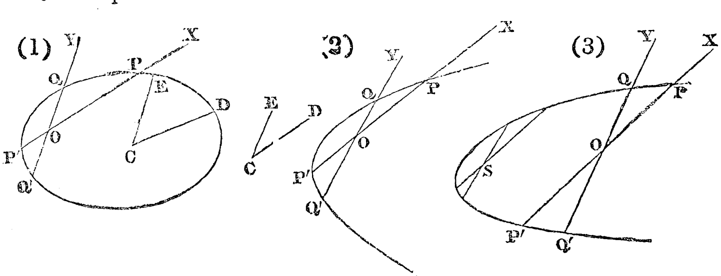
293. We shall conclude the subject of conic sections with the following theorem.

If through any point within or without a conic section two straight lines making a given angle with each other, be drawn to meet the curve, the rectangle contained by the segments of the one will be in a constant ratio to the rectangle contained by the segments of the other.

Case 1. The ellipse and hyperbola.

Let C D, C E be two semi-diameters parallel to the chords $P O P'$, $Q O Q'$; then, wherever chords parallel to these be drawn, we shall always have the following proportion:

The rectangle P O, $O P'$: the rectangle Q O, O Q :: the square on C D : the square on C E.



Let O be the origin of oblique axes O X, O Y: then the equation to the curve will be of the form

$$a\,y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0.$$

Let $x = 0$; $\therefore\ a\,y^2 + d\,y + f = 0$, and the product of the roots being $\frac{f}{a}$, we have

$$\text{The rectangle } Q\,O,\ O\,Q' = \frac{f}{a};$$

$$\text{Similarly the rectangle } P\,O,\ O\,P' = \frac{f}{c};$$

$\therefore$ the rectangle $Q\,O$, $O\,Q'$: the rectangle $P\,O$, $O\,P'$ $:: \frac{f}{a} : \frac{f}{c} :: c : a$

Now, let the origin be transferred to the centre without changing the direction of the axes, then the form of the equation is

$$a\,y^2 + b\,x\,y + c\,x^2 + f' = 0 \quad (81).$$

Let $x = 0$; $\therefore$ the square on $C\,E = \frac{-f'}{a}$; and the square on $C\,D = \frac{-f'}{c}$; $\therefore$ the square on $C\,E$: the square on $C\,D$; $:: c : a$;

$\therefore$ the rectangle $Q\,O$, $O\,Q'$: the rectangle $P\,O$, $O\,P'$ :: the square on $C\,E$: the square on $C\,D$.

In the hyperbola fig. (2), $C\,E$ and $C\,D$ do not meet the curve; but in order to show that these lines are semi-diameters, let the axis of y be carried round till it becomes conjugate to $C\,D$, then the formulas for transformation in (55) become for $\theta = 0$,

$$y = y'\,\frac{\sin.\,\theta'}{\sin.\,\omega},\quad x = x' + y'\,\frac{\sin.\,(\omega - \theta')}{\sin.\,\omega}.$$

If these values of x and y be substituted in the general central equation above, and it be reduced to the conjugate form by putting $b' = 0$, the transformed equation is of the form $a'\,y^2 + c\,x^2 + f' = 0$, where c and f' are not changed, and $-\frac{f'}{c}$ is the square on the semi-diameter along the axis of x (86); hence the theorem is true for the hyperbola.

Case 2. The Parabola fig. (3.)

As before, we have the rectangle $P\,O$, $O\,P$: the rectangle $Q\,O$, $O\,Q'$ $:: c : a$.

Let P and Q be the parameters to the chords $P\,O\,P'$ and $Q\,O\,Q'$; transfer the origin to the focus, the axes remaining parallel to $P\,O$, and $Q\;O$, by which transformation c and a are not altered.

Now in this case, the chords passing through the focus, we have the rectangle $P\,S$, $S\,P'$: the rectangle $Q\,S$, $S\,Q'$; $:: \frac{p}{4}P : \frac{p}{4}Q$ (254) and also as $c : a$; hence the rectangle $P\,O$, $O\,P'$: the rectangle $Q\,O$, $O\,Q'$ $:: c : a :: P : Q$.

294. If the point O be without the curves, and the points $P\,P'$ coincide as well as Q and Q', or the lines become tangents, we have for the ellipse and hyperbola,

The square on O P : the square on O Q :: the square on C D : the square on C E;

or O P : O Q :: C D : CE.

For the parabola;

The square on O P : the square on O Q :: S P : S Q;

hence it may be proved that, if a polygon circumscribe an ellipse, the algebraical product of its alternate segments are equal. And the same theorem will apply to tangents about an hyperbola; the tangents commencing from any point in the asymptotes.

CHAPTER XII.

ON CURVES OF THE HIGHER ORDERS.

295. Having completed the discussion of lines of the second order, we should naturally proceed to the investigation of the higher orders; but the bare mention of the number of those in the next or third order (for they amount to eighty) is quite sufficient to show that their complete investigation would far exceed the limits of an elementary treatise like the present. Nor is it requisite: we have examined the sections of the cone at great length, because, from their connexion with the system of the world, every property of these curves may be useful; but it is not so with the higher orders; generally speaking they possess but few important qualities, and may be considered more as objects of mathematical curiosity than of practical utility.

The third order is chiefly remarkable from its investigation having been first undertaken by Newton. Of the eighty species now known, seventy-two were examined by him; eight others, which escaped his searching eye, have since been discovered.

Those who wish to study these curves, may refer to Newton's "Enumeratio Linearum tertii Ordinis;" or to the work of Stirling upon the same subject.

Of the fourth order there are above five thousand species, and the number in the higher orders is so enormous as to preclude the possibility of their general investigation in the present state of analysis.

A systematic examination of curves being thus impossible, all that we can do is to give a selection, taking care that amongst them shall be found all the algebraical or transcendental curves which are most remarkable either for their utility or history.

We shall generally introduce them as examples of indeterminate problems, that is, of problems leading to final equations, containing two variables. We shall then trace the loci of those equations, and explain, when necessary, anything relating to the construction or properties of the curves.

It would be useless to give any general rules for the working of these questions; those given for determinate problems will here serve equally well; but, in both cases, experience is the only sure guide. In the solution of these problems we shall not always follow the same, nor even the easiest, method; but we shall endeavour to vary the manner, so that an attentive observer may learn how to act in any particular case.

We commence with problems leading to loci of the second order.

296. Given the straight line A B ($= a$) to find the point P without A B, so that A P : P B :: m : 1.

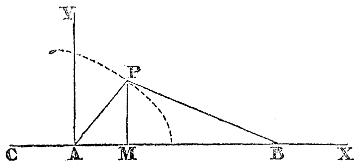
Let A be the origin of rectangular co-ordinates, A X and A Y the axes, A M $= x$, M P $= y$, and $\therefore$ M B $= a - x$,

then A P : P B :: m : 1

$$\text{or } \sqrt{x^2 + y^2} : \sqrt{(a-x)^2 + y^2} :: m : 1;$$

$$\therefore x^2 + y^2 = m^2 (a - x)^2 + m^2 y^2,$$

$$\text{or } (1 - m^2) y^2 + (1 - m^2) x^2 + 2 m^2 a x - m^2 a^2 = 0,$$



$$\text{or } y^2 + \left(x + \frac{m^2 a}{1 - m^2}\right)^2 = \frac{m^2 a^2}{(1 - m^2)^2}.$$

This equation shows that there are an infinite number of points satisfying the conditions of the problem, all situated on the circumference of a circle (66).

To draw this circle; in A x take A C $= \dfrac{m^2 a}{1 - m^2}$, and with centre C and radius $\dfrac{m a}{1 - m^2}$ describe a circle; this is the required locus.

If $m = 1$, reverting to the original equation we have $x = \dfrac{a}{2}$, which is the equation to a straight line drawn through the bisection of A B, and parallel to A Y.

297. If perpendiculars be drawn to two lines given in position from a point P, and the distance between the feet of the perpendiculars be a constant quantity a, required the locus of P.

Let the intersection of the given lines be the origin of rectangular axes, take one of the lines for the axis of x, and let $y = \alpha x$ be the equation to the other; then the equation to the line passing through P $(x' y')$, and perpendicular to the line $y = \alpha x$, is $y - y' = -\dfrac{1}{\alpha}(x - x')$; then from these two equations the co-ordinates of the point where their loci meet, that is, the co-ordinates of the foot of the perpendicular are readily obtained; and then the final equation found, by art. 29, is $y'^2 + x'^2 = a^2 \dfrac{(1 + \alpha^2)}{\alpha^2}$, which belongs to a circle whose centre is at the intersection of the lines.

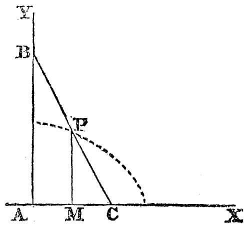
298. A given straight line B C moves between two straight lines, A B, A C, so that its extremities B C are constantly on those lines; to find the curve traced out by any given point P in B C.

Let the lines A B, A C be the axes of y and x,

$AM = x$, $BP = a$,

$MP = y$, $PC = b$,

and let B A C be a right angle;



then $AM : BP :: MC : PC$,

$$x : a :: \sqrt{b^2 - y^2} : b$$

$$\therefore b^2 x^2 = a^2 b^2 - a^2 y^2,$$

$$\text{or } a^2 y^2 + b^2 x^2 = a^2 b^2,$$

which is the equation to an ellipse whose centre is A and axes $2a$, $2b$. If a ladder be placed against a wall, and its foot drawn along the ground at right angles to the wall, any step will trace out a quarter of an ellipse, and the middle step will trace out a quadrant of a circle.

If the co-ordinate axes be inclined at an angle θ, we have

$$AB = \frac{a+b}{b}\,y, \text{ and } AC = \frac{a+b}{a}\,x,$$

$$\text{Whence } a^2 y^2 + b^2 x^2 - 2ab\cos.\theta . xy - a^2 b^2 = 0,$$

which is the equation to an ellipse (76).

It is easy to see that from this problem arises a very simple mechanical method of describing the ellipse.

If a straight line B C of variable length move between two straight lines A B, A C, so that the triangle A B C is constant, the curve traced out by a point P which divides B C in a given ratio is an hyperbola.

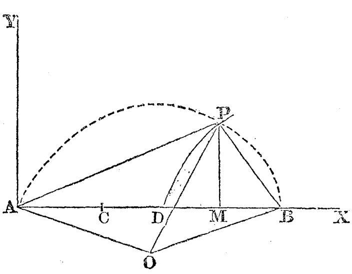
299. Given the line A B ($= c$) to find a point P without A B, such that drawing P A and P B, the angle P B A may be double of P A B.

Let A be the origin; A X, A Y the rectangular axes:

The equation to A P is $y = \alpha x$, (1)

and that to B P is $y = \alpha' (x - c)$;

$$\text{but } \alpha' = \tan. PBX = -\tan. PBA = -\tan. 2PAB = -\frac{2\alpha}{1-\alpha^2};$$



$$\therefore y = \frac{-2\alpha}{1-\alpha^2}(x - c) \quad (2)$$

Eliminating α between the equations (1) and (2), we have $y^2 = 3x^2 - 2cx$, hence the locus of P is an hyperbola; comparing its equation with the equation $y^2 = \frac{b^2}{a^2}(x^2 - 2ax)$, we find the axes to be $\frac{2c}{3}$ and

$\frac{2c}{\sqrt{3}}$, and the centre at C where A C $= \frac{c}{3}$.

By this hyperbola, a circular arc may be trisected; for if A P B be the arc to be trisected, describe the hyperbola D P as above, and let the curves intersect in P; then if O be the centre of the circle, the angle A O P = 2 A B P = 4 P A B = 2 P O B, or the arc P B is one-third of B P A.

This problem may also be thus solved:

$$\text{Let } A M = x,\ M P = y, \text{ and angle } P A B = \theta;$$

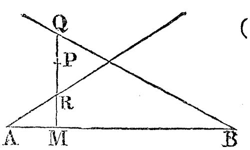
Then $\tan.\theta = \frac{y}{x}$, and $\tan. 2\theta = \frac{y}{c - x}$ but $\tan. 2\theta = \frac{2\tan.\theta}{1 - (\tan.\theta)^2}$,

$$\therefore \frac{y}{c - x} = \frac{\frac{2y}{x}}{1 - \frac{y^2}{x^2}}, \text{ or } y^2 = 3x^2 - 2cx.$$

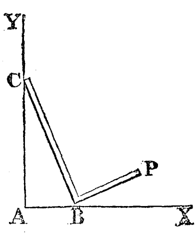
On examination it will be seen that the above two methods of solution are identical.

300. The following problems give loci of the second order.

1. From the given points A and B, (fig. 1,) two straight lines given in position are drawn, M R Q is a common ordinate to these lines, and M P is taken in M R Q a mean proportional to M Q and M R; required the locus of P.



(1)



(2)

2. A common carpenter's square C B P, (fig. 2,) moves in the right angle X A Y, so that the point C is always in A Y, and the right angle B in the line A X; required the locus of P.

3. If the base and difference of the angles at the base of a triangle be given, the locus of the vertex is an equilateral hyperbola.

4. To find a point P, from which, drawing perpendiculars on two given straight lines, the enclosed quadrilateral shall be equal to a given square.

301. Let A Q A′ be an ellipse, A A′ the axis major, Q Q′ any ordinate, join A Q and A′ Q′; required the locus of their intersection P.

Let C be the origin of rectangular co-ordinates.

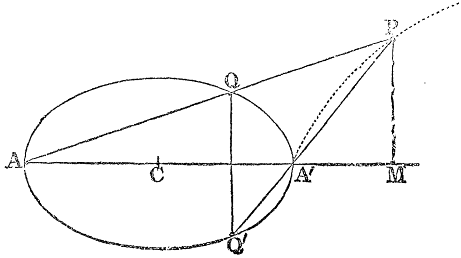
$$C M = x,\ M P = y,\ C N = x',\ \text{and } N Q = y'.$$

Then the equation to A Q is $y = \alpha x + c$

which at A is $0 = -\alpha a + c$;

$$\therefore\ y = \alpha (x + a),$$

At Q it becomes $y' = \alpha\,(x' + a)$ $\therefore\ \alpha = \dfrac{y'}{x' + a}$;



Hence the equation to A Q is $y = \dfrac{y'}{x' + a}\,(x + a)$, (1)

And similarly that to A′Q′ is $y = \dfrac{-\,y'}{x' - a}\,(x - a)$, (2)

Also $a^2\,y'^2 + b^2\,x'^2 = a^2\,b^2$; (3)

Eliminating x' and y' between (1) and (2), we have

$$x' = \frac{a^2}{x} \text{ and } y' = \frac{a\,y}{x}.$$

Substituting in (3) we obtain the final equation

$$a^2\,\frac{a^2\,y^2}{x^2} + \frac{b^2\,a^4}{x^2} = a^2\,b^2,$$

$$\text{or } a^2\,y^2 - b^2\,c^2 = -\,a^2\,b^2,$$

which is the equation to an hyperbola, whose centre is C, and transverse axis 2 a.

The method of elimination used in this problem is of great use; the principle admits of a clear explanation. We have the equations to A Q and A′ Q′; putting x and y the same for both equations intimates that x and y are the co-ordinates C M and M P in one particular case of intersection; but the elimination of x' and y' intimates that x and y are also always the co-ordinates of intersection, and therefore that the resulting equation belongs to the locus of their intersection.

302. To find the locus of the centres of all the circles drawn tangential to a given line A X, and whose circumferences pass through a given point Q ($a\ b$).

Let S Q M be one of these circles, referred to rectangular axes A x, A y.

x, y the co-ordinates of its centre P,

x', y' . . . . any point on its circumference.

Then the equation to S Q M is

$$(y' - y)^2 + (x' - x)^2 = r^2; \quad (65)$$

but passing through Q, it becomes

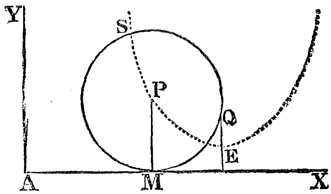
$$(b - y)^2 + (a - x)^2 = r^2,$$

and, being tangential to A X, we have $r = y$,

$$\therefore\ (b - y)^2 + (a - x)^2 = y^2,$$

$$\text{or } x^2 - 2\,a\,x - 2\,b\,y + a^2 + b^2 = 0.$$

This is the equation to a parabola (78).



It may be put in the form $(x - a)^2 = 2\,b\left(y - \frac{b}{2}\right)$. Hence if we transfer the origin to the point E $\left(a, \frac{b}{2}\right)$, we have the equation $x^2 = 2\,b\,y$, and the curve is referred to its vertex E, which is the centre of the least circle.

If, instead of the circle passing through a given point, it touch a given circle, a parabola is again the locus of P.

303. Let A B, B C, C D, and D A (fig. 1, p. 148) be four straight lines given in position, to find the locus of a point P, such, that drawing the lines P E, P F, P G, and P H making given angles with A B, B C, C D, and D A, we may have the rectangle P E, P F = the rectangle P G, P H.

Let O be the origin of rectangular axes O X, O Y; x and y the coordinates of P; β, β', β'' and β''' the cosecants of the angles which the lines P E, P F, &c., make with A B, B C, &c. Then the equation

to A B being $y' = \alpha\,x' + b$ we have $\text{P E} = \dfrac{y - \alpha\,x - b}{\sqrt{1 + \alpha^2}}\,\beta$ (49)

to B C . . $y' = \alpha'\,x' + b'$. . . $\text{P F} = \dfrac{y - \alpha'\,x - b'}{\sqrt{1 + \alpha'^2}}\,\beta'$

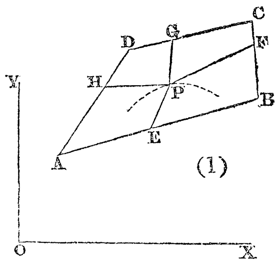
to D C . . $y' = \alpha''\,x' + b''$ . . $\text{P G} = \dfrac{y - \alpha''\,x - b''}{\sqrt{1 + \alpha''^2}}\,\beta''$

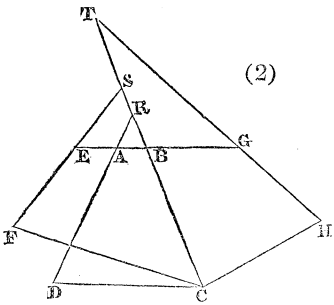
to A D . . $y = \alpha'''\,x' + b'''$ . . $\text{P H} = \dfrac{y - \alpha'''\,x - b'''}{\sqrt{1 + \alpha'''^2}}\,\beta'''$;

$$\therefore \text{ by the question } \frac{y - \alpha\,x - b}{\sqrt{1 + \alpha^2}}\;\frac{y - \alpha'\,x - b'}{\sqrt{1 + \alpha'^2}}\;\beta\,\beta'$$

$$= \frac{y - \alpha''\,x - b''}{\sqrt{1 + \alpha''^2}}\;\frac{y - \alpha'''x - b'''}{\sqrt{1 + \alpha'''^2}}\;\beta''\,\beta''.$$

This equation being evidently of two dimensions, the locus of P is a conic section, the particular species of which depends on the situation of the given lines.





This problem may be expressed much more generally. Suppose 3, 4, 5 or a greater number of lines to be given in position, required a point from which, drawing lines to the given lines, each making a given angle with them, the rectangle of two lines thus drawn from the given point may have a given ratio to the square on the third, if there are three; or to the rectangle of the two others, if there are four: or again, if there are five lines, that the parallelopiped composed of three lines may have a given ratio to the parallelopiped of the two remaining lines, together with a third given line, or to the parallelopiped composed of the three others, if there are six: or again, if there are seven, that the algebraical product of four may have a given ratio to the algebraical product of the three others and a given line, or to the four others, if there are eight, and so on.

This was a problem which very much perplexed the ancient geometricians. Pappus says, that neither Euclid nor Apollonius could give a solution. He himself knew that when there are only three or four lines the locus was a conic section, but he could not describe it, much less could he tell what the curve would be when the number of lines were more than four. When the number of lines were seven or eight, the ancients could scarcely enunciate the problem, for there are no figures beyond solids, and without the aid of algebra, it is impossible to conceive what the product of four lines can mean.

It was this problem which Descartes successfully attacked, and which, most probably, led him to apply algebra generally to geometry. The following solution is that given by Descartes, with a few abbreviations:

A B, A D, E F and G H (fig. 2) are the given lines, C the required point from which are drawn the lines C B, C D, C F and C H making given angles C B A, C D A, C F E, and C H G. A B ($= x$) and B C ($= y$) are the principal lines to which all the others will be referred. Suppose the given lines to meet C B in the points R, S, T, and A B in the points A, E and G. Let A E $= c$ and A G $= d$.

Then since all the angles of the triangle A B R are known, we have B R $= \alpha$.A B $= \alpha x$; $\therefore$ C R $= \alpha x + y$ and C D $= \beta (\alpha x + y)$; also B S $= \alpha'$.B E $= \alpha' (c + x)$; $\therefore$ C S $= y + \alpha' (c + x)$ and C F $= \beta' \{y + \alpha' (c + x)\}$; also B T $= \alpha''$.B G $= \alpha'' (d - x)$; $\therefore$ C T $= y + \alpha'' (d - x)$ and C H $= \beta''.\{y + \alpha'' (d - x)\}$; then since the rectangle C B, C F = the rectangle C D, C H, we have the equation

$$y\beta' \{y + \alpha'(c + x)\} = \beta(\alpha x + y)\beta'' \{y + \alpha''(d - x)\}.$$

This equation Descartes showed to belong to a conic section which he described. He also gave the following numerical example:

Let E A = 3, A G = 5, A B = B R, B S = $\frac{1}{2}$ B E, G B = B T,

C D = $\frac{3}{2}$ C R, C F = 2 C S, C H = $\frac{2}{3}$ C T, the angle A B R = 60°,

and the rectangle C B, C F = the rectangle C D, C H. By the above method he found the equation to be

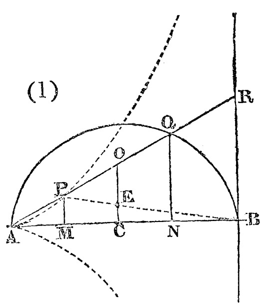
$$y^2 + xy + x^2 - 2y - 5x = 0;$$

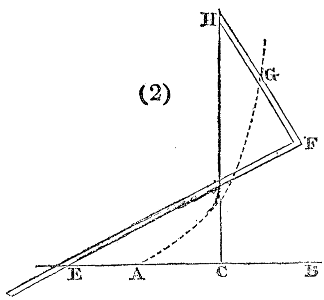
which he showed belonged to a circle. Taking the expressions in art. (72) we have the co-ordinates of the centre $\frac{8}{3}$ and $-\frac{1}{3}$, and the radius $= \frac{\sqrt{19}}{3}$.

304. Let A Q B be a semi-circle of which A B is the diameter, B R an indefinite straight line perpendicular to A B, A Q R a straight line meeting the circle in Q and B R in R; take A P = Q R; required the locus of P.

Let A be the origin of rectangular axes, and A B the axis of x.

A B = $2a$, A M = x, M P = y, and draw Q N parallel to M P;





then since A P = Q R, we have A M = B N,

and A M : M P :: A N : N Q;

that is, $x : y :: (2a - x) : \sqrt{(2a - x)x}$; (65)

$$\therefore y^2 = \frac{x^3}{2a - x} \text{ and } y = \pm \sqrt{\frac{x^3}{2a - x}}.$$

The following table gives the corresponding values of x and y:

	1	2	3	4	5	6
Values of x	0	a	$< 2a$	$2a$	$> 2a$	–
Values of y	0	a	possible	∞	impos.	impos.

From (1) the curve passes through the origin, from (2) it bisects the semi-circular arc A Q B, from (3) there are possible values of y for all values of x less than $2a$, from (4) there is an infinite ordinate at B, or B R is an asymptote to the curve: from these values we thus obtain an infinite arc proceeding from A to meet the asymptote B R. Again, from (5) for any value of x greater than $2a$, y is impossible, or no part of the curve is found to the right of the asymptote; and from (6) no part of the curve is on the left of A. Also, for every value of x there are two of y equal and opposite; hence there is a branch below A B similar to the one above it.

Diocles, a mathematician of the sixth century, invented this curve, which he called the Cissoid, from a Greek word signifying "ivy," because this curve climbs up its asymptote like ivy up a tree. He employed it in solving the celebrated problem of the insertion of two mean proportionals between given extremes.

Before his time, Pappus had reduced the problem to this case:

Let B C, C E be the two extremes, and A Q B a circle whose centre is C and radius C B; draw an indefinite straight line B E P through E and then draw the straight line A P O Q meeting B E and C E produced, and also meeting the circle at Q in such a manner that O Q = O P, then C O will be the first of the two mean proportionals. But the point P could not be directly found: hence, Diocles invented this curve to determine a series of points which will solve the problem for any length of C E: for example, suppose that B C, C E and the cissoid be drawn, join B E meeting the curve in P, then since O R = O A and Q R = A P we have O Q = O P.

From the definition of the curve it can be readily described by points; but as this is only a tentative process at best, and therefore not geometrically correct, Newton invented a very simple instrument for describing the curve by continued motion:

Let C H (fig. 2, p. 149) be a straight line parallel to B R; take A E = A C and let E F H be a common carpenter's square, the side F E being of indefinite length, and F H = A B; move this square so that the longer leg F E always passing through E, the extremity H of the other slides along C H, the middle point G of F H traces out the cissoid.

To obtain the polar equation to this curve:

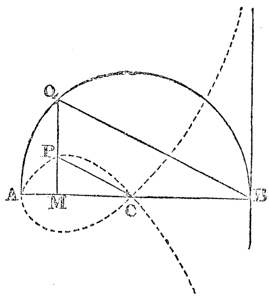
$$\text{Let } y = r \sin. \theta \text{ and } x = r \cos. \theta;$$

Substitute these values in the equation $y^2 = \frac{x^3}{2a-x}$

$$\therefore r^2 (\sin.\theta)^2 = \frac{r^3 (\cos.\theta)^3}{2a - r\cos.\theta}; \text{ whence } r = 2a \sin.\theta . \tan\theta.$$

Ex. If a perpendicular be drawn from the vertex of a parabola to a tangent, the locus of their intersection is the cissoid.

305. If C be a point in the diameter A B of the circle A Q B, and M Q any ordinate, join B Q, and draw C P parallel to B Q, meeting M Q in P required the locus of P.



Let $AM = x$,

$MP = y$,

$AB = a$,

$AC = b$;

then $BM : MQ :: CM : MP$,

$$\text{or } (a-x) : \sqrt{ax - x^2} :: (b-x) : y,$$

$$\therefore y = \pm (b-x)\sqrt{\frac{x}{a-x}}.$$

Hence the following table of values :

	1	2	3	4	5	6
Values of x	0	b	a	$< a$	$> a$	$-$
Values of y	0	0	$\pm\infty$	pos.	imp.	imp

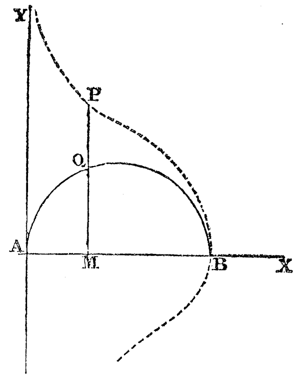
From (1) and (2) the curve passes through A and C ; from (3) the ordinate at B is an asymptote to the curve; from (4) there are two arcs between A and C, also two between C and B ; from (5) and (6) no part of the curve extends to the right of B or the left of A.

If $b = 0$, the oval between A and C disappears, and the curve is the cissoid of Diocles.

If b is negative, or the point C on the left of A, the curve consists of two branches proceeding from A to the asymptote through B, and the point C, though not on the curve, yet essentially belongs to it. This insulated point is called a conjugate point. The theory of such points will be fully explained in the treatise on the Differential Calculus.

Ex. A point Q is taken in the ordinate M P of the parabola, always equidistant from P, and from the vertex of the parabola; required the locus of Q.

306. M Q is an ordinate to the semicircle A Q B, and M Q is produced to P, so that M P : M Q :: A B : A M to find the locus of P.



Let A B X and A Y be the rectangular axes.

$A M = x$,

$M P = y$,

$A B = 2a$;

Then M P : M Q :: A B : A M,

$$\text{or } y : \sqrt{2ax - x^2} :: 2a : x,$$

$$yx = \pm 2a\sqrt{2ax - x^2},$$

$$\therefore y = \pm 2a\sqrt{\frac{2a - x}{x}};$$

	1	2	3	4	5
Values of x	0	$2a$	$< 2a$	$> 2a$	neg.
Values of y	$\pm \infty$	0	pos.	imp.	imp.

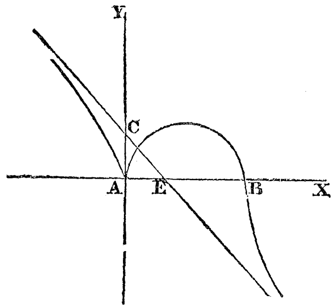
From (1) we have the ordinate at the origin infinite, and therefore an asymptote to the curve; from (2) the curve cuts the axis at B; from (3) the curve extends between A and B; from (4) no part of the curve is beyond B; from (5) no part is to the left of A.

This curve is called the Witch, and is the invention of an Italian lady, Maria Gaetana Agnesi, Professor of Mathematics in the University of Bologna, A.D. 1748.

307. In the circle the square on the ordinate is equal to the rectangle under the segments of the diameter; required the form of the curve on which the curve upon the ordinate is equal to the parallelopiped, of which the base is the square on one segment, and the altitude is the other segment, or $y^3 = x^2(2a - x)$.

Let A be the origin Y, A X, A the rectangular axes, and $A B = 2a$.

Let $x = 0$ or $= 2a$, $\therefore\ y = 0$; hence the curve passes through A and B; for $x < 2a$, y is positive; but when x is $> 2a$, y increases negatively to infinity, since the third root of a negative quantity is negative and possible. Again, y is positive for all negative values of x, and increases to ∞; also for each value of x, there is only one real value of y, the other two roots of an equation $y^3 \pm 1 = 0$, being always impossible.



Expanding the equation we have

$$y = -x\sqrt[3]{1 - \frac{2a}{x}} = -x\left\{1 - \frac{1}{3}\frac{2a}{x} - \frac{4a^2}{9x^2} + \text{\&c.}\right\};$$

$\therefore$ the equation to the asymptote is $y = -x + \frac{2a}{3}$ (195).

In A Y take A C $= \frac{2a}{3}$, and in A X take A E $= \frac{2a}{3}$, join C E, this line produced is an asymptote to the curve.

Ex. Find the locus of the equation, $y^3 + x^3 = a^3$; and of the equation $y^3 = a^2 x - x^3$.

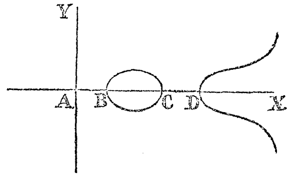
308. To trace the curve whose equation is $a y^2 = x^3 + m x^2 + n x + p$.

Case (1). Suppose the roots of this equation to be real and unequal, and to be represented by the letters a, b, and c, of which a is less than b and b less than c, then the equation is of the form

$$y = \pm\sqrt{\left\{\left(\frac{x-a}{a}\right)(x-b)\ (x-c)\right\}}.$$

	1	2	3	4	5	6	7	8	9	10
Values of x	0	a	$< a$	$> a < b$	b	$> b < c$		$> c$	∞	$-$
Values of y	imp.	0	imp.	pos.	0	imp.	0	pos.	$\pm\infty$	imp.

Let A be the origin, A X, A Y, the axes; A B $= a$, A C $= b$, and A D $= c$;



From (2) (5) and (7) the curve passes through B, C, and D; from (3) and (6) no part of the curve is found between A and B, or C and D; from (4) there are two branches between B and C; from (8) and (9) the curve proceeds from D to ∞, and from (10) no part of the curve is on the left of A.

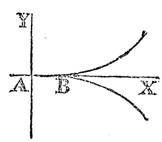
If the roots had been negative, the curve would have the same form, but would be rather differently situated with regard to the origin.

Case (2). If two roots be equal, the equation is $y = (x-c)\sqrt{\dfrac{x-a}{a}}$,

or $y = \pm (x - a)\sqrt{\dfrac{x - c}{a}}$; in the former case the figure is nearly the same as above, when the points C and D coincide; in the latter, supposing the points B and C to coincide, or the oval to become a conjugate point.

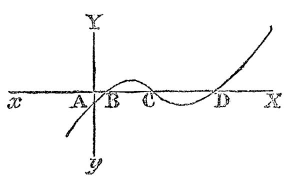
Case (3). If two of the roots be impossible, we have only the bell-shaped part of the curve from D.

Case (4). If the three roots be equal, the equation is $a y^2 = (x - a)^3$. The figure now consists of two branches proceeding from B with their convexity towards the axis. This curve is called the semi-cubical parabola; its equation is the most simple when the origin is at the vertex B; that is, putting x instead of $x - a$, when $a y^2 = x^3$.



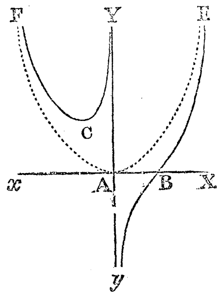
This curve is remarkable as being the first curve which was rectified, that is, the length of any portion of it was shown to be equal to a number of the common rectilinear unit.

309. The equation $a^2 y = x^3 + m x^2 + n x + p$, can be traced exactly as in the last article: the accompanying figure applies to the case when the three roots are positive, real, and unequal. If two of them be equal, one of the semi-ovals disappears; if three are equal, both disappear: in this case the equation is of the form $a^2 y = (x - a)^3$, or $a^2 y = x^3$, if the origin be transferred to B; the curve is then called the cubical parabola.



310. If the equation be $a x y = x^3 + m x^2 + n x + p$, the axis of y is an asymptote, and there is a branch in the angle Y A x; the rest

of the curve is like that in the last figure, supposing the lower branch from B to come to A y as the asymptote, the form will vary as the roots vary. We shall take the case where $y = \frac{x^3 - a^3}{a\,x}$.



	1	2	3	4	5	6	7
Values of x	0	a	$< a$	$> a$	∞	$-$	$-\infty$
Values of y	∞	0	$-$	$+$	∞	$+$	$+\infty$

From (1) Ay is an asymptote; from (2) the curve cuts the axis at B (A B $= a$); from (3, 4, 5) it is below the axis of x from A to B, and above from B to ∞; from (6) and (7) we have the branch F C Y.

This curve is called the trident, from its form. This curve enables us to point out the difference between what are called parabolic and hyperbolic branches of a curve: B y and C Y are hyperbolic, because they admit of a straight line Y y for the asymptote; but B E and C F are parabolic, because they admit of a parabolic asymptote, represented by the dotted curve F A E,

whose equation is $y = \frac{x^2}{a}$ (196).

Ex. Find the locus of the equation $x^2 y - y - x^3 - x = 0$.

If the equation be $x\,y^2 + a^2\,y = x^3 + m\,x^2 + n\,x + p$, the form of the curve will depend on the nature of the roots of the equation $x^4 + m\,x^3 + n\,x^2 + p\,x + \frac{a^4}{4} = 0$; there will be no difficulty in any particular case. Generally the equation to the asymptotes is $y = \pm\,(x + \frac{1}{2}\,m)$; and the axis of y is an asymptote.

311. If the terms x^3 and $m\,x^2$ are wanting, the equation is

$$x y^2 + a^2 y = n x + p,$$

$$\therefore\ y = \frac{-a^2 \pm \sqrt{\{a^4 + 4 p x + 4 n x^2\}}}{2 x}$$

If the denominator of this expression had been constant, the equation would have belonged to an ellipse, hyperbola, or parabola, according as n was negative, positive or nothing; hence if such constant quantity be replaced by the variable quantity $2x$, the conic section becomes "hyperbolized" by having an infinite branch proceeding to the axis of y as an asymptote.

For the nine figures corresponding to the values of p, see Newton, Enum. Lin. Tert. Ord.

From the last article it appears that all curves of the third order have infinite branches; and this must necessarily be the case, for every equation of an odd degree has at least one real root, so that there is always one real value of y corresponding to any real value of x.

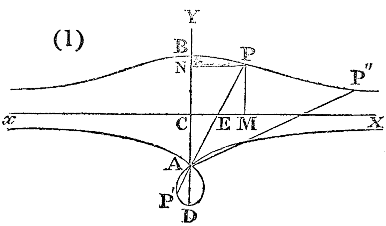
312. The conchoid of Nicomedes.

Let X x (fig. 1) be an indefinite straight line, A a given point, from which draw the straight line A C B perpendicular to X x, and also any number of straight lines A E P, A E P″, &c.; take E P always equal to C B, then the locus of P is the conchoid.

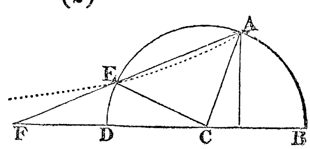
If in E A we take E P′ = E P the locus of P′ is called the inferior conchoid; both conchoids form but one curve, that is, both are expressed by the same equation.

C B is called the modulus, and X x the base or rule;

(1)



(2)



Let $AC = a$, $CM = x$,

$CB = b$, $MP = y$,

then $EP : PM :: AP : AN$,

or $b : y :: \sqrt{x^2 + (a+y)^2} : a + y$,

$$\therefore\ y^2 x^2 + y^2 (a+y)^2 = b^2 (a+y)^2,$$

$$\therefore\ x^2 = (b^2 - y^2)\left(\frac{a+y}{y}\right)^2$$

$$\therefore\ x = \pm \frac{a+y}{y}\sqrt{b^2 - y^2}.$$

We have three cases according as b is $> a$, $= a$, or $< a$.

Case 1. $b > a$.

		2	3	4	5	6	7	8
Values of y	0	b	$< b$	$> b$	$-a$	$-b$	$< -a$	$> -a, < -b$
Values of x	∞	0	pos.	imp.	0	0	pos.	pos.

From (1) X x is an asymptote; from (2) the curve passes through B; from (3) and (4) the curve extends from the asymptote upwards to B and no higher; hence the branch B P P$''$. Again from (5) and (6) the curve passes through A and D if C D $= b$; from (7) there is a branch A $\times$ extending from A to the asymptote; and from (8) the curve exists between A and D; the double value of x gives the same results along C x.

Case 2. $b = a$; in the table of values put $b = a$, and omit (8); thus the figure will be the same as the preceding, with the exception of the oval A P$'$ D, which vanishes by the coincidence of A and D.

Case 3. $b < a$; in the table of values put b for a in (7), and for (8) write "if y is $> - b$, x is impossible;" the upper part of the curve is not altered, but the point D falls between A and C; from (8) no part of the curve is between D and A; but from (5) A is a point not on the curve, but belonging to it, and called a conjugate point. In this case the lower curve is similar to the upper one.

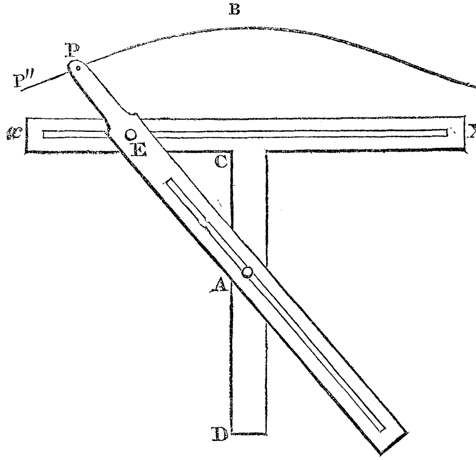
The generation of the conchoid gives a good idea of the nature of an asymptote, for the line E P must always be equal to C B, and this condition manifestly brings the curve continually nearer to C X, as at P$''$, so that the curve, though never actually coinciding with C X, approaches nearer to it than by any finite distance.

This curve was invented by Nicomedes, a Greek geometrician, who flourished about 200 years B.C. He called it the Conchoid, from a Greek word signifying "a shell;" it was employed by him in solving the problems of the duplication of the cube, and the trisection of an angle.

To show how the curve may be applied to the latter problem, let B C A (fig. 2) be the angle to be trisected; draw A E F meeting the circle in E, and the diameter produced in F, and so that the part E F equal the radius C A, then it is directly seen that the arc D E is one-third of B A.

Now it is not possible by the common geometry, that is, with the straight line and circle alone, to draw the line A E F, so that E F shall be equal to C A (the tentative process, though easy, being never considered geometrically correct), and for a long time the ancient geometricians would not hear of any other mathematical instruments than the ruler and compasses; hence the problem was quite insuperable: finding at last that this was the case, they began to invent some curves to assist in the solution of this and other problems: of these curves, the most celebrated is the conchoid of Nicomedes. It may be thus applied to the present problem. Let A be the pole of the inferior conchoid, B F the asymptote or base, and A C the modulus, the intersection of the curve with the circle evidently gives the required point E. The superior conchoid may also be used for the same purpose.

Unless the curve could be described by continued motion, the solution would be incomplete. Nicomedes therefore invented the following simple machine for describing it. Let x X be a straight ruler with a groove cut in it; C D is another ruler fixed at right angles to x X; at A there is a fixed pin, which is inserted in the groove of a third ruler A E P; in A P is a fixed pin at E, which is inserted into the groove of x X; P E is any given length; then, by the constrained motion of the ruler P E A, a pencil at P will trace out a conchoid, and another pencil fixed in E A would trace out the inferior curve.



This curve was formerly used by architects; the contour of the shaft of a column being the portion B P P'' of a conchoid.

The polar equation to the conchoid is thus found:

Let A (fig. 1, page 156) be the pole, A P $= r$, P A B $= \theta$;

$$\therefore\ y + a = r \cos.\ \theta, \text{ and } x = r \sin.\ \theta.$$

Substituting these values in the equation, and reducing, we arrive at the polar equation $r = a \sec.\ \theta + b$.

The polar equation may, however, be much more easily obtained from the definition of the curve. We have

$$r = \mathrm{AP} = \mathrm{AE} + \mathrm{EP} = \mathrm{AC} \sec.\ \mathrm{CAE} + \mathrm{CB} = a \sec.\ \theta + b.$$

313. The following method of obtaining the equation to the conchoid will be found applicable to many similar problems.

Let any number of lines, A E P, fig. 1, be drawn cutting C X in different points E, &c.; from each of these points E as centre, and with radius b describe a circle cutting the line A E P in P and P'; the locus of the point P is the conchoid.

Let A be the origin of the rectangular co-ordinates.

A B the axis of y, and A X parallel to C X in the figure.

Let the general equation to the line A E P be $y = \alpha x$, where α is indeterminate;

Then $y' = a$, and $x' = \frac{a}{\alpha}$ are the equations to the point E;

The equation to the circle which has the point E for its centre and radius b, is

$$(y - y')^2 + (x - x')^2 = b^2,$$

$$\text{or} \quad (y - a)^2 + \left(x - \frac{a}{\alpha}\right)^2 = b^2.$$

And eliminating α between this equation, and that to the line A E P, we have the final equation to the curve,

$$(y - a)^2 + \left(x - \frac{a\,x}{y}\right)^2 = b^2,$$

$$\text{or } (y - a)^2 \frac{x^2 + y^2}{y^2} = b^2.$$

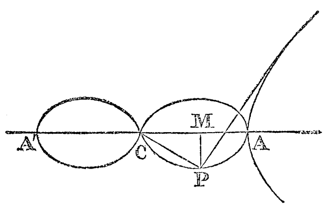
In general if the line C X be a curve whose equation is $y = f(x)$, the co-ordinates of the point E are found by eliminating x and y from the equations $y = \alpha\, x$, and $y = f(x)$; hence we find $x = f'(\alpha)$, and $y = \alpha f'(\alpha)$, and the equation to the circle is

$$\{y - \alpha f'(\alpha)\}^2 + \{x - f'(\alpha)\}^2 = b^2,$$

And the general equation to the curve is

$$\left\{y - \frac{y}{x} f'\left(\frac{y}{x}\right)\right\}^2 + \left\{x - f'\left(\frac{y}{x}\right)\right\}^2 = b^2.$$

314. A perpendicular is drawn from the centre of an hyperbola upon a tangent, find the locus of their intersection.



The equation to the tangent is

$$a^2\, y\, y' - b^2\, x\, x' = -a^2\, b^2. \qquad (1)$$

The equation to the perpendicular on it from the centre is

$$y = -\frac{a^2}{b^2}\frac{y'}{x'}\, x. \qquad (2)$$

In order to get the equation to their intersection, we must eliminate x' and y' from these two equations and that to the hyperbola; from (1) and (2) we find

$$x' = \frac{a^2\, x}{x^2 + y^2},\; y' = \frac{-\,b^2\, y}{x^2 + y^2}.$$

Substituting in the equation $a^2\, y'^2 - b^2\, x'^2 = -\,a^2\, b^2$, we have

$$(x^2 + y^2)^2 + b^2\, y^2 - a^2\, x^2 = 0,$$

which is an equation of the fourth degree.

We shall only investigate the figure in the case when $b = a$, that is when the hyperbola is equilateral, in which case the equation is $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$,

$$\therefore y^4 + (2x^2 + a^2) y^2 + x^4 - a^2 x^2 = 0,$$

$$\text{and } y = \pm \sqrt{\left\{ -\left(x^2 + \frac{a^2}{2}\right) \pm a \sqrt{2x^2 + \frac{a^2}{4}} \right\}}.$$

If the sign of the interior root be negative, y is impossible; hence we shall only examine the equation

$$y = \pm \sqrt{\left\{ -\left(x^2 + \frac{a^2}{2}\right) + a \sqrt{2x^2 + \frac{a^2}{4}} \right\}};$$

here y is impossible, if $x^2 + \frac{a^2}{2}$ is $> a \sqrt{2x^2 + \frac{a^2}{4}}$,

$$\text{if } x^4 + a^2 x^2 + \frac{a^4}{4} \text{ is } > 2a^2 x^2 + \frac{a^4}{4}$$

$$\text{if } x^4 \text{ is } > a^2 x^2,$$

$$\text{if } x \text{ is } > \pm a;$$

hence we have the following table :

	1	2	3	4
Values of x	0	$\pm a$	$< \pm a$	$> \pm a$
Values of y	0	0	pos.	imp.

From (1) the curve passes through C; from (2) it passes through A and A′; from (3) it has two branches from C to A and from C to A′; from (4) it does not extend beyond A and A′.

We may judge yet more nearly of the form of these ovals, for the tangent at the vertex of the hyperbola being perpendicular to the axis, the oval will cut the axis at A at a right angle; and again at C in an angle of 45°, because the tangent nearly coinciding with the asymptote, the perpendicular on it makes an angle of 45° with the axis ultimately.

This curve was invented by James Bernouilli; it is called the Lemniscata, and forms one of a series of curves corresponding to different values of b.

To find the polar equation to the lemniscata,

$$\text{Let } y = r \sin. \theta, \text{ and } x = r \cos. \theta;$$

hence the equation $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$ becomes $r^2 = a^2 \cos. 2\theta$.

Any curve that is of the form of this figure is called a lemniscata.

315. In the following example the curve may be easily traced by points.

Let a circle be described with centre C and any radius C Q; draw the ordinate Q M, and in Q C take Q P = Q M; the locus of P is a lemniscata.

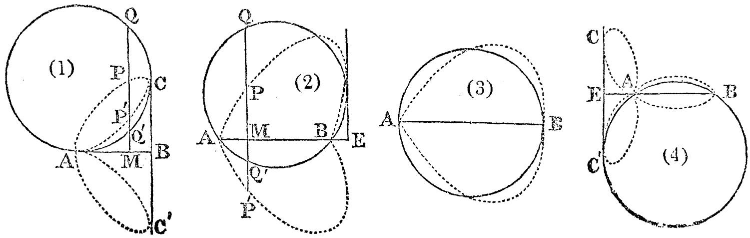
Again, if in M Q we take M R = a third proportional to M Q and C M, the locus of R is another lemniscata whose equation is

$$x^4 - a^2 x^2 + a^2 y^2 = 0.$$

The equation $a^2 (y - a)^2 = (x - a)^2 (2\,a\,x - x^2)$ belongs to the same curve referred to a different origin.

Ex. Trace the locus of the equation $y^2 = x^2 \dfrac{a^2 + x^2}{a^2 - x^2}$.

316. A M, fig. 1, is a tangent to a circle A C Q, M Q an ordinate to the abscissa A M; M P is taken a mean proportional between A M and M Q; required the locus of **P**.



Let A M $= x$, and M P $= y$, be the rectangular co-ordinates of P, and let the radius of the circle $= b$,

then the square on M P = the rectangle A M, M Q.

To find M Q, we have the equation to the circle

$$(y - y')^2 + (x - x')^2 = r^2,$$

$$\text{or } y^2 - 2\,b\,y + x^2 = 0, \text{ since } x' = 0, \text{ and } y' = r = b,$$

$$\therefore \text{M Q} = b \pm \sqrt{b^2 - x^2},$$

$$\therefore \text{M P or } y = \pm \sqrt{\{b\,x \pm x \sqrt{b^2 - x^2}\}}.$$

Since $b^2 - x^2$ is $< b^2$, there are four values of y to each positive value of $x < b$, and no value of y to x negative; hence if A B $= b$, fig. 1, the straight line C B C′ perpendicular to A B is a limit to the curve, and when $x = b$, the ordinate to the curve is equal to the extreme ordinate of the circle, that is, to the tangent B C.

Between $x = 0$, and $x = b$, we have four values of y, which give the two dotted ovals of fig. (1).

To make the question more general we shall suppose the line A B to be a chord of the circle, figs. (2) (3) (4).

Then if b and a are the co-ordinates to the centre of the circle, and A the origin, the equation to the curve will be

$$y = \pm \sqrt{\{b\,x \pm x \sqrt{b^2 + 2\,a\,x - x^2}\}},$$

and we have four cases depending on the values of b and a; hence we have four curves of different forms, yet partaking of the same character and generation.

Case (1). $a = 0$, fig. (1) already discussed.

Case (2). a and b positive, fig. (2). A E $= a + \sqrt{a^2 + b^2}$.

Case (3). $b = 0$, fig. (3).

Case (4). b negative, fig. (4), the equation is

$$y = \pm \sqrt{\{- b\,x \pm x \sqrt{b^2 + 2\,a\,x - x^2}\}}.$$

There are two values of y for x positive, and $< 2a$; but four values for x negative, and $< \sqrt{a^2 + b^2} - a$, that is, $<$ A E.

The gradual transition of one curve to another is apparent, but that the same problem should produce such very different curves as (2) and (4) requires some explanation.

In fig. (1) P and P′ are determined by mean proportionals between A M and M Q, and also between A M and M Q′. Moreover P may be in Q M produced as well as in M Q, thus we have the double oval, fig. (1.) On the left of A the abscissas A M will be negative, and the ordinates M Q positive; hence no possible mean proportional can exist, or no part of the curve can be on the left of A.

In fig. (2) A M and M Q determine the points P and P′; but A M and M Q′ give only an imaginary locus.

Fig. (3) requires no comment.

In fig. (4) the reasoning on fig. (2) will explain the positive side of A; on the left of A the abscissa and both ordinates are negative; therefore two mean proportionals can be found, or four points in the curve for each abscissa.

Such curves may be invented at pleasure, by taking the parabola or other curves for the base instead of the circle.

Ex. To find the locus of the equation $y^4 + 2axy^2 - ax^3 = 0$.

317. To find a point P′, such that drawing straight lines to two given points S and H, we may have the rectangle S P, H P constant.

Join the points S and H, and bisect S H in C; let C be the origin of rectangular axes, S H $= 2a$, C M $= x$, M P $= y$ and let the rectangle S P, H P, $= ab$.

Then since S M $= a + x$, and H M $= a - x$, we have

$$\{y^2 + (a + x)^2\}\{y^2 + (a - x)^2\} = a^2 b^2,$$

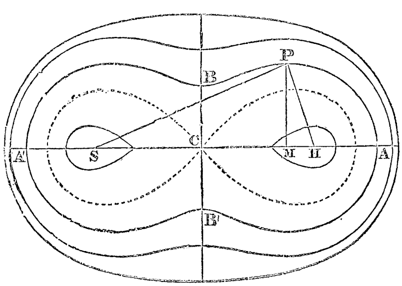
$$\text{or } (y^2 + x^2 + a^2 + 2ax)(y^2 + x^2 + a^2 - 2ax) = a^2 b^2,$$

$$\text{or, } \{y^2 + x^2 + a^2\}^2 - 4a^2x^2 = a^2b^2;$$

$$\text{hence } y = \pm\sqrt{\{-(a^2 + x^2) + a\sqrt{b^2 + 4x^2}\}}.$$

$$\text{Let } y = 0, \therefore x = \pm\sqrt{a(a \pm b)} \qquad (1).$$

$$\text{Let } x = 0, \therefore y = \pm\sqrt{a(b - a)} \qquad (2).$$



1. Let a be less than b.

Then from (1) we have the points A and A′, and from (2) we have the points B and B′.

Also by comparing the values of y in the original equation and in equation (2) we shall find that M P is greater than C B as long as x is greater than $\sqrt{2\,a\,(2\,a - b)}$; thus the form of the curve must be like that of the figure A P B A′ B′ A.

As b increases, the oval becomes flatter at the top, and takes the form of the outer curves.

2. Let $a = b$, then we have the dotted curve passing through C; also since the equation becomes $(x^2 + y^2)^2 = 2\,a^2\,(x^2 - y^2)$ the locus is in this case the lemniscata of Bernouilli.

3. Let a be greater than b.

Then from (1) we have two values of x, and from (2) an impossible value of y; hence the curve must consist of the two small oval figures round S and H.

As b decreases, the little ovals decrease; and when $b = 0$, we have the points S and H themselves for the locus.

These curves are called the ovals of Cassini, that celebrated astronomer having imagined that the path of a planet was a curve like the exterior one in the above figure.

The equation $(y^2 + x^2)^2 = b^2\,y^2 + a^2\,x^2$, found in art. (123), gives a figure like that in case 1.

318. There are some cases in which it is useful to introduce a third variable; for example, if the equation be $y^4 + x^2\,y^2 + 2\,y^3 + x^3 = 0$, it requires the solution of an equation of three or four dimensions, in order to find corresponding values of x and y; to avoid this difficulty, assume $x = u\,y$,

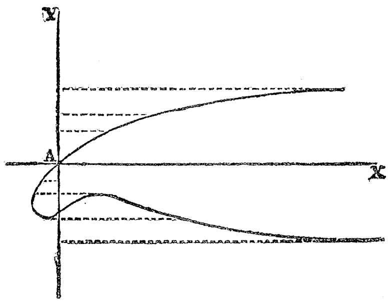
$$\therefore\ y^4 + u^2\,y^4 + 2\,y^3 - u^3\,y^3 = 0,$$

$$\text{or,}\quad y + u^2\,y + 2 - u^3 = 0,$$

$$\therefore\ y = \frac{u^3 - 2}{u^2 + 1},\ \text{and } x = u.\frac{u^3 - 2}{u^2 + 1};$$

from these equations we can find a series of corresponding values for x and y.

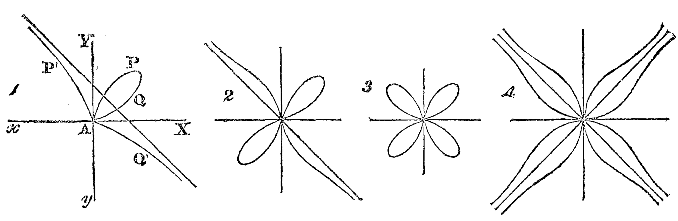
$u = -3$	$y = -2\frac{9}{10}$,	$x = 8\frac{7}{10}$
$u = -2$	$= -2$	$= 4$
$= -1$	$= -1\frac{1}{2}$	$= 1\frac{1}{2}$
$= -\frac{1}{2}$	$= -1\frac{7}{10}$	$= \frac{17}{20}$
$= 0$	$= -2$	$= 0$
$= \frac{1}{2}$	$= -1\frac{1}{2}$	$= -\frac{3}{4}$
$= 1$	$= -\frac{1}{2}$	$= -\frac{1}{2}$
$= 1\frac{1}{2}$	$= \frac{11}{26}$	$= \frac{33}{52}$
$= 2$	$= 1\frac{1}{5}$	$= 2\frac{2}{5}$
$= 3$	$= 2\frac{1}{2}$	$= 7\frac{1}{2}$
$= 4$	$= 3\frac{11}{17}$	$= 14\frac{10}{17}$
&c.	&c.	&c.



Also when $y = 0$, $x = 0$, hence the curve passes through A. Let A X, A Y be the axes; along the axis of y take values equal to those in the table for y; and from the points thus determined draw lines equal to the corresponding values in the table for x (these are the dotted lines in the figure); by this method we obtain a number of points in the curve sufficient to determine its course.

This example is taken from the "Analyse des Lignes Courbes, by G. Cramer. Geneva. 1750," a work which will be found extremely useful in the study of algebraical curves.

319. To trace the curve whose equation is $y^5 - 5\,a\,x^2\,y^2 + x^5 = 0$.



Let x be very small $\therefore$ x^5 being exceedingly small may be omitted, and the equation becomes $y^5 = 5\;a\,x^2\,y^2$, or $y^3 = 5\;a\,x^2$, which is the equation to a semi-cubical parabola P A P′ fig. (1.); and if y be very small, we have $x^3 = 5\;a\,y^2$, which gives the parabola Q A Q′; hence near the origin the curve assumes the forms of the two parabolic branches. Again when x is infinitely great, x^2 may be neglected in comparison with x^5 and the equation becomes $y^5 = -\,x^5$, $\therefore y = -\,x$; hence for x positive, we have an infinite branch in the angle X A y, and for x negative an infinite branch in the angle Y A x.

To find the asymptote:

$$y^5 = -\,x^5 + 5\,a\,x^2\,y^2,$$

$$= -\,x^5\left(1 - 5\,\frac{a\,y^2}{x^3}\right),$$

$$\therefore\; y = -\,x\left(1 - 5\,a\,\frac{y^2}{x^3}\right)^{\frac{1}{5}},$$

$$= -\,x\left\{1 - a\,\frac{y^2}{x^3} - 2\,a^2\frac{y^4}{x^6} - \text{, \&c.}\right\}$$

$$= -x + a\left(\frac{y}{x}\right)^2 + 2a^2\frac{y^2}{x^5} +, \&c.$$

$$= -x + a + 2\frac{a^2}{x^3} +, \&c. \text{ when } y = -x:$$

Therefore the equation to the asymptote is $y + x = a$; this being drawn and the branches A P′, A Q′ produced towards it, we have nearly a correct idea of the curve.

If the equation be $y^5 - 5a^2x^2y + x^5 = 0$, the curve will be traced in the same manner, fig. (2).

If the equation be $y^6 - a^2x^2y^2 + x^6 = 0$, we have fig. (3);

And the equation $y^6 - a^2x^2y^2 - x^6 = 0$ will give fig. (4).

Ex. Find the locus of the equation $y^4 - 4a^2xy - x^4 = 0$.

For the above method of tracing curves of this species, see a treatise on the Differential Calculus, by Professor Miller. Cambridge, 1832.

320. B C is a straight line of given length ($2b$), having its extremities always in the circumferences of two equal circles, to find the locus of the middle point P of the line B C.

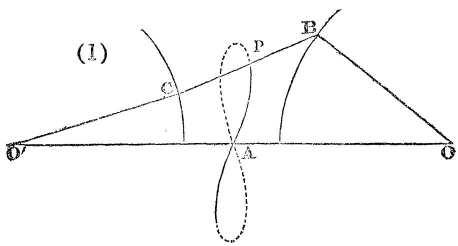
Let the line joining the centres O, O′ of the circles be the axis of x, and let the origin of rectangular axes be at A, the bisecting point of O O′.

Let $x\,y$ be the co-ordinates of B.

$x'\,y'$ . . . . . C.

X Y . . . . . P.

(1)



$$AO = AO' = a,$$

$$OB = O'C = c,$$

$$\left.\begin{array}{lll} \text{the equation to B is} & y^2 + (x-a)^2 = c^2 & (1) \\ \text{to C is} & y'^2 + (x' + a^2)^2 = c^2 & (2) \\ \text{also } (y-y')^2 + (x-x')^2 = 4b^2 & & (3) \\ & 2Y = y + y' & (4) \\ & 2X = x + x' & (5) \end{array}\right\}$$

From these five equations we must eliminate the four quantities y, x, y' and x'; from (1) and (2)

$$y^2 - y'^2 + x^2 - x'^2 - 2a(x + x') = 0,$$

$$\text{or } (y - y')Y + (x - x')X - 2aX = 0 \qquad (6),$$

from (4) and (5) $y^2 + y'^2 + x^2 + x'^2 + 2yy' + 2xx' = 4Y^2 + 4X^2$

from (3) $y^2 + y'^2 + x^2 + x'^2 - 2yy' - 2xx' = 4b^2$,

from (1) and (2) $2y^2 + 2y'^2 + 2x^2 + 2x'^2 - 4a(x - x') = 4c^2 - 4a^2$,

∴ by substitution $4a(x - x') = 4(Y^2 + X^2 + b^2 + a^2 - c^2)$,

$$\text{or } (x - x') = \frac{Y^2 + X^2 + m^2}{a}, \text{ if } m^2 = a^2 + b^2 - c^2,$$

and from (6)

$$y - y' = \{2a - (x - x')\} \frac{X}{Y} = \left(2a - \frac{Y^2 + X^2 + m^2}{a}\right) \frac{X}{Y}$$

Substituting these values of $x - x'$ and $y - y'$ in (3), we have

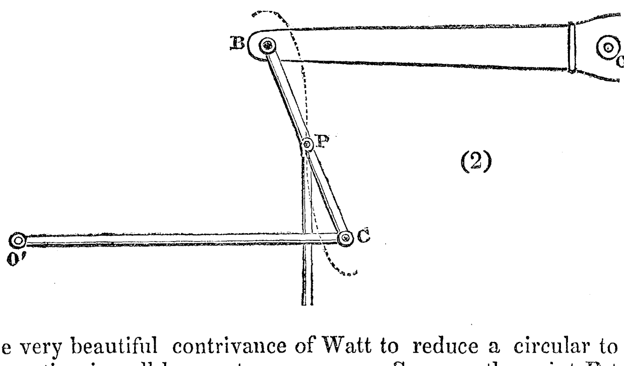
$$\left\{2a - \frac{Y^2 + X^2 + m^2}{a}\right\}^2 \frac{X^2}{Y^2} + \left\{\frac{Y^2 + X^2 + m^2}{a}\right\}^2 = 4b^2,$$

$$\text{or } 4a^2X^2 - 4(Y^2 + X^2 + m^2)X^2 + \left(\frac{Y^2 + X^2 + m^2}{a}\right)^2 (X^2 + Y^2) = 4b^2Y^2$$

$$\therefore \ a^2X - b^2Y^2 - X^2(X^2 + Y^2 + m^2) + (X^2 + Y^2)\left(\frac{X^2 + Y^2 + m^2}{2a}\right)^2 = 0.$$

This equation, being of the sixth dimension, and the highest terms being both positive, the curve must be limited in every direction: when X is very small, there are four values of Y; also when $X = 0$, we have $Y = 0$; hence the curve is a species of double oval, or lemniscata.

If the circles be unequal, and P be any point in the line B C, the curve will be of the same nature, but the investigation is much longer.



(2)

The very beautiful contrivance of Watt to reduce a circular to a rectilinear motion is well known to every one. Suppose the point B to be the extremity of an engine-beam, moveable about its centre O, this beam is required to moved a piston-rod always in the same vertical position; it is plain that this motion cannot be obtained by fixing the piston-rod to B, or to any point in O B. Suppose now, a beam O′ C, called the radius-rod, to move about a centre O′, and join the extremities B, C, by a bar B C; the extremity of the piston-rod is fixed to the middle of the beam B C, and its motion, according to the above demonstration, is in a portion of the curve, such as the dark part of the lemniscate in the first figure, and consequently the rod itself continues much more in the same vertical line than if attached to B. The comparative lengths of the rods necessary to render the motion as nearly vertical as possible are stated in most works on the steam-engine, and in the Mechanics' Magazine. For a more complete but very different method of finding the equation to the above curve, see "Prony, Hydraulique."

321. We have no space for the discussion of any higher algebraic curves, if it were necessary; but in fact we have not the means: it must

have been already seen that many of the preceding curves have not been drawn with mathematical exactness; for unless we took the trouble of tracing them by points, we could not easily determine their curvature; we shall therefore pass to the consideration of the general equation of the nth dimension, and then proceed to the intersection of algebraic curves*.

322. The general equation of the nth degree, with all its terms complete, is

$$y^n + (a x + b) y^{n-1} + (c x^2 + d x + e) y^{n-2} + \ldots + f x^n + g x^{n-1} + h x^{n-2} + \ldots + k x + l = 0;$$

it contains all the possible combinations of x and y, so that the sum of the exponents in no one term exceeds n.

The number of terms is $1 + 2 + 3 + \ldots + (n + 1)$, or is the sum of an arithmetic progression, whose first term and common difference is unity, and the number of terms is $n + 1$; therefore the sum of this series is $\frac{(n+2)\ (n+1)}{2}$.

The number of independent constants is (dividing by the co-efficient of y^n if necessary) one less than the number of terms in the equation, that is,

$$= \frac{(n+2)\ (n+1)}{2} - 1 = \frac{n\ (n+3)}{2}$$

323. An algebraic curve of the nth degree may pass through as many given points as it has arbitrary constants, that is, through $\frac{n\ (n+3)}{2}$ points, for giving to x and y their values at each one of the given points, we have $\frac{n\ (n+3)}{2}$ different equations, by which the values of the constants may be determined. For example,

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* We must refer our readers to our treatise on the Differential Calculus for information on the curvature of lines. It must not, however, be imagined that algebraic geometry is incapable of exhibiting the form of curves; the following method of determining the curvature is an instance to the contrary.

Let y_1, y_2, and y_3 be three consecutive ordinates, at equal distances from each other; then drawing a corresponding figure, it will be seen that the curve is concave or convex to the axis, according as y_2 is $>$ or $< \frac{y_1 + y_3}{2}$; as an example, take the cubical parabola, whose equation is $a^2 y = x^3$, then the curve is convex, if $2 x^3$ is $< (x-1)^3 + (x+1)^3$ is $> 2 x^3 + 6 x$, which it is, and therefore the curve is convex. The distances at which the ordinates are drawn from each other must depend on the constants in the equation.

Again, to determine the angle at which a curve cuts the axis of x, transfer the origin to that point; then the tangent to the curve at that point and the curve itself make the same angle with the axis; but the value of the tangent of the angle which the tangent to the curve makes with the axis is then $\frac{y}{x} = \frac{0}{0}$, which may be any value whatever: for example, let $a^2 y = x$ $\therefore \frac{y}{x} = \frac{x^2}{a^2} = 0$ when $x = 0$, therefore the curve coincides with the axis at the origin. Again, take the example in art. 307 $y^3 = x^2 (2a - x)$; at A we have $\frac{y^3}{x^3} = \frac{2a - x}{x} = \frac{1}{0}$, and at B we have $\frac{y}{2a - x} = \frac{x^2}{y^2} = \frac{4 a^2}{0}$; hence the curve cuts the axis of x in both cases at an angle of 90°.

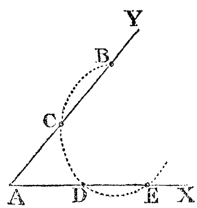
The general equation to the conic sections, dividing by the co-efficient of y^2, is

$$y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0,$$

in which there are five co-efficients, and therefore a conic section may pass through five given points; substituting the co-ordinates of the given points separately for x and y we obtain five equations from which the constants can be determined, and thence we have the particular curve required; it will be an ellipse, hyperbola, or parabola, according as $b^2 - 4c$ is negative, positive, or nothing. (79.)

324. The elimination is long, but the trouble may be much lessened by assuming one of the given points for the origin, and two lines drawn from the origin to other two given points for the axes.

For example, if it be required to pass a conic section through four given points B C D E, join B C and D E, and let them meet in A; let A B be the axis of y and A D the axis of x,



Let $AC = y_1$, $AB = y_2$,

$AD = x_1$, $AE = x_2$;

Assume the equation to be

$$y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0;$$

we have for $x = 0$, $y_1^2 + d\,y_1 + f = 0$

and $y_2^2 + d\,y_2 + f = 0$;

$$\therefore \quad d = -(y_1 + y_2), \text{ and } f = y_1\,y_2.$$

Similarly for $y = 0$, $e = -c\,(x_1 + x_2)$, and $f = c\,x_1\,x_2$;

equating the values of f, we have $c = \dfrac{y_1\,y_2}{x_1\,x_2}$.

Substituting and dividing by $y_1\,y_2$, we have

$$\frac{y^2}{y_1\,y_2} + \frac{b}{y_1\,y^2}\,x\,y + \frac{x^2}{x_1\,x_2} - \frac{y_1 + y_2}{y_1\,y_2}\,y + \frac{x_1 + x_2}{x_1\,x_2}\,x + 1 = 0,$$

an equation involving only one unknown co-efficient b.

There are some restrictions depending on the situation of the given points; thus no more than two can be in the same straight line, or else the conic section degenerates into two straight lines.

The five given points are the same as five conditions expressed analytically; four are sufficient if the curve is to be a parabola; for $b^2 - 4c = 0$, is equivalent to one. If the curve has a centre, whose position is given, three other conditions suffice, because we may assume the equation to be $y^2 + b\,x\,y + c\,x^2 + f = 0$. If the position of two conjugate diameters be given, only two more conditions are requisite.

Newton, in his Universal Arithmetic, gives excellent methods for describing, by continued motion, a conic section passing through five given points.

325. If it be required to pass a curve, whose species is not given, through a number of given points, we may with advantage assume the equation to be of the form

$$y = a + b\,x + c\,x^2 + d\,x^3 +, \text{ \&c.}$$

The elimination of the constants is more regular, and therefore easier in this equation than in any other: such curves are called parabolic (the three first terms giving the common parabola) and consist of a series of sinuosities, such as in (309), which are easily traced. For the elimination of the constants, see Lagrange, or Lardner's Algebraic Geometry, art. 617.

326. We saw in article (79) that the general equation of the second order sometimes gave straight lines for the loci; such will be the case whenever any equation is reducible into rational factors of the first degree; so that we must not always conclude that an equation of the nth order has a curve of the nth order for its locus. If the equation be reducible into factors of lower degrees, there will be a series of lines corresponding to those factors; thus if an equation of the 4th degree be composed of one factor of two degrees, and two factors of the 1st degree, the loci are a conic section and two straight lines; and hence a general equation of any order embraces under it all curves of inferior orders: if any of the factors be impossible, their loci are either points, or imaginary.

If the sum of the indices of x and y be the same in every term, the loci are either straight lines or points; for an equation of this species will have the form

$$y^n + a\,y^{n-1}\,x + b\,y^{n-2}\,x^2 \;\ldots\ldots + l\,x^n = 0,$$

$$\text{or}\quad \left(\frac{y}{x}\right)^n + a\left(\frac{y}{x}\right)^{n-1} + b\left(\frac{y}{x}\right)^{n-2} \ldots\ldots + l = 0,$$

let the roots of this equation be α, β, γ, &c., then the equation will be

$$\left(\frac{y}{x} - \alpha\right)\left(\frac{y}{x} - \beta\right)\left(\frac{y}{x} - \gamma\right)\ldots = 0,$$

each factor of which being $= 0$, its corresponding locus is evidently a straight line; if the roots of the equation be impossible, the corresponding loci are points.

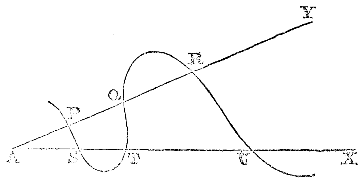
Ex. $y^2 - 2\,x\,y \sec.\alpha + x^2 = 0$. The locus consists of two straight lines whose equations are $y = x\,\dfrac{1 \pm \sin.\alpha}{\cos.\alpha} = x \tan.\left(45^\circ \pm \dfrac{\alpha}{2}\right)$ and therefore the lines pass through the origin, and are inclined to the axis of x at angles of $45^\circ \pm \dfrac{\alpha}{2}$.

327. Since the general equation includes all equations below it, the properties of the curve of n dimensions will generally be true for the lower orders, and also for certain combinations of the lower orders; thus, a property of a line of the third degree will be generally true for a conic section, or for a figure consisting of a conic section and a straight line, or for three straight lines. Moreover the lower orders of curves have generally some analogy to the higher curves, and hence the properties of inferior orders often lead to the discovery of those of the superior.

328. From the application of the theory of equations to curves, an immense number of curious theorems arise, which may be seen in the

works of Waring and Maclaurin: we have only room for two or three of the most important.

If two straight lines, A X, A Y cut a curve of n dimensions, in the points P Q R, &c., S T U, &c., so that A P, A Q, A R, &c. $= y_1, y_2, y_3$, &c. respectively, and A S, A T, A U, &c. $= x_1, x_2, x_3$, &c. respectively, then if A X and A Y move parallel to themselves, we shall always have $y_1 . y_2 . y_3$. &c. : $x_1 . x_2 . x_3$. &c., in a constant ratio.



Let the equation to the curve be referred to the origin A, and to axes A X, A Y, by means of the transformation of co-ordinates, and suppose the equation to be

$$y^n + (a x + b) y^{n-1} + \ldots c x^n + d x^{n-1} + \ldots k x + l = 0.$$

$$\text{Let } y = 0 \quad \therefore \; c x^n + d x^{n-1} + \ldots . k x + l = 0. \qquad (1)$$

$$x = 0 \quad \therefore \; y^n + b y^{n-1} + \ldots . k y + l = 0. \qquad (2)$$

The roots of (1) are A S, A T, A U, &c.; $\therefore x_1 . x_2 . x_3$. &c. $= \dfrac{l}{c}$.

The roots of (2) are A P, A Q, A R, &c.; $\therefore y_1 . y_2 . y_3$. &c. $= l$.

$$\therefore y . y_2 . y_3, \&c. : x_1 . x_2 . x_3, \&c. :: c : 1.$$

Now the transformation of the axes, parallel to themselves, never alters the co-efficients of y^n and x^n; hence the above ratio is constant for any parallel position of A X and A Y.

Article 293 is an example of this theorem.

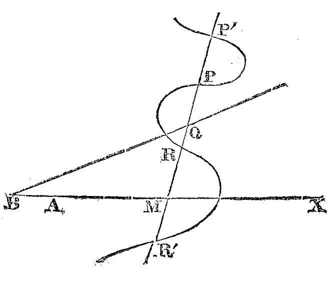
329. A diameter was defined in (76) to be a straight line, bisecting a system of parallel chords; more generally it is a line, such that if any one of its parallel chords be drawn, meeting the curve in various points, the sum of the ordinates on one side shall equal the sum on the other; thus, in the figure, if P Q + P′ Q + &c. = R Q + R′ Q = &c., and the same be true for all lines parallel to P R, then B Q is a diameter.

To find the equation to the diameter B Q let the equation to the curve, referred to A X and a parallel to P Q, &c.

$$y^n + (a x + b) y^{n-1} + (c x^2 + d x + e) y^{n-2} +, \&c. = 0.$$

Let M Q $= u$, and P Q $= y'$, $\therefore y = y' + u$,
by substitution we have

$$y'^n + (a x + b + n u) y'^{n-1} + \{c x^2 + d x + e + \overline{n-1}\, u . \overline{a x + b} + n . \frac{n-1}{2} u^2\} y'^{n-2} +, \&c. = 0$$



By the definition the sum of the values of y' must equal nothing, and that sum is the co-efficient of the second term in the last equation with its sign changed,

$$\therefore\ a x + b + n u = 0,$$

$$\text{or} \quad u = -\frac{a x + b}{n},$$

and this is the equation to the diameter B Q.

Again, by the same reasoning, the equation

$$c x^2 + d x + e + \overline{n-1}\, u \,.\, \overline{a x + b} + n \,.\, \frac{n-1}{2} u^2 = 0$$

is that to a conic section drawn so that the sum of the products of the values of y, taken two and two together, shall equal nothing.

We might proceed on with the co-efficient of the fourth term.

These curves are sometimes called curvilinear diameters.

330. The method of finding the centre, if any, of a curve, is given in (81); the operation is too long to apply it to a general equation of high dimensions, and therefore we shall take an example among the lines of the third order as fully illustrating the subject.

Let the equation be $x y^2 + e y = a x^3 + b x^2 + c x + d$, under which form are comprehended most of the curves of the third order.

Let $x = x + m$, $y = y + n$; the transformed equation is

$$x y^2 + 2 n x y + m y^2 + (2 n m + e) y - a x^3 - (3 a m + b) x^2$$
$$+ (n^2 - 3 a m^2 - 2 b m - c) x + m n^2 + e n - a m^3 - b m^2$$
$$- c m - d = 0;$$

in order that the curve may have a centre, the 2nd, 3rd, 6th, and last or constant term must each $= 0$; $\therefore n = 0$, $m = 0$, $b = 0$, $d = 0$, so that the corresponding curve has a centre, which is the origin, only when the co-efficients b and d are wanting.

CHAPTER XIII.

ON THE INTERSECTION OF ALGEBRAIC CURVES.

331. The intersection of a straight line with a line of the nth order is found by eliminating y from the two equations; hence the resulting equation in terms of x will be of the nth order, and therefore may have n real roots; thus there may be n intersections: there may be less, since some of the roots of the resulting equation may be equal to one another, or some impossible.

Generally speaking, a curve of n dimensions may be cut by a straight line drawn in some direction in n points; but the curve, in its most general form, must be taken; otherwise certain points as conjugate and multiple points, must be considered as evanescent ovals or evanescent branches of the curve, and thus a line passing through such points is equivalent to two or more intersections.

332. The intersections of two lines of the mth and nth orders are found also by eliminating y from both; hence the resulting equation may be of the mnth order, or there may be $m\,n$ intersections; there are often less, for not all the real roots of the equation $X = 0$ will give points of intersection: for example, if we eliminate y from the equations

$$y^2 = 2\,a\,x - x^2 \text{ and } y^2 = 2\,a\,(x - b) \text{ we find } x = \sqrt{2\,a\,b};$$

hence, apparently, there is always an intersection corresponding to the abscissa $\sqrt{2\,a\,b}$; but this is not the case; for then $y^2 = 2\,a\,(\sqrt{2ab} - b)$, and therefore y is impossible, if b is $> 2\,a$, which is evident on drawing the two curves; hence after the abscissa is found, we must examine the corresponding ordinates in each curve; if they are not real, there can be no intersection corresponding to such abscissa.

If we have the two equations $y^2 + 2\,x = 0$, $y^2 + 4\,x^2 - 10\,x - 16 = 0$, the elimination y gives the abscissas of intersection $x = 4$ and $x = -1$, the second of which alone determines a point of intersection.

333. In finding the intersections of lines, we often fall upon a final equation of an order higher than the second, or arrive at an equation whose roots are of a form not readily constructed; to avoid this difficulty a method is often used which consists in drawing a line which shall pass through all the required points of intersection, and thus determine their situation.

Let $y = f\,(x)$* (1), and $y = \phi\,(x)$ (2), be the equations to two lines, then at the point of intersection they have the same ordinates and abscissas; or calling X and Y the co-ordinates of the point of intersection, we have simultaneously $Y = f\,(X)$ and $Y = \phi\,(X)$; hence $f\,(X) = \phi\,(X)$, from which equation X and Y might be obtained, and their values constructed

But since	$Y = f\,(X)$	(3)
	and $Y = \phi\,(X)$	(4)
we have by addition	$2\,Y = f\,(X) + \phi\,(X)$	(5)
or by multiplication	$Y^2 = f\,(X)\;.\;\phi\,(X)$	(6)
or generally	$Y = F\,\{f(X), \phi\,(X)\}$	(7)

F implying any function arising from the addition, subtraction, multiplication, &c. of (3) and (4).

Now any one of these equations gives a true relation between the co-ordinates X and Y of the point of intersection of (1) and (2); but by supposing X and Y to vary, it will give a relation between a series of points, of which the required point of intersection is certainly one; that is, drawing the locus of (5) or (6) or (7), it must pass through the required point of intersection of (1) and (2).

It is manifest that if one of the equations (5), (6), or (7), be a circle

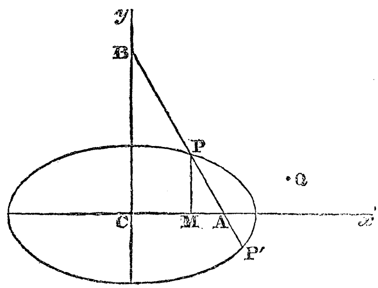
* The symbols $F\,(x)$, $f\,(x)$, $\phi\,(x)$, serve to denote different functions of x, that is, indicate expressions into which the same quantity x enters, but combined in different ways with given quantities. But $f\,(x)$, $f\,(y)$, indicate similar formulæ for both x and y; thus, if $f\,(x) = 2\,a\,x - x^2$, then $f\,(y) = 2\,a\,y - y^2$, or $2\,b\,y - y^2$.

or straight line, it will be much easier to draw this circle or straight line than to find the intersection by means of elimination.

Also we may often find the intersection of (1) and (2), when one of them is a given curve, by drawing the locus of the other, and this method is the simplest when that other is a straight line.

We shall give a few examples to illustrate the subject.

334. From a given point Q without an ellipse, to draw a tangent to it.



Let the co-ordinates of Q be m and n, and let X and Y be the co-ordinates of the point P, where the required tangent meets the curve.

Then by (111) the equation to the tangent through P is

$$a^2 y \mathrm{Y} + b^2 x \mathrm{X} = a^2 b^2,$$

and since this passes through Q we have

$$a^2 n \mathrm{Y} + b^2 m \mathrm{X} = a^2 b^2 \qquad (1)$$

$$\text{and} \quad a^2 \mathrm{Y}^2 + b^2 \mathrm{X}^2 = a^2 b^2. \qquad (2)$$

From (1) and (2) we might, by elimination, find X and Y, and their constructed values would be the co-ordinates C M, M P of the required point.

Now (1) is not the equation to any straight line, but only gives the relation between C M and M P; but if we suppose X and Y to vary, it will give the relation between a series of points, of which P is certainly one; and therefore, if the line whose equation is (1) be drawn, it must pass through P, and consequently, with the ellipse (2), will completely fix the situation of P.

To draw the line (1),

Let $\mathrm{X} = 0$; $\therefore \mathrm{Y} = \frac{b^2}{n}$; Let $\mathrm{Y} = 0$; $\therefore \mathrm{X} = \frac{a^2}{m}$;

in C y take C B $= \frac{b^2}{n}$, and in C x take C A $= \frac{a^2}{m}$; join B A; B A produced is the locus of (1), and it cuts the ellipse in two points P and P; hence if Q P and Q P′ be joined, they are the tangents required.

The same method may be employed in drawing tangents to the parabola and hyperbola.

To take the more general case, let $a y^2 + c x^2 + d y + e x = 0$ (1) be the equation to the curves of the second order referred to axes parallel to conjugate diameters.

Then the equation to a tangent at a point $x' y'$ is

$$a y y' + c x x' + \frac{d}{2}(y + y') + \frac{e}{2}(x + x') = 0,$$

or $(2ay'+d)\,y+(2cx'+e)\,x+dy'+ex'=0$ (2).

Let this tangent pass through a point $m\,n$, then (2) becomes

$$(2ay'+d)\,n+(2cx'+e)\,m+dy'+ex'=0 \qquad (3),$$

or, $(2an+d)\,y'+(2cm+e)\,x'+dn+em=0$ (4).

Now let x' and y' in (4) be considered variable, and construct the straight line, which is the locus of (4); this with the curve itself, determines the position of the secant line which joins the two points on the curve, whence tangents are drawn to the point $m\,n$.

335. Again, suppose the secant line (4) to pass through a given point $m'\,n'$; Then the equation (4) becomes

$$(2an+d)\,n'+(2cm+e)\,m'+dn+em=0 \qquad (5),$$

and of course the point $m\ n$, whence tangents were originally drawn, must have a particular position corresponding to each secant line passing through $m'\,n'$; if therefore we make m and n variable in (5) we shall have the equation to the locus of the point $m\,n$

$$(2an'+d)\,n+(2cm'+e)\,m+dn'+em'=0$$

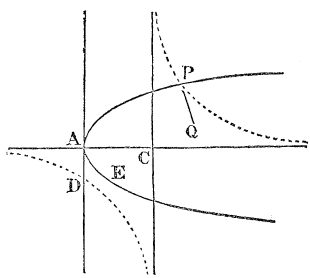
where m and n are the variable co-ordinates.

Hence we have the following theorem: if from any point secants be drawn to a line of the second order, and from the two points where each of these secants intersect the curve, tangents be drawn meeting each other, the locus of all such points of concourse is a straight line.

336. To draw a normal to a parabola from a point Q $(a, b,)$ not on the curve.

Let $y^2=4mx$, be the equation to the curve, and let X and Y be the co-ordinates of the required point, then the equation to the tangent at the point X Y, is by (232)

$$Yy=2m(X+x),$$



and therefore that to the normal at X Y is

$$y-Y=-\frac{Y}{2m}(x-X),$$

and since it passes through $(a\,b)$ we have

$$b-Y=-\frac{Y}{2m}(a-X),$$

or, $XY-(a-2m)\,Y-2mb=0,$ (1)

also, $Y^2=4mX$ . . . . . (2)

The elimination of X gives $Y^3-4m\,(a-2m)\,Y-8m^2b=0$ (3), an equation whose roots would give the three required ordinates.

To avoid this equation we shall construct the locus of (1), which is the equation to an equilateral hyperbola. The axis of x is one asymptote (198), and the other is parallel to the axis of y, and at a distance $AC=a-2m$ from A; the equation to the hyperbola referred to its centre C and asymptotes is $XY=2mb$; moreover the hyperbola cuts the axis of y in the point D, where $AD=\dfrac{2mb}{2m-a}$; hence this hyperbola (the dotted curve in the figure) may be constructed.

We have drawn the figure, so that there shall be only one intersection of the curves, and hence only one normal is drawn from Q. If the curves touched, as at E, there would be two normals; and if the hyperbola cut the parabola in the lower branch, there would be three normals drawn from Q. These cases correspond respectively to the equation (3), having one real root; three real roots of which two are equal; and, lastly, three real and unequal roots.

337. We must particularly observe that, in the construction of loci, those are to be selected which admit of the easiest description, and of all curves the circle is to be preferred; hence, in the present case, we must look carefully to see if it is possible, by any combination of (1) and (2), to obtain the equation to the circle; for by 333 this will pass through the required normal points.

Multiply (1) by Y, then

$$XY^2 - (a - 2m)\,Y^2 - 2mb\,Y = 0,$$

$$\text{or,}\quad X\,.\,4mX - (a - 2m)\,4mX - 2mb\,Y = 0;$$

$$\therefore\ X^2 - (a - 2m)\,X - \frac{b}{2}\,Y = 0,$$

$$\text{and } Y^2 - 4mX = 0, \text{ from (2)}$$

$$\therefore \text{ by addition }\quad Y^2 + X^2 - (a + 2m)\,X - \frac{b}{2}\,Y = 0$$

which is the equation to a circle, the co-ordinates of whose centre are $\frac{a}{2} + m$ and $\frac{b}{4}$, and whose radius is $\sqrt{\left\{\left(\frac{a}{2} + m\right)^2 + \frac{b^2}{16}\right\}}$. Although this circle passes through the vertex of the parabola, yet that point is not one of the required intersections, but merely arises from the multiplication of (1) by Y.

If the parabola and circle be drawn, the latter in various situations according to the position of Q, we shall see, as before, that there will be one, two or three intersections: such practice will be found very useful.

The problem of drawing a normal to an ellipse is of the same nature, only in this case there may be four intersections.

338. The intersection of curves has been employed in the last articles to avoid the resolution of equations resulting from elimination, but the principle may be extended, so as to render curves generally subservient to the solution of equations; for as two equations combined produce one whose roots give the intersection of their loci, so that one may in its turn be separated into two, whose loci can be drawn, and their intersection will determine the roots of the one.

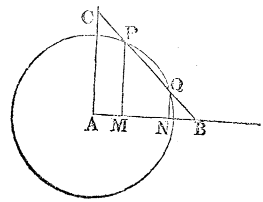
This method, known by the name of "the construction of equations," was much used by mathematicians before the present methods of approximation were invented; it is even now useful to a certain extent, and therefore we proceed to explain it.

Let there be two equations: $y + x = a$ (1), $y^2 + x^2 = b^2$ (2), by elimination we find

$$x^2 - ax + \frac{a^2 - b^2}{2} = 0 \quad (3).$$

We already know that the roots of (3) are the abscissas to the points of intersection of the loci (1) (2); but, conversely, it is manifest that the roots of (3) can be determined by drawing the loci of (1) and (2), and measuring the abscissas of intersection.

Hence if it be required to exhibit geometrically the roots of (3), let it be decomposed into the two equations (1) and (2), and let C P Q B be the locus of (1), and the circle E P Q of (2), having the same origin and axes: draw the ordinates M P, N Q, then A M and A N are the roots of (3).



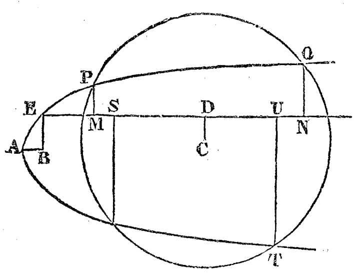
The method consists in parting any given equation into two others, and then drawing the loci of those two; and as it is obvious that there are a great many equations which, when combined together, may produce the given equation, so we may construct a great many loci, whose intersections will give the required roots: thus, in the above case, the equation (3) may be resolved into the two $x^2 = a\,y$, and $a\,y - a\,x + \frac{a^2 - b^2}{2} = 0$, and the corresponding parabola and straight line being drawn, their intersections will give the roots of (3).

In general the roots of an equation can be found by the intersection of any two species of curves whose indices, multiplied together, are equal to the index of the equation: thus, a straight line and a curve of the third order will give the solution of an equation of the third order; and any two conic sections, except two circles, will give the roots of an equation of the fourth order.

339. As equations of the third and fourth order are of frequent recurrence in mathematical researches, we proceed to the solution of the complete equation of the fourth order,

$$y^4 + p\,y^3 + q\,y^2 + r\,y + s = 0.$$

Here the circle and parabola, as curves of easy description, ought to be chosen, and assuming the equation to the parabola a slight artifice will give us that to the circle.



$$\text{Let } y^2 + \frac{p}{2}\,y = x \quad (1);$$

$$\therefore\ y^4 + p\,y^3 + \frac{p^2}{4}\,y^2 = x^2$$

$$\text{but}\quad y^4 + p\,y^3 + q\,y^2 + r\,y + s = 0,$$

$$\therefore \text{ by subtraction } \left(q - \frac{p^2}{4}\right)y^2 + x^2 + r\,y + s = 0,$$

$$\text{or from (1), } \left(q - \frac{p^2}{4}\right)\left(x - \frac{p}{2}\,y\ \right) + x^2 + r\,y + s = 0\,;$$

$$\text{or } x^2 + \left(q - \frac{p^2}{4}\right)x + \left(r - \frac{p\,q}{2} + \frac{p^3}{8}\right)y + s = 0,$$

$$\text{and from (1), } y^2 + \frac{p}{2}\,y - x \qquad = 0,$$

$$\therefore\ y^2 + x^2 + \left(r + \frac{p^3}{8} + \frac{p}{2} - \frac{p\,q}{2}\right)y + \left(q - 1 - \frac{p^2}{4}\right)x + s = 0 \quad (2).$$

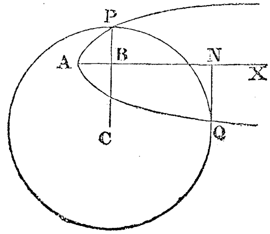
The locus of (1) is the parabola A E Q, the origin being at E $\left(\text{BE} = \frac{p}{4}\right)$, and the co-ordinates rectangular. The locus of (2) is the circle Q P R; the co-ordinates E D, D C of the centre, and the radius are readily determined from (2). The roots of the equation are drawn as if two, P M, Q N were positive, and other two R S, T U were negative. If the circle touch the parabola, two roots are equal; the cases of three or four equal roots can only be discussed by the principles of osculation, but as two roots are sufficient to depress the equation to one of the second order, we need not here consider those cases. If there be only two intersections, two roots are impossible; and if there be no intersection, all four roots are impossible.

340. In practice the operation is shortened by first taking away the second term of the equation; for example, to construct the roots of the equation

$$x^4 + 8\,x^3 + 23\,x^2 + 32\,x + 16 = 0. \qquad (1).$$

Let $x = y - 2$, and the reduced equation is

$$y^4 - y^2 + 4\,y - 4 = 0. \qquad (2).$$



Let $y^2 = x$ (3);

$\therefore$ by substitution $x^2 - x + 4\,y - 4 = 0$,

$\therefore$ by addition $y^2 + x^2 + 4\,y - 2\,x - 4 = 0$,

or, $(y + 2)^2 + (x - 1)^2 = 9$ (4).

Let P A Q be the parabola (3), whose parameter is unity, the co-ordinates of the centre C of the circle (4) are A B = 1, and B C = − 2, the radius = 3. Describing this circle, the ordinates B P and Q N are the possible roots of (2); measuring these values we shall find P B = 1, and Q N = − 2; hence the possible roots of (2) are 1 and − 2, and therefore those of (1) are − 1 and − 4.

341. The construction of equations of the third order is involved in that

of the fourth order. Take away the second term, if necessary, multiply the resulting equation $Y = 0$ by y, and then proceed precisely as in the last article. The circle will always pass through the vertex of the parabola, but this intersection gives the root $y = 0$, introduced by multiplication, and has therefore nothing to do with the roots of the given equation. This circumstance of the circle passing through the vertex of the parabola, is singularly convenient, as it entirely saves the trouble of calculating the radius to decimal places, which is often necessary in the preceding cases.

Ex. 1. $x^3 - 6x^2 - x + 6 = 0$. Let $x = y + 2$;

$$\therefore y^3 - 13y - 12 = 0,$$

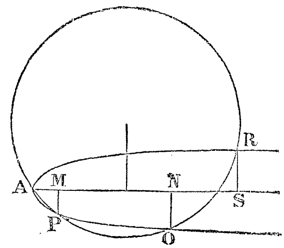
$$\text{or, } y^4 - 13y^2 - 12y = 0.$$

$$\text{Let } \quad y^2 - x = 0 - (1)$$

$$\therefore x^2 - 13x - 12y = 0,$$

$$\therefore y^2 - 12y + x^2 - 12x = 0,$$

$$\text{or, } (y-6)^2 + (x-6)^2 = 72 \quad (2).$$



The three roots of y, as given by the figure, are 4, -1 and -3; hence the values of x are 6, -1 and -1.

Ex. 2. $4y^3 + 6y - 5 = 0$. There is one possible root nearly $= \frac{1}{\sqrt{2}}$.

Ex. 3. $4y^3 - 3y + 1 = 0$.

There never can be any difficulty in constructing the loci of these equations; having once drawn a parabola, whose parameter is unity, with tolerable exactness, it will serve for the construction of any number of such equations.

As another example, we take the following question.

342. To find two mean proportionals between two given lines a and b,

Let y and x be the required lines;

$$\text{then } \quad a : y :: y : x, \quad \therefore y^2 = a x \qquad (1),$$

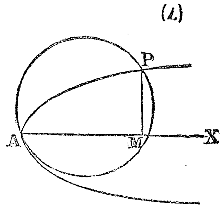
$$y : x :: x : b, \quad \therefore x^2 = b y \qquad (2),$$

$$\therefore y^4 = a^2 x^2 = a^2 b y, \text{ or } y^3 - a^2 b = 0;$$

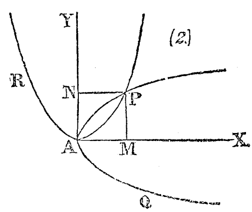
but by addition of (1) and (2), $\quad y^2 - by + x^2 - ax = 0,$

$$\text{or, } \left(y - \frac{b}{2}\right)^2 + \left(x - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4} \quad (3).$$

(1)



(2)



Let PAQ be the parabola (1), then the intersection of the circle (3) will give MP and AM, the two mean proportionals required.

The other roots of the equation $y^3 - a^2 b = 0$ are impossible.

This problem was one of those so much celebrated by the ancient mathematicians. Menechme, of the school of Plato, was the first who gave a solution of it: his method being particularly ingenious, as well as being the first instance known of the application of geometrical loci to plain problems, is well worth insertion.

With a parameter a, draw the parabola P A Q (fig. 2), and on A Y perpendicular to A X describe the parabola P A R with parameter b.

Then the rectangle a, A M or a, N P is equal to the square on M P;

$$\therefore\ a,\ MP \text{ and } NP \text{ are in continued proportion.}$$

Again, the rectangle b, A N or b, M P is equal to the square on N P;

$$\therefore\ MP,\ NP, \text{ and } b, \text{ are in continued proportion;}$$

hence we have at the same time the two proportions

$$a : MP :: MP : NP \text{ and } MP : NP :: NP : b;$$

$$\therefore\ a,\ MP,\ NP, \text{ and } b, \text{ are in continued proportion.}$$

Menechme also gave a second solution depending on the intersection of a parabola and hyperbola.

343. To find a cube which shall be double of a given cube.

Let a be a side of the given cube, then the equation to be solved is

$$y^3 = 2\,a^3, \text{ or } y^4 - 2\,a^3\,y = 0,$$

Let $y^2 = a\,x$ (1), $\therefore\ a^2\,x^2 - 2\,a^3\,y = 0$, or, $x^2 - 2\,a\,y = 0$;

$$\therefore \text{ by addition, } y^2 - 2\,a\,y + x^2 - a\,x = 0 \quad (2);$$

The loci of (1) and (2) being drawn, the ordinate P M of their intersection is the side of the required cube.

This problem, like the former, occupied the attention of the early geometricians; they soon discovered that its solution is involved in the preceding one; for if $b = 2\,a$, the resulting equations are the same.

In this manner a cube may be found which shall be m times greater than a given cube.

344. We may thus find any number of mean proportionals between two given quantities a and b.

For if y be the first of the mean proportionals, they will form the following progression:

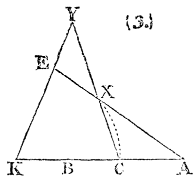
$$a,\ y,\ \frac{y^2}{a},\ \frac{y^3}{a^2},\ \frac{y^4}{a^3},\ \&c.$$

Let there be four mean proportionals, then the sixth term of the progression being b we have $\dfrac{y^5}{a^4} = b$, or $y^5 - a^4\,b = 0$.

Describe the parabola whose equation is $y^2 = a\,x$, and then draw the locus of the equation $y\,x^2 - a^2\,b = 0$. The last curve consists of an hyperbolic branch in each of the angles Y A X, Y A x, and therefore the ordinate corresponding to the real root is readily found.

345. Newton constructed equations by means of the conchoid of Nicomedes: he justly observes that those curves are to be preferred whose mechanical description is the easiest; and he adds, that of all curves, the conchoid next to the circle is, in this respect, the most simple. See the instrument in (312). The following is one of the many examples given in the Universal Arithmetic.

(3.)



Let the equation be $x^3 + q x + r = 0$, draw a straight line K A, of any length n. In K A take K B $= \frac{q}{n}$, and bisect B A in C; with centre k and radius K C describe a circle, in which inscribe the straight line C X $= \frac{r}{n^2}$; join A X, and between the lines C X and A X produced, inscribe E Y equal to C A, so that, when produced, it passes through the point K.

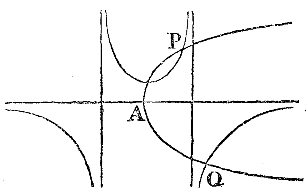
A geometrical proof follows to show that, from this construction, the equation for the length of X Y is $x^3 + q x + r = 0$, so that X Y is a root of the equation.

The conchoid is employed to insert the line E Y between C X and C A.

Let K be the pole, A X E the base, and C A the modulus; then the common description of the curve determines the point Y on the line C X Y, such that E Y = C A.

346. With regard to the higher equations, there is not much advantage in constructions, since it is extremely difficult to draw the curves with sufficient exactness. The method, however, is so far useful as enabling us to detect the number of impossible roots in any equation, as we can generally trace the curves with sufficient accuracy to determine the number of intersections, though not the exact points of intersection.

Ex. $y^5 - 3y + 1 = 0$.
Let $y^2 = x$ . . . . . (1),
$\therefore y x^2 - 3y + 1 = 0$, (2)
or $y = \frac{1}{3 - x^2}$;



the locus of (1) is a parabola P A Q, that of (2) is a curve of the third order, and there are three intersections; and, therefore, three possible roots, two positive, and one negative.

347. There is some uncertainty in the employment of curves in finding roots; we stated in (332), that real roots may correspond to imaginary intersections; so, on the contrary, imaginary intersections, or what is the same, the absence of intersections, does not always prove the absence of real roots; for example, if to prove the equation $x^4 + 15x + 14 = 0$ we assume $y^2 = x^3$ (1), and therefore $x y^2 + 15x + 14 = 0$ (2), the loci of (1) and (2) will not intersect, but yet two roots are possible. The error was in choosing a curve (1), which proceeds only in the positive direction, when from the form of the equation it is apparent that there are negative roots. Taking the circle and common parabola for the loci, as in (340), we shall find the roots to be -1 and -2. Hence, in general, to ascertain real roots it will be advisable to try more than two curves.

CHAPTER XIV.

TRANSCENDENTAL CURVES.

348. It was stated in art. (23), that those equations which cannot be put into a finite and rational algebraical form with respect to the variables, are called Transcendental; of this nature are the equations $y = \sin. x$ and $y = a^x$. In Chapter XII. we have obtained the equations to curves, generally from some distinct Geometrical property of those curves; but there are many curves whose equations thus obtained cannot be expressed in the ordinary language of algebra; that is, the equation resulting from the description or generation of the curve is dependent upon Trigonometrical or Logarithmical quantities; these curves, from the nature of their equations, are called Transcendental.

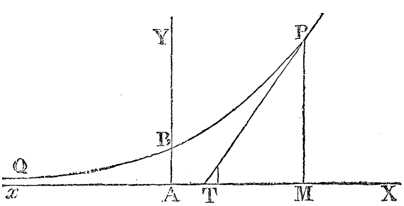
We shall here investigate the equations and the forms of the most celebrated of these curves, and mention a few of the remarkable properties belonging to them, although they can be only fully investigated by the higher calculus.

349. In this class will be found some curves, as the Cardioide, whose equations may be expressed in finite algebraic terms; but these examples are only particular cases of a species of curves decidedly Transcendental, and which cannot be separated from the rest without injury to the general arrangement.

Some of the Transcendental class have been called Mechanical curves, because they can be described by the continued motion of a point; but this name as a distinction is erroneous, for it is very probable that all curves may be thus described by a proper adjustment of machinery.

THE LOGARITHMIC CURVE.

350. The curve Q B P, of which the abscissa A M is the logarithm of the corresponding ordinate M P, is called the Logarithmic curve.



Let A M $= x$, M P $= y$, then $x = \log. y$, that is, if a be the base of the system of logarithms, $y = a^x$.

To examine the course of the curve we find when $x = 0$, $y = a^0 = 1$; as x increases from 0 to ∞, y increases from 1 to ∞; as $-x$ increases to ∞, y decreases from 1 to 0. In A Y take A B = the linear unit, then the curve proceeding from B to the right of A B, recedes from the axis of x, and on the left continually approaches that axis, which is therefore an asymptote.

This curve was invented by James Gregory; Huyghens discovered that if P T be a tangent meeting A X in T, M T is constant and equal to the modulus $\left(\frac{1}{\log. a}\right)$ of the system of logarithms. Also that the whole area M P Q x extending infinitely towards x is finite, and equal to twice the triangle P M T, and that the solid described by the revolution of the same area about X x is equal to $1\frac{1}{2}$ times the cone, by the revolution of P T M about X x.

That such areas and solids are finite is curious, but not unintelligible to those who are accustomed to the summation of decreasing infinite series.

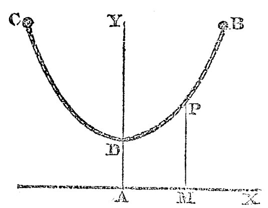
If the equation be $y = a^{-x}$, the curve is the same, but placed in the opposite direction with regard to the axis of y.

351. The equation to the curve called the Catenary, formed by suspending a chain, or string, between two points B and C, is

$$y = \tfrac{1}{2}(e^{x} + e^{-x})$$

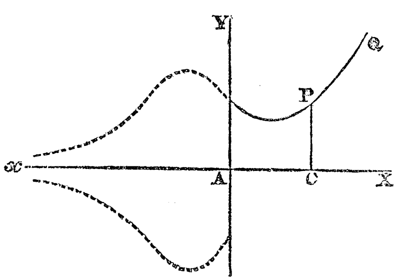
where A M $= x$, M P $= y$,

and A D $= 1$.



This equation cannot be obtained by the ordinary algebraical analysis; but it is evident that the curve may be traced from this equation, by adding together the ordinates of two logarithmic curves corresponding to the equations $y = e^{x}$ and $y = e^{-x}$.

352. Trace the locus of the equation $y = a^{\frac{1}{x}}$. (Fig. 1.)



353. To trace the curve whose equation is $y = x^{x}$. Let $x = 0$ $\therefore y = 1$; let $x = 1$ $\therefore y = 1$; and between $x = 0$ and $x = 1$, we have y less than 1; also x increases from 1 to ∞, y increases to infinity.

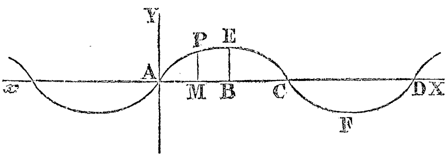
hence if A B = 1 (fig. 2,) and A C = 1, we have the branch B P Q corresponding to positive values of x.

Let x be negative $\therefore y = (-x)^{-x} = \frac{1}{(-x)^x}$; now if we take for x three consecutive values, as 2, $2\frac{1}{2}$, 3, it is evident that y will be positive, impossible, or negative; hence the curve must consist of a series of isolated points above and below the axis A x.

For further information on this subject see a very interesting memoir by M. Vincent, in the fifteenth volume of the "Annales des Math." M. Vincent calls such discontinuous branches by the name "Branches Ponctuées;" and he also shows, that in the common logarithmic curve there must be a similar branch below the axis of x, corresponding to fractional values of x with even denominators.

THE CURVE OF SINES.

354. The curve A P E C, of which the ordinates M P, B E are the sines of the corresponding abscissas A M, A B, is called the Curve of Sines.



Let A M = x, M P = y, then the equation is $y = \sin. x$,

$$y = r \sin. \frac{x}{r}.$$

	1	2	3	4	5
Values of x	0	$\frac{\pi r}{2}$	πr	$\frac{3 \pi r}{2}$	$2 \pi r$
Values of y	0	r	0	$- r$	0

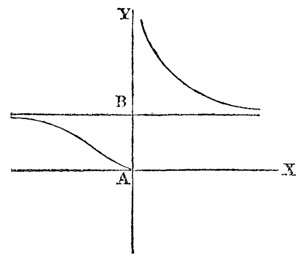
Take A B = $\frac{\pi r}{2}$, A C = πr, A D = $2 \pi r$; then from (1) the curve cuts the axis at A; from (2), if B E = r, the curve passes through E, and this is the highest point of its course, because between (1) and (2) y increases, and between (2) and (3) y decreases; the curve cuts the axis again in C; from C, y increases negatively until it equals $- r$, and then decreases to 0, so that we have a second branch C F D equal and similar to the first. Beyond D the values of y recur, and the curve continues the same course *ad infinitum;* also since $\sin. (-x) = - \sin. x$ there is a similar branch to the left of A.

This curve may be supposed to arise from the development of circular arcs into a straight line X x, ordinates being drawn corresponding to the sines of these circular arcs.

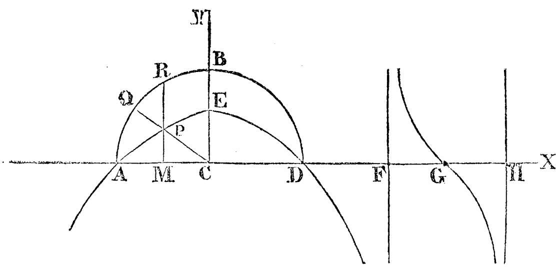
In a similar manner the curve of cosines, of versed sines, of tangents, &c., may readily be investigated.

If the ordinates of the curve of sines be increased or diminished in a given ratio, the resulting curve ($y = m$ sin. x) is the curve formed by the simple vibration of a musical chord: hence this curve is called the Harmonic Curve.

355. The accompanying figure belongs to the curve whose equation is $y = x$ tan. x. Such curves are useful in finding the roots of an equation as x tan. $x = a$; for, supposing the curve to be described, in A Y take A B $= a$, and from B draw a line parallel to A X; then the ordinates corresponding to the points of intersection of this straight line with the curve are the values of y, that is, of x tan. x.



THE QUADRATRIX.



356. Let C be the centre of a circle A Q B D; let the ordinate M R move uniformly from A to B C, and in the same time let the radius C Q, turning round C, move from C A to C B; then the intersection P of C Q and R M traces out a curve called the Quadratrix.

Let A be the origin, A M $= x$, M P $= y$, A C $= r$, angle A C Q $= \theta$,

Then A M : A C :: A Q : A B,

$$x : r :: r\theta : \frac{\pi r}{2} \therefore \theta = \frac{\pi x}{2r}.$$

But M P $=$ M C tan. θ,

$$\therefore y = (r - x) \text{ tan. } \frac{\pi x}{2r}, \text{ which is the equation to the curve.}$$

When $x = 0$, $y = 0$; $\therefore$ the curve passes through A; as x increases from 0 to r, y increases, because the tangent increases faster than the angle; when $x = r = $ A C, $y = \frac{0}{0}$, the real value of which found by the Differential Calculus is $\frac{2r}{\pi}$; hence if C E $= \frac{2r}{\pi}$, the curve passes through E; as x increases beyond r the tangent diminishes but is negative, and so is $r - x$; $\therefore y$ is positive and diminishes until it finally becomes 0, when $x = 2r = $ A D; when x is greater than $2r$ the tangent is positive, therefore y is negative and increases; when $x = 3r$, the tangent $= \infty$; $\therefore y = -\infty$; this gives an asymptote through F. As x increases beyond $3r$ the tangent decreases but is negative; hence y is positive; when $x = 4r$, $y = 0$, when $x = 5r$, $y = -\infty$, and between $x = 4r$ and $5r$, y is negative: therefore we have the branch between the asymptotes at F and H, and proceeding onwards we should find a series of branches like the last. The part of the curve to the left of A is the same as that to the right of D.

This curve was invented most probably by a Greek mathematician of the name of Hippias, a cotemporary of Socrates; his object was to trisect an angle, or generally to divide an angle into any number of equal parts, and this would be done if the curve could be accurately drawn; thus to trisect an angle A C Q, draw the quadratrix and the ordinate M P, trisect the line A M in the points N and O, draw the ordinates N S, O T to the quadratrix. Then from the equation $\theta = \frac{\pi x}{2r}$, we shall see that C S and C T trisect the angle A C Q.

This curve was afterwards employed by Dinostratus to find the area or quadrature of the circle, and hence its name: supposing the point E to be determined by mechanical description we have the value of π given by the equation C E $= \frac{2r}{\pi}$, and therefore the ratio of the circumference to the diameter of the circle would be known.

There is another quadratrix, that of Tschirnhausen, which is generated by drawing two lines through Q and M parallel respectively to A C and B C, and finding the locus of their intersection; its equation will be

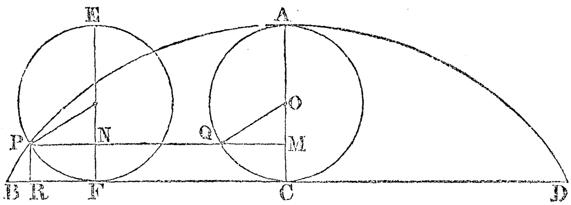
$$y = r \cos.\left(\frac{\pi}{2} - \theta\right) = r \sin.\theta = r \sin.\frac{\pi x}{2r}.$$

THE CYCLOID.

357. If a circle E P F be made to roll in a given plane upon a straight line B C D, the point in the circumference which was in contact with B at the commencement of the motion, will, in a revolution of the circle, describe a curve B P A D, which is called the cycloid.

This is the curve which a nail in the rim of a carriage-wheel describes in the air during the motion of the carriage on a level road; hence the generating circle E P F is called the wheel. The curve derives its name from two Greek words signifying "circle formed."

The line B D which the circle passes over in one revolution is called the base of the cycloid; if A Q C be the position of the generating circle in the middle of its course, A is called the vertex and A C the axis of the curve. The description of the curve shows that the line B D is equal to the circumference of the circle, and that B C is equal to half that circumference. Hence also if E P F be the position of the generating circle, and P the generating point, then every point in the circular arc P F having coincided with B F, we have the line B F = the arc P F, and F C = the arc E P or A Q ;



Draw P N Q M parallel to the base B D.

Let A be the origin of rectangular axes,

A C the axis of x, and O the centre of the circle A Q C.

Let A M $= x$, A O $= a$,

M P $= y$, angle A O Q $= \theta$;

then by the similarity of position of the two circles, we have

$$PN = QM, \text{ and } PQ = NM;$$

$\therefore$ M P = P Q + Q M = N M + Q M = F C + Q M = arc A Q + Q M

that is, $y = a\,\theta + a \sin.\,\theta = a\,(\theta + \sin.\,\theta)$ (1)

$x = a - a \cos.\,\theta = a \text{ vers.}\,\theta$ (2)

The equation between y and x is found by eliminating θ between (1) and (2)

$$\cos.\,\theta = \frac{a - x}{a} \quad \therefore \sin.\,\theta = \frac{\sqrt{2\,a\,x - x^2}}{a}$$

and $y = a\,\theta + a \sin.\,\theta$

$$= a \cos.^{-1}\left(\frac{a - x}{a}\right) + \sqrt{2\,a\,x - x^2}$$

But we can obtain an equation between x and y from (1) alone ; that is, from the equation, M P = arc A Q + Q M.

For arc A Q = a circular arc whose radius is a and versed sine x

$$= a \left\{ \text{a circular arc whose radius is unity and vers. sin. } \frac{x}{a} \right\}$$

$$= a \text{ vers.}^{-1}\frac{x}{a}$$

$$\therefore y = a \text{ vers.}^{-1}\frac{x}{a} + \sqrt{2\,a\,x - x^2}$$

If the origin is at B, B R = x and R P = y, the equations are

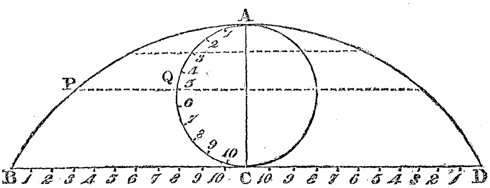
$$x = a\theta - a \sin. \theta$$
$$y = a - a \cos. \theta.$$

We shall not stop to discuss these equations, as the mechanical description of the curve sufficiently indicates its form.

The cycloid, if not first imagined by Galileo, was first examined by him; and it is remarkable for having occupied the attention of the most eminent mathematicians of the seventeenth century.

Of the many properties of this curve the most curious are that the whole area is three times that of the generating circle, that the arc A P is double of the chord of A Q, and that the tangent at P is parallel to the same chord. Also that if the figure be inverted, a body will fall from any point P on the curve to the lowest point A in the same time; and if a body falls from one point to another point, not in the same vertical line, its path of quickest descent is not the straight line joining the two points, but the arc of a cycloid, the concavity or hollow side being placed upwards.

358. Given the base of a cycloid to trace the curve.

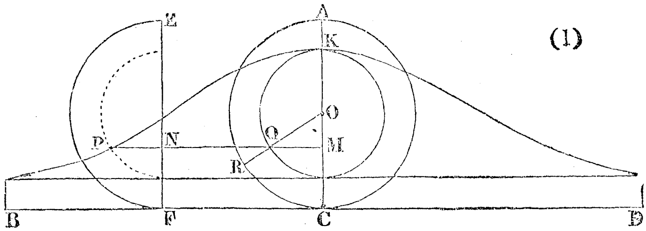


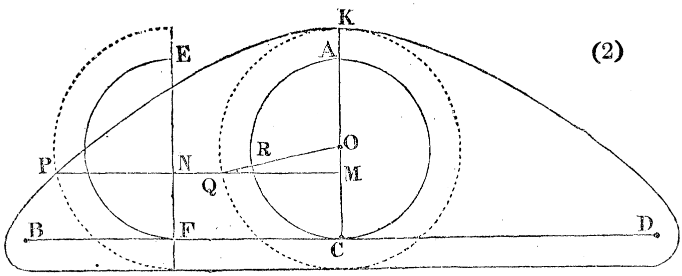
Let the base B D be divided into twenty-two equal parts, and let them be numbered from B and D towards the middle point C; from C draw the perpendicular line C A equal to 7 of these parts; and on A C describe a circle A Q C. Along the circumference mark off the same number of equal parts, either by measurement or by applying the line B C to the circle C A. In the figure the point Q is supposed to coincide with the end of the fifth division from the top.

Then the arc C Q being equal to the length C 5 measured on the base, if P Q be drawn parallel to the base, and equal to the remainder of the base, that is, to B 5 or A Q, it is evident that P is a point in the cycloid, and thus any number of points may be found.

The ratio of the circumference to the diameter of a circle is generally taken as in this case to be as 22 to 7.

359. Instead of the point P being on the circumference of the circle, it may be anywhere in the plane of that circle, either within or without the circumference. In the former case the curve is called the prolate cycloid or trochoid, (fig. 1,) in the latter case the curtate or shortened cycloid, (fig. 2.)





B D is the base on which the generating circle A R C rolls, O the centre of the generating circle, P the describing point when that circle is at F. Draw P N Q M parallel to the base.

Let A be the origin of rectangular axes,

$AM = x$, $MP = y$, $AO = a$, $KO = m\,a$, angle $AOR = \theta$, then

$$MP = MN + PN = MN + QM = FC + QM = \text{arc } AR + QM$$

$$\text{and } AM = AO + OM;$$

$$\therefore\ y = a\,\theta + m\,a \sin.\,\theta$$

$$\text{and } x = a \text{ vers. } \theta.$$

These are the equations to the prolate, curtate, or common cycloid, according as m is less than, greater than, or equal to, unity.

If the vertex K of the curve be the origin of co-ordinates in figs. (1) and (2,) we have $KO = a$, and $AO = m\,a$: also $MP = FC + QM = \text{arc } AR + QM$

$$\therefore\ y = m\,a\,\theta + a \sin.\,\theta$$

$$= m \text{ vers.}^{-1} \frac{x}{a} + \sqrt{2\,a\,x - x^2}$$

The curve whose equations are $y = a\,\theta$, and $x = a$ vers. θ is called the companion to the cycloid.

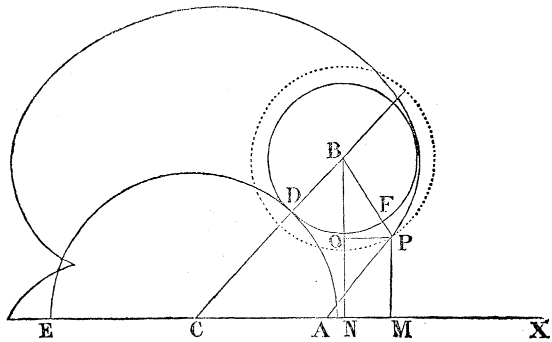
360. The class of cycloids may be much extended by supposing the base on which the circle rolls to be no longer a straight line, but itself a curve: thus let the base be a circle, and let another circle roll on the circumference of the former; then a point either within or without the

circumference of the rolling circle will describe a curve called the epitrochoid; but if the describing point is on the circumference, it is called the epicycloid.

If the revolving circle roll on the inner or concave side of the base, the curve described by a point within or without the revolving circle is called the hypotrochoid; and when the point is on the circumference it is called the hypocycloid.

To find the equation to the epitrochoid.

Let C be the centre of its base E D, and B the centre of the revolving circle D F in one of its positions: C A M the straight line passing through the centres of both circles at the commencement of the motion; that is, when the generating point P is nearest to C or at A.



Let C A be the axis of x,

$$CM = x, \quad MP = y,$$

$$CD = a, \quad DB = b,$$

$$BP = mb, \text{ angle } ACB = \theta$$

Draw B N parallel to M P, and P Q parallel to E M. Then, since every point in D F has coincided with the base A D, we have $DF = a\theta$, and angle $DBF = \frac{a\theta}{b}$; also angle FBQ = angle FBD − angle QBD

$$= \frac{a\theta}{b} - \left(\frac{\pi}{2} - \theta\right) = \frac{a+b}{b}\theta - \frac{\pi}{2}$$

Now $CM = CN + NM = CB \cos. BCN + PB \sin. PBQ$

$$= (a+b)\cos.\theta + mb\sin.\left(\frac{a+b}{b}\theta - \frac{\pi}{2}\right)$$

And $MP = BN - BQ = (a+b)\sin.\theta - mb\cos.\left(\frac{a+b}{b}\theta - \frac{\pi}{2}\right)$

$$\left.\begin{array}{l} \text{or } x = (a+b)\cos.\theta - mb\cos.\frac{a+b}{b} \\ \text{and } y = (a+b)\sin.\theta - mb\sin.\frac{a+b}{b}\theta \end{array}\right\} \quad (1)$$

The equations to the epicycloid are found by putting b for $m\,b$ in (1.)

$$\left.\begin{aligned} \therefore\ x &= (a + b)\ \cos.\ \theta - b \cos.\ \frac{a+b}{b}\,\theta \\ \text{and } y &= (a + b)\ \sin.\ \theta - b \sin.\ \frac{a+b}{b}\,\theta \end{aligned}\right\}\ (2)$$

The equations to the hypotrochoid may be obtained in the same manner as the system (1), or more simply by putting $-b$ for b in the equations (1.)

$$\left.\begin{aligned} \therefore\ x &= (a - b) \cos.\ \theta + m\,b \cos.\ \frac{a-b}{b}\,\theta \\ \text{and } y &= (a - b) \sin.\ \theta - m\,b \sin.\ \frac{a-b}{b}\,\theta \end{aligned}\right\}\ (3)$$

The equations to the hypocycloid are found by putting $-\,b$ for both b and $m\,b$ in system (1.)

$$\left.\begin{aligned} \therefore\ x &= (a - b) \cos.\ \theta + b \cos.\ \frac{a-b}{b}\,\theta \\ \text{and } y &= (a - b)\ \sin.\ \theta - b\ \sin.\ \frac{a-b}{b}\,\theta \end{aligned}\right\}\ (4)$$

We have comprehended all the systems in (1), but each of them might be obtained from their respective figures.

361. The elimination of the trigonometrical quantities is possible, and gives finite algebraic equations whenever a and b are in the proportion of two integral numbers. For then $\cos.\ \theta$, $\cos.\ \frac{a+b}{b}\,\theta$, $\sin.\ \theta$, &c., can be expressed by trigonometrical formulas, in terms of $\cos.\ \phi$ and $\sin.\ \phi$, where ϕ is a common submultiple of θ and $\frac{a+b}{b}\,\theta$; and then $\cos.\ \phi$ and $\sin.\ \phi$ may be expressed in terms of x and y. Also since the resulting equation in $x\,y$ is finite, the curve does not make an infinite series of convolutions, but the wheel or revolving circle, after a certain number of revolutions, is found, having the generating point exactly in the same position as at first, and thence describing the same curve line over again.

For example, let $a = b$, the equations to the epicycloid become

$$x = a\ (2 \cos.\ \theta - \cos.\ 2\,\theta)$$
$$y = a\ (2 \sin.\ \theta - \sin.\ 2\,\theta)$$
$$\therefore\ x = a\ (2 \cos.\ \theta - 2\ (\cos.\ \theta)^2 + 1)$$
$$\text{and } y = 2\,a \sin.\ \theta\ (1 - \cos.\ \theta).$$

From the first of these equations we find $\cos.\ \theta$, and then from the second we have $\sin.\ \theta$, adding the values of $(\cos.\ \theta)^2$ and $(\sin.\ \theta)^2$ together and reducing, we have

$$(y^2 + x^2 - 3\,a^2)^2 = 4\,a^4 \left(3 - \frac{2\,x}{a}\right)$$
$$\text{or } \{x^2 + y^2 - a^2\}^2 - 4\,a^2\,\{(x - a)^2 + y^2\} = 0.$$

This curve, from its heart-like shape, is called the cardioide.

Let A be the origin; that is, for x put $x + a$ in the last algebraical equation, and then by transformation into polar co-ordinates, the equation to the cardioide becomes

$$r = 2a(1 - \cos.\ \phi).$$

362. If $b = \frac{a}{2}$ the equations (4) to the hypocycloid become

$$x = a \cos.\ \theta$$
$$\text{and } y = 0;$$

and the hypocycloid has degenerated into the diameter of the circle A C E.

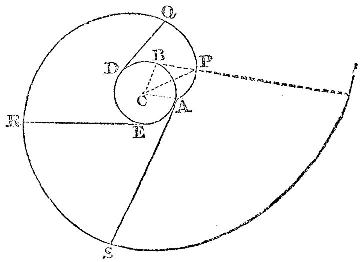
In the same case the equations to the hypotrochoid become

$$x = \frac{a}{2}(1 + m) \cos.\ \theta$$

$$y = \frac{a}{2}(1 - m) \sin.\ \theta;$$

which by the elimination of θ give the equation to an ellipse, whose axes are $a(1 + m)$ and $a(1 - m)$.

363. If a thread coinciding with a circular axis be unwound from the circle, the extremity of the thread will trace out a curve called the involute of the circle.



Thus suppose a thread fixed round the circle A B C D; then if it be unwound from A, the extremity in the hand will trace out the curve A P Q R S; the lines B P, D Q, C R, A S, which are particular positions of the thread, are also tangents to the circle, and each of them is equal to the length of the corresponding circular arc measured from A.

The curve makes an infinite number of revolutions, the successive branches being separated by a distance equal to the circumference of the circle.

To find the equation to the involute.

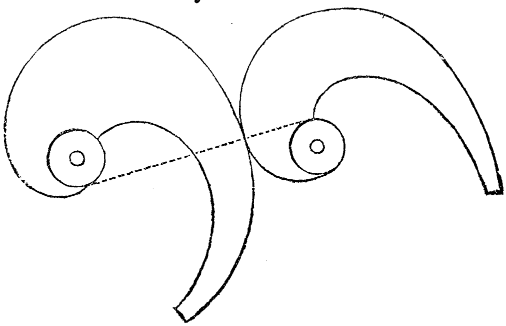
Let C A $= a$, C P $= r$, and angle A C P $= \theta$; then from the triangle B C P, we have B C $=$ P C cos. P C B, or angle P C B $= \cos.^{-1} \frac{a}{r}$;

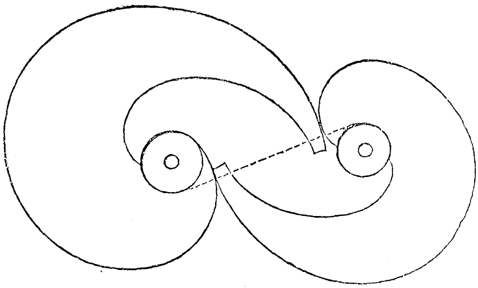
$$\therefore \text{B P} = \text{B A} = a\left(\cos.^{-1} \frac{a}{r} + \theta\right)$$

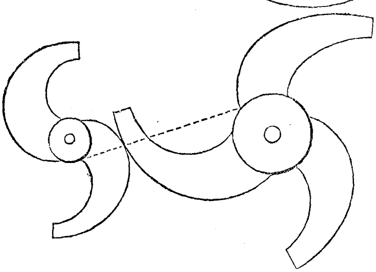
$$\text{or } \sqrt{(r^2 - a^2)} = a\left(\cos.^{-1} \frac{a}{r} + \theta\right)$$

$$\therefore \theta = \frac{\sqrt{\{r^2 - a^2\}}}{a} - \cos.^{-1} \frac{a}{r}.$$

The involute of the circle is usefully employed in toothed wheels ; for there is less waste of power in passing from one tooth to another when they are of this form than in any other case.







In the figures (2) and (3) we have examples of two equal wheels which have each two teeth ; and by turning one wheel the other wheel will be kept in motion by means of the continual contact of the teeth. The dotted line of contact is, by the property of the involute, a common tangent to the two wheels ; this dotted line is the constant line of contact, and the force is the same in every part of a revolution.

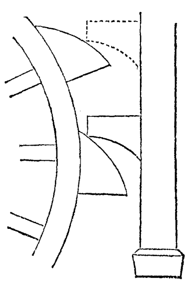


Fig. (3) is another example ; and by making the teeth smaller and more numerous we shall have toothed wheels always in contact, and therefore giving no jar or shake to the machinery.

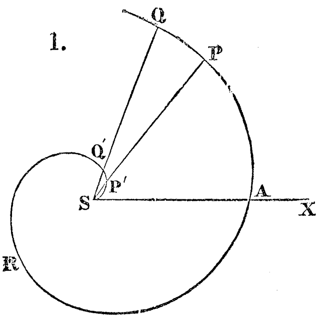
Again, in raising a piston or hammer, the involute of the circle is the best form for the teeth of the turning-wheel, as the force acts on the piston entirely in a vertical direction.

ON SPIRALS.

364. There is one class of transcendental curves which are called spirals, from their peculiar twisting form. They were invented by the ancient geometricians, and were much used in architectural ornaments. Of these curves, the most important as well as the most simple, is the spiral invented by the celebrated Archimedes.

This spiral is thus generated: Let a straight line S P of indefinite length move uniformly round a fixed point S, and from a fixed line S X, and let a point P move uniformly also along the line S P, starting from S, at the same time that the line S P commences its motion from S X, then the

1.



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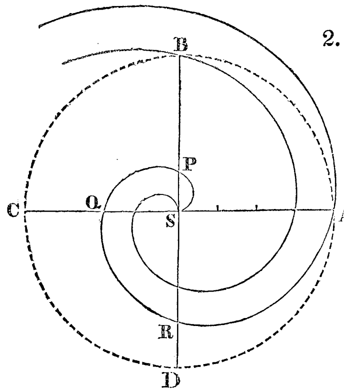


point will evidently trace out a curve line S P Q R A, commencing at S, and gradually extending further from S. When the line S P has made one revolution, P will have got to a certain point A, and S P still continuing to turn as before, we shall have the curve proceeding on regularly through a series of turnings, and extending further from S.

To examine the form and properties of this curve, we must express this method of generation by means of an equation between polar co-ordinates.

$$\text{Let } S P = r,\ S A = b,\ A S P = \theta;$$

then since the increase of r and θ is uniform, we have

$$S P : S A :: \text{angle } A S P : \text{four right angles} :: \theta : 2\pi$$

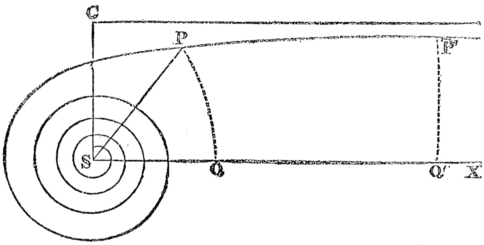
$$\therefore r = \frac{b\,\theta}{2\,\pi} = a\theta, \text{ if } a = \frac{b}{2\,\pi}.$$

From this equation it appears that when S P has made two revolutions or $\theta = 4\,\pi$, we have $r = 2\,b$, or the curve cuts the axis S X again at a distance 2 S A; and similarly after 3, 4, n revolutions it meets the axis S X at distances 3; 4, n times S A. Archimedes discovered that the area S P Q R A is equal to one-third of the area of the circle described with centre S and radius S A.

365. The spiral of Archimedes is sometimes used for the volutes of the capitals of columns, and in that case the following description by points is useful.

Let a circle A B C D, fig. (2), be described on the diameter C S A, and draw the diameter B D at right angles to C A; divide the radius S A into four equal parts, and in S B take S P $= \frac{1}{4}$ S A, in S C take S Q $= \frac{1}{2}$ S A, and in S D take S R $= \frac{3}{4}$ S A; then from the equation to the curve these points belong to the spiral; by subdividing the radius S A and the angles in each quadrant we may obtain other points as in the figure. In order to complete the raised part in the volute, another spiral commences from S B.

366. The spiral of Archimedes is one of a class of spirals comprehended in the general equation $r = a\,\theta^n$. Of this class we shall consider the cases where $n = -1$, and $n = -\frac{1}{2}$.



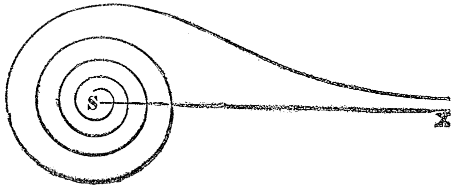
Let $n = -1 \quad \therefore r = a\theta^{-1}$

Let S be the pole, S X the axis from which the angle θ is measured, S P $= r$.

When $\theta = 0$, $r = \infty$; as θ increases, r decreases very rapidly at first and more uniformly afterwards; as θ may go on increasing ad infinitum r also may go on diminishing ad infinitum without ever actually becoming nothing: hence we have an infinite series of convolutions round S: Describe a circular arc P Q with centre S and radius S P, then P Q $= r\theta = a$; and since this value of a is the same for all positions of P, we must have P Q $=$ P′ Q′ $=$ the straight line S C at an infinite distance, and therefore the curve must approach to an asymptote drawn through C parallel to S X.

This curve is called the reciprocal spiral from the form of its equation, since the variables are inversely as each other, or the hyperbolic spiral, from the similarity of the equation to that of the hyperbola referred to its asymptotes ($x\,y = k^2$).

367. Let $n = -\frac{1}{2}$; $\therefore r = a\,\theta^{-\frac{1}{2}}$ or $r^2\theta = a^2$. This curve, called the lituus or trumpet, is described as in the figure; proceeding from the asymptote S X, it makes an infinite series of convolutions round S.



368. If in the equation $r\theta = a$, we always deduct the constant quantity, b, we have the equation $(r - b)\theta = a$; this curve commences its course like the reciprocal spiral; but as θ increases we have $r - b$ approximating to nothing, or r approximating to b; hence the spiral, after an infinite number of convolutions, approaches to an asymptotic circle, whose centre is S, and radius b.

369. Trace the spiral whose equation is $\theta\sqrt{ar - r^2} = b$; this curve has an infinite number of small revolutions round the pole, and gradually extends outwards to meet an asymptotic circle whose radius is a.

370. The spiral whose equation is $(r - a)^2 = b^2\theta$ commences its course from a point in the circumference of the circle whose radius is a, and extends outwards round S in an infinite series of convolutions. This curve is formed by twisting the axis of the common parabola round the circumference of a circle, the curve line of the parabola forming the spiral.

371. The curve whose equation is $r = a^\theta$ is called the logarithmic spiral, for the logarithm of the radius vector is proportional to the angle θ. Examining all the values of θ from 0 to $\pm\infty$ we find that there are an infinite series of convolutions round the pole S. This curve is also called the equiangular spiral, for it is found by the principles of the higher analysis that this curve cuts the radius vector in a constant angle.

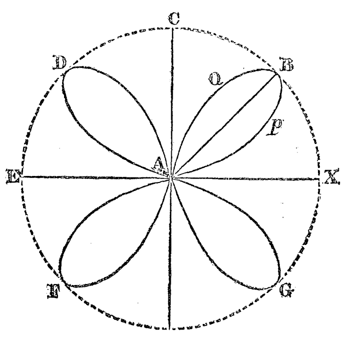
Descartes, who first imagined this curve, found also that the whole length of the curve from any point P to the pole was proportional to the radius vector at P.

372. It will often happen that the algebraical equation of a curve is much more complicated than the polar equation; the conchoid art. 312 is an example. In these cases it is advisable to transform the equation from algebraical to polar co-ordinates, and then trace the curve from the polar equation.

For example, if the equation be $(x^2 + y^2)^{\frac{3}{2}} = 2axy$, there would be much difficulty in ascertaining the form of the curve from this equation; but let $x = r\cos.\theta$ and $y = r\sin.\theta$ (61)

$$\therefore r^3 = 2ar^2\cos.\theta\sin.\theta,$$

$$\text{or } r = a\sin.2\theta.$$



Let A be the origin of polar co-ordinates; A X the axis whence θ is measured; with centre A and radius a describe a circle B C D. Then for $\theta = 0$ we have $r = 0$, as θ increases from 0 to 45°, r increases from

0 to a; hence the branch A P B. Again, as θ increases from 45° to 90°, sin. 2θ diminishes from 1 to 0; $\therefore$ r diminishes, and we trace the branch B Q A. As θ increases from 90° to 180°, sin. 2θ increases and decreases as before; hence the similar oval in the second quadrant. By following θ from 180° to 360°, we shall have the ovals in the third and fourth quadrant: and since the sine of an arc advances similarly in each quadrant of the circle, we have the four ovals similar and equal.

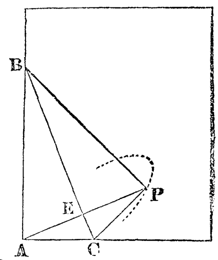
In this case we have paid no regard to the algebraical sign of r; we have considered θ to vary from 0 to 360°, which method we prefer to that of giving θ all values from 0 to 180°, and then making the sign of r to vary.

If the equation had been $(x^2 + y^2)^2 = 2a^2 x y$, we should have found two equal and similar ovals in the first and third quadrant.

The locus of the equation $r = a$ (cos. θ — sin. θ) is the same kind of figure differently situated with respect to the lines A X and A C.

The equation to the lemniscata $r^2 = a^2$ cos. 2θ art. (314), may be similarly traced.

373. In many indeterminate problems we shall find that polar co-ordinates may be very usefully employed. For example,



Let the corner of the page of a book be turned over into the position B C P, and in such a manner that the triangle B C P be constant, to find the locus of P.

Let A P $= r$, angle P A C $= \theta$, and let the area A B C $= a^2$; then since the triangles A B E, P B E are equal, we have A E $= \frac{r}{2}$, and the angle A E B a right angle $\therefore$ A E $=$ A C cos. θ, and A E $=$ A B cos. $\left(\frac{\pi}{2} - \theta\right) =$ A B sin. θ $\therefore \frac{r^2}{4} = \frac{a^2}{2}$ sin. θ cos. θ, or $r^2 = a^2$ sin. 2θ. Hence the locus is an oval A P B Q as in the last figure.

If a point be taken in the radius vector S P of a parabola so that its distance from the focus is equal to the perpendicular from the focus on the tangent, the locus of the point is the curve whose equation is $r = a$ sec. $\frac{\theta}{2}$.

PART II.

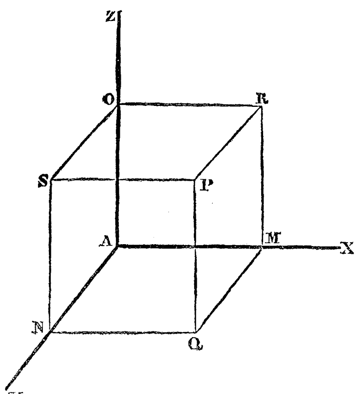
APPLICATION OF ALGEBRA TO SOLID GEOMETRY.

CHAPTER I.

INTRODUCTION.

374. In the preceding part of this Treatise lines and points have always been considered as situated in one plane, and have been referred to two lines called axes situated in that plane. Now we may readily imagine a curve line, the parts of which are not situated in one plane; also, if we consider a surface, as that of a sphere, for example, we observe immediately that all the points in such a surface cannot be in the same plane; hence the method of considering figures which has been hitherto adopted cannot be applied to such cases, and therefore we must have recourse to some more general method for investigating the properties of figures.

375. We begin by showing how the position of a point in space may be determined.



Let three planes Z A X, Z A Y, and X A Y, be drawn perpendicular to each other, and let the three straight lines A X, A Y, A Z be the intersections of these planes, and A the common point of concourse.

From any point P in space draw the lines P Q, P R, and P S respectively perpendicular to the planes X A Y, Z A X, and Z A Y; then the position of the point P is completely determined when these three perpendicular lines are known.

Complete the rectangular parallelopiped A P, then P Q, P R, and P S are respectively equal to A O, A N, and A M.

These three lines A M, A N, and A O, or more commonly their equals A M, M Q, and Q P, are called the co-ordinates of P, and are denoted by the letters x, y, and z respectively.

The point A is called the origin.

The line A X is called the axis of x, the line A Y is called the axis of y, and the line A Z is called the axis of z.

The plane X A Y is called the plane of $x y$, the plane Z A X is called the plane of $z x$, and the plane Z A Y is called the plane of $z y$ *.

From P we have drawn three perpendicular lines, P Q, P R, and P S, on the three co-ordinate planes. The three points, Q, R, and S are called the projections of the point P on the planes of $x y$, $x z$, and $z y$ respectively.

The method of projections is so useful in the investigation and description of surfaces, that we proceed to give a few of the principal theorems on the subject so far as may be required in this work.

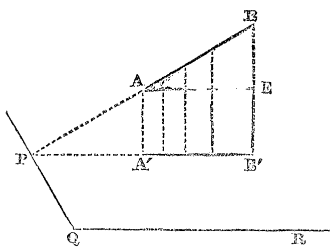
PROJECTIONS.

376. If several points be situated in a straight line, their projections on any one of the co-ordinate planes are also in a straight line.

For they are all comprised in the plane passing through the given straight line, and drawn perpendicular to the co-ordinate plane; and as the intersection of any two planes is a straight line, the projections of the points must be all in one straight line.

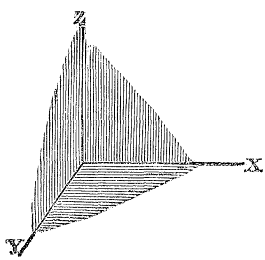
This plane, which contains all the perpendiculars drawn from different points of the straight line, is called the projecting plane; and its intersection with the co-ordinate plane is called the projection of the straight line.

377. To find the length of the projection of a straight line upon a plane.



Let A B be the line to be projected on the plane P Q R; produce A B to meet this plane in P; draw A A′ and B B′ perpendicular to the plane, and meeting it in A′ and B′. Join A′ B′; then A′ B′ is the projection of A B.

* This system of co-ordinate planes may be represented by the sides and floor of a room, the corner being the origin of the axes, the plane X Y is then represented by the floor of the room, and the two remaining planes by the two adjacent sides of the room.

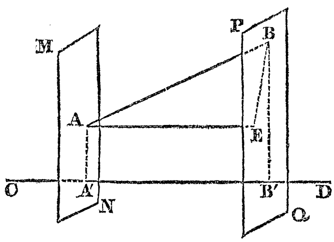


Since A B and A′ B′ are in the same plane, they will meet in P. Let the angle B P B′ or the angle of the inclination of A B to the plane $= \theta$, and in the projecting plane A B′ draw A E parallel to A′ B′, then

$$A'B' = AE = AB \cos. BAE = AB \cos. \theta$$

The same proof will apply to the projection of a straight line upon another straight line, both being in the same plane.

378. To find the length of the projection of a straight line upon another straight line not in the same plane.



Let A B be the line to be projected; C D the line upon which it is to be projected. From A and B draw lines A A′ and B B′ perpendicular to C D, then A′ B′ is the projection of A B.

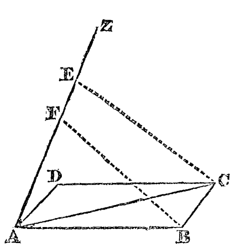
Through A and B draw planes M N and P Q perpendicular to C D. These planes contain the perpendicular lines A A′ and B B′.

From A draw A E perpendicular to the plane P Q, and therefore equal and parallel to A′ B′; join B E; then the triangle A B E having a right angle at E, we have $A'B' = AE = AB \cos. BAE$, and angle B A E is equal to the angle θ of inclination between A B and C D; hence

$$A'B' = AB \cos. \theta.$$

Also any line equal and parallel to A B has an equal projection A′ B′ on C D, and the projection of A B on any line parallel to C D is of the same length as A′ B′.

379. The projection of the diagonal of a parallelogram on any straight line is equal to the sum of the projections of the two sides upon the same straight line.



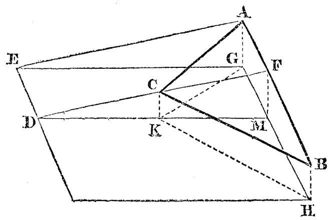
Let A B C D be a parallelogram, A Z any straight line through A inclined to the plane of the parallelogram. From C and B draw perpendiculars C E and B F upon A Z, then A E is the projection of A C upon A Z or $AE = AC \cos. CAZ$; and A F is the projection of A B upon

A Z or A F = A B cos. B A Z. Also F E is the projection of B C or A D upon A Z or F E = B C cos. D A Z

$$\text{and } AE = AF + FE;$$

hence the projection of A C = the sum of projections of A B and B C.

380. To find the projection of the area of any plane figure on a given plane E D G H.



Let A B C be a triangle inclined to the given plane E D G H at an angle θ; draw A E, C D, perpendicular to the intersection E D of these planes; then the triangle A B C and its projection G K H have equal bases A B, G H, but unequal altitudes C F, K M;

$$\therefore \text{ area } ABC : GKH :: CF : KM :: DF : DM :: 1 : \cos. \theta$$

$$\text{or area } GKH = ABC \cos. \theta;$$

and this being true for any triangle, is true for any polygon, and therefore ultimately for any plane area.

CHAPTER II.

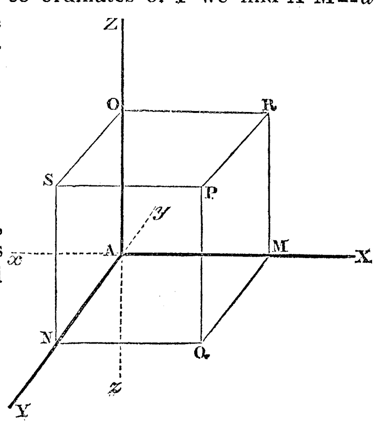
THE POINT AND STRAIGHT LINE.

381. We have already explained how the position of a point in space is determined by drawing perpendicular lines from it upon three fixed planes called the co-ordinate planes. If, then, on measuring the lengths of these three perpendicular lines or co-ordinates of P we find A M $=a$, A N $=b$, and A O $=c$, we have the position of a point P completely determined by the three equations $x=a$, $y=b$ and $z=c$; and as these are sufficient for that object, they are called the equations to the point P.

This point may also be defined as in Art. (25) by the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=0,$$

since the only values that render this expression real are $x=a$, $y=b$, and $z=c$.



382. The algebraical signs of the co-ordinates x, y, and z, are determined as in Plane Geometry, by the directions of the co-ordinate lines: thus A O is positive or negative according as it is drawn from A along A Z or A z, that is, according as it is above or below the plane of $x\,y$: and so on for the other lines: hence we have the following values of co-ordinates for a point in each of the eight compartments into which space is divided by the co-ordinate planes.

$+x+y+z$ a point P situated in the angle	X A Y Z
$+x-y+z$ . . .	X A y Z
$-x-y+z$ . . .	x A y Z
$-x+y+z$ . . .	x A Y Z
$+x+y-z$ . . .	X A Y z
$+x-y-z$ . . .	X A y z
$-x-y-z$ . . .	x A y z
$-x+y-z$ . . .	x A Y z

383. A point also may be situated in one of the co-ordinate planes, in which case the co-ordinate perpendicular to that plane must $= 0$; thus, if the point be in the plane of $x\,y$, its distance z from this plane must $= 0$: hence the equations to the point in the plane of $x\,y$ are

$$x = a,\ y = b,\ z = 0$$
$$\text{or } (x-a)^2 + (y-b)^2 + z^2 = 0.$$

If the point be in the plane of $x\ z$, the equations are

$$x = a,\ y = 0,\ z = c$$

And if the point be in the plane of $y\ z$

$$x = 0,\ y = b,\ z = c.$$

Also, if the point be on the axis of x, its distance from the planes $x\,y$ and $y\,z = 0$, therefore the equations to such a point are

$$x = a,\ y = 0,\ z = 0;$$

and so on for points situated on the other axes.

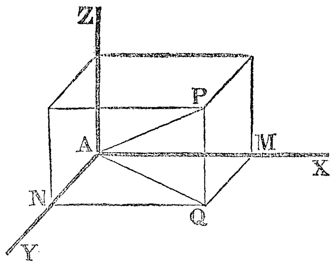
384. The points Q, R, and S, in the last figure, are the projections of the point P on the co-ordinate planes; on referring each of these points to the axes in its own plane, we have

The equations to Q on $x\,y$ are $x = a,\ y = b$
R on $x\,z$ are $x = a,\ z = c$
S on $y\,z$ are $y = b,\ z = c$

Hence we see that the projections of the point P on two of the co-ordinate planes being known, the projection on the third plane is necessarily given: thus, if S and R are given, draw S N and R M parallel to A Z, also N Q and M Q respectively parallel to A X and A Y, and the position of Q is known.

385. To find the distance A P of a point from the origin of co-ordinates A.

Let A X, A Y, and A Z be the rectangular axes; A M $= x$, M Q $= y$, and P Q $= z$, the co-ordinates of P.



The square on A P = the square on A Q + the square on P Q

= the squares on A M, M Q + the square on P Q

or $d^2 = x^2 + y^2 + z^2$.

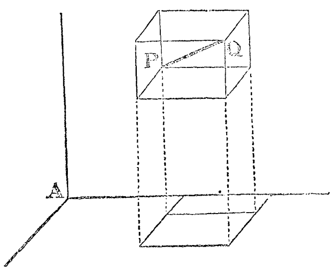
386. Let α, β, γ, be the angles which A P makes with the axis of x, y, and z, respectively;

$$\text{then } x = \text{AM} = \text{AP} \cos \text{PAM} = d \cos \alpha$$
$$y = \text{MQ} = \text{AN} = \text{AP} \cos. \text{PAN} = d \cos. \beta$$
$$z = \text{PQ} = \text{AP} \sin. \text{PAQ} = d \cos. \gamma$$
$$\therefore d^2 = x^2 + y^2 + z^2 = d^2 (\cos. \alpha)^2 + d^2 (\cos. \beta)^2 + d^2 (\cos. \gamma)^2$$
$$\therefore (\cos. \alpha)^2 + (\cos. \beta)^2 + (\cos. \gamma)^2 = 1.$$

387. Again $d^2 = x^2 + y^2 + z^2 = x\, d \cos. \alpha + y\, d \cos. \beta + z\, d \cos. \gamma$

$$\therefore d = x \cos. \alpha + y \cos. \beta + z \cos. \gamma.$$

388. To find the distance between two points, let the co-ordinates of the points P and Q be respectively $x\ y\ z$ and $x_1\ y_1\ z_1$; then the distance between these points is the diagonal of a parallelopiped, the three contiguous sides of which are the differences of the parallel co-ordinates; hence, by the last article we have



$$d^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

If d_1 and d_2 be the distances of the points $x_1\ y_1\ z_1$ and $x_2\ y_2\ z_2$ respectively from the origin, the above expression may be put in the form

$$d^2 = d_1^2 + d_2^2 - 2\,(x_1\, x_2 + y_1\, y_2 + z_1\, z_2).$$

THE STRAIGHT LINE.

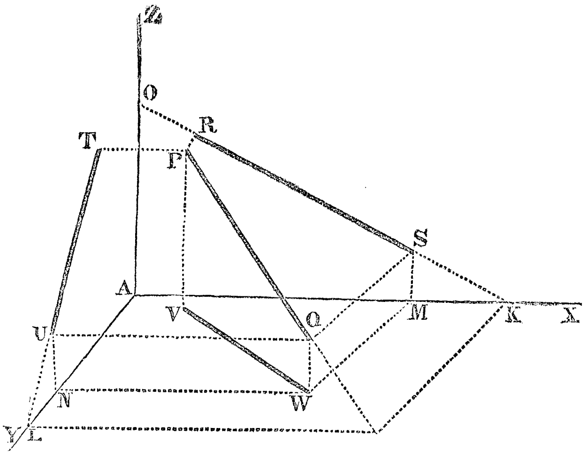
389. A straight line may be considered as the intersection of two planes, and therefore its position will be known if the situation of these planes is known; hence it may be determined by the projecting planes, and the situation of these last is fixed by their intersections with the co-ordinate planes, that is, by the projections of the line upon the co-ordinate planes; hence, the position of a straight line is geometrically fixed by knowing its projections; and it is also algebraically determined by the equations to those projections taken conjointly. Taking the axis of z as the axis of abscissas the equation to the projection on the plane $x\ z$ is of the form

$x = \alpha z + a$ (31), and the equation to the projection on the plane of $y z$ is $y = \beta z + b$.

As these two equations fix the position of the straight line in space, they are, taken together, called the equations to a straight line.

390. To illustrate this subject we shall let P Q be a portion of the straight line, R S its projection on $x z$, T U its projection on $y z$, V W its projection on $x y$;

And let $x = \alpha z + a$, be the equation to R S, and $y = \beta z + b$, be the equation to T U:



then any point Q in the projecting plane P Q R S has the same values of z and x that its projection S has, that is, the co-ordinates A M and M S are the same as N W and W Q; hence there is the same relation between them in each case; and therefore, the equation $x = \alpha z + a$ expresses not only the relation between the x and z of all the points in R S, but also of all the points in the plane P Q R S.

Similarly the equation $y = \beta z + b$ not only relates to T U, but also to all the points in the plane T U Q P.

Therefore, the system of the two equations exists for all the points in the straight line P Q, the intersection of the two projecting planes, and for this line only; hence, the equations to the straight line P Q are

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\}.$$

The elimination of z between these two equations gives

$$\frac{1}{\alpha}(x - a) = \frac{1}{\beta}(y - b)$$

$$y - b = \frac{\beta}{\alpha}(x - a)$$

and this is the relation between the co-ordinates A M and M W of the projection W of any point Q in the line P Q; and therefore, this last equation is that to the projection V W on the plane $x y$.

391. In the equations $x = \alpha z + a$ and $y = \beta z + b$, a is the distance

of the origin from the intersection of R S with A X, or $a =$ A K; similarly $b =$ A L.

Let $x = 0 \therefore z = -\frac{a}{\alpha} =$ A O $\therefore$ A K $= -\alpha$ A O, but A K $=$ A O tan. A O K $= -$ A O tan. Z O R $\therefore \alpha$ is tangent of the angle which R S makes with A Z, and similarly β is tangent of the angle which U T makes with A Z.

392. The straight line will assume various positions according to the algebraical signs of a, b, α and β: however, it would be of very little use to go through all the cases arising from these changes of sign, especially as they offer nothing of consequence, and no one case presents any difficulty. We shall only consider the cases where the absolute value of a, b, α and β is changed.

Let $a = 0$ and $b = 0$, then $x = \alpha z$ and $y = \beta z$, and the two projections pass through the origin, and therefore the line itself passes through the origin; the equation to the third projection is $y = \frac{\beta}{\alpha} x$.

Let $a = 0$ then $x = \alpha z$ and $y = \beta z + b$, the projection on $x z$ passing through the origin, the line itself must pass through the axis A Y perpendicular to $x z$: similarly, if $b = 0$, the equations $x = \alpha z + a$, $y = \beta z$ belong to a line passing through the axis of x, and if the equations are $y = \alpha x$, $y = \beta z + b$, the straight line passes through the axis of z: this last case may be represented by supposing (in the last figure) W V to pass through A, then the equation to V W is of the form $y = \alpha x$, and the equation to O T U is $y = \beta z + b$; now, if two planes be drawn, one through T U perpendicular to $y z$, and the other through V W perpendicular to $x y$, both planes pass through the point O, and therefore the line itself must pass through O.

393. Let $\beta = 0 \therefore x = \alpha z + a$, $y = b$, the line is in a plane parallel to $x z$ and distant from it by the quantity b. If the last figure be adapted to this case we should have U T perpendicular to A Y, and therefore P Q equal and parallel to R S situated in the plane W N U Q perpendicular to $x y$.

Let $\alpha = 0 \therefore x = a$, $y = \beta z + b$, the line is in a plane parallel to to $y z$.

Similarly $z = c$, $y = \alpha' x + a'$ belong to a line in a plane parallel to $x y$.

394. A straight line may also be situated in one of the co-ordinate planes as in the plane of $y z$; for example, the equations to such a line are $y = \beta z + b$, $x = 0$. If the line be in the plane of $x z$ the equations are $x = \alpha z + a$, $y = 0$; and if the line be in the plane of $x y$ the equations become $y = \alpha' x + a'$ $z = 0$.

395. If the straight line be perpendicular to one of the co-ordinate planes, as $x y$ for example; α and β must each equal 0, and therefore the equations to this line are

$$x = a, y = b, z = \frac{0}{0}.$$

Similarly the equations to a line perpendicular to $x z$ are

$$x = a, y = \frac{0}{0}, z = c$$

and the equations to a line perpendicular to $y\,z$ are

$$x = \frac{0}{0},\ y = b,\ z = c.$$

396. To find the point where a straight line meets the co-ordinate planes:

Let $x = \alpha z + a$ and $y = \beta z + b$ be the equations to the line; when it meets the plane of $x\,y$ we have $z = 0$ $\therefore$ $x = a$, $y = b$ are the equations to the required point.

Similarly $z = -\frac{b}{\beta}$, $x = -\frac{\alpha}{\beta}\,b + a$ are the equations to the point where the line meets the plane of $x\,z$, and $z = -\frac{a}{\alpha}$, $y = -\frac{\beta}{\alpha}\,a + b$ are the equations to the point where the line pierces the plane of $z\,y$.

397. There are four constant quantities in the general equations to a straight line, and if they are all given, the position of the line is completely determined; for we have only to give to one of the variables as z a value z', and we have

$$x = \alpha z + a = \alpha z' + a = x' \text{ and } y = \beta z + b = \beta z' + b = y';$$

or, x' and y' are also necessarily determined; hence, taking A M $= x'$, (see the last figure,) and drawing M W ($= y'$) parallel to A Y, and lastly, drawing from W a perpendicular W Q $= z'$, the point Q thus determined belongs to the line; and similarly, any number of points in the line are determined, or the position of the line is completely ascertained. Again, the straight line may be subject to certain conditions, as passing through a given point, or being parallel to a given line; or, in other words, conditions may be given which will enable us to determine the quantities α, β, a and b, supposing them first to be unknown; in this manner arises a series of Problems on straight lines similar to those already worked for straight lines situated in one plane (40, 50).

PROBLEMS ON STRAIGHT LINES.

398. To find the equations to a straight line passing through a given point:

Let the co-ordinates of the given point be x_1, y_1 and z_1, and let the equations to the straight line be $x = \alpha z + a$, $y = \beta z + b$.

Now since this line passes through the given point, the projections of the line must also pass through the projections of the point; hence the projection $x = \alpha z + a$ passing through x_1 and z_1, we have $x_1 = \alpha z_1 + a$,

$$\therefore x - x_1 = \alpha (z - z_1)$$

$$\text{and similarly } y - y_1 = \beta (z - z_1)$$

hence these are the equations required: α and β being indeterminate, there may be an infinite number of straight lines passing through the given point.

If the given point be in the plane of $x\,y$, we have $z_1 = 0$,

$$\therefore \left.\begin{aligned} x - x_1 &= \alpha z \\ y - y_1 &= \beta z \end{aligned}\right\}$$

If the given point be on the axis of x, we have $z_1 = 0$ and $y_1 = 0$,

$$\left.\begin{aligned} x - x_1 &= \alpha z \\ y &= \beta z \end{aligned}\right\}$$

And the equation would assume various other forms according to the position of the given point.

399. To find the equations to a straight line passing through two given points, $x_1\, y_1\, z_1$ and $x_2\, y_2\, z_2$.

Since the line passes through the point $x_1\, y_1\, z_1$ its equations are

$$x - x_1 = \alpha (z - z_1)$$
$$y - y_1 = \beta (z - z_1).$$

And since the line also passes through $x_2\, y_2 .\, z_2$ the last equations become

$$x_2 - x_1 = \alpha (z_2 - z_1)$$
$$y_2 - y_1 = \beta (z_2 - z_1)$$

$$\therefore \alpha = \frac{x_2 - x_1}{z_2 - z_1} \text{ and } \beta = \frac{y_2 - y_1}{z_2 - z_1};$$

hence the equations to the required line are

$$x - x_1 = \frac{x_2 - x_1}{z_2 - z_1} (z - z_1)$$

$$y - y_1 = \frac{y_2 - y_1}{z_2 - z_1} (z - z_1).$$

These equations will assume many various forms dependent on the position of the given point, for example: If the first point be in the plane of $y\, z$, and the second in the axis of x, we have $x_1 = 0$; $y_2 = 0$, $z_2 = 0$

$$\therefore x = \frac{x_2}{-z_1} (z - z_1)$$

$$y - y_1 = \frac{y_1}{z_1} (z - z_1).$$

If the second point be the origin, we have $x_2\, y_2\, z_2$ each $= 0$,

$$\therefore x - x_1 = \frac{x_1}{z_1} (z - z_1) = \frac{x_1}{z_1} z - x_1$$

$$y - y_1 = \frac{y_1}{z_1} (z - z_1) = \frac{y_1}{z_1} z - y_1;$$

hence the equations to a point passing through the origin are

$$x = \frac{x_1}{z_1} z, \text{ and } y = \frac{y_1}{z_1} z.$$

And these equations may be also obtained by considering that the projections pass through the origin, and therefore their equations are of the form $x = \alpha z$, $y = \beta z$, and the first passing through $x_1\, z_1$ we have

$\alpha = \dfrac{x_1}{z_1}$, and similarly $\beta = \dfrac{y_1}{z_1}$.

400. To find the equation to a straight line parallel to a given straight line.

Since the lines are parallel their projecting planes on any one of the co-ordinate planes are also parallel, and therefore the projections themselves parallel; hence, if the equations to the given line are

$$x = \alpha z + a, \; y = \beta z + b,$$

the equations to the required line are

$$x = \alpha z + a', \; y = \beta z + b'.$$

∴ If the straight line pass also through a given point $x_1\, y_1\, z_1$ its equations are

$$x - x_1 = \alpha\,(z - z_1), \; y - y_1 = \beta\,(z - z_1).$$

401. To find the intersection of two given straight lines.

Two straight lines situated in one plane must meet in general, but this is not necessarily the case if the lines be situated anywhere in space; hence there must be a particular relation among the constant quantities in the equation in order that the lines may meet: to find this relation, let the equations to the lines be

$$\left.\begin{aligned} x &= \alpha z + a \\ y &= \beta z + b \end{aligned}\right\} \qquad \left.\begin{aligned} x &= \alpha' z + a' \\ y &= \beta' z + b' \end{aligned}\right\}$$

For the point of intersection the projected values of x, y and z must be the same in all the equations; hence

$$\alpha z + a = \alpha' z + a' \text{ and } z = \frac{a' - a}{\alpha - \alpha'}$$

$$\text{and } \beta z + b = \beta' z + b' \text{ and } z = \frac{b' - b}{\beta - \beta'},$$

$$\therefore \frac{a' - a}{\alpha - \alpha'} = \frac{b' - b}{\beta - \beta'};$$

$$\text{or, } (a' - a)\;(\beta' - \beta) = (b' - b)\;(\alpha' - \alpha).$$

And this is the relation which must exist amongst the constants in order that the two lines may meet.

Having thus determined the necessary relation among the constants, the co-ordinates of intersection are given by the equations

$$z = \frac{a' - a}{\alpha - \alpha'} \text{ or } = \frac{b' - b}{\beta - \beta'}$$

$$y = \beta z + b = \beta\,\frac{b' - b}{\beta - \beta'} + b = \frac{\beta b' - \beta' b}{\beta - \beta'}$$

$$x = \alpha z + a = \alpha\,\frac{a' - a}{\alpha - \alpha'} + a = \frac{\alpha a' - \alpha' a}{\alpha - \alpha'}.$$

402. To find the angles which a straight line (l) makes with the co-ordinate axes; and thence with the co-ordinate planes:

Let the equations to the given line be

$$x = \alpha z + a$$
$$y = \beta z + b;$$

the equations to the parallel line through the origin are

$$x = \alpha z, \; y = \beta z;$$

also let r be the distance of any point (x, y, z) in this last line from the origin:

$$\therefore r^2 = x^2 + y^2 + z^2$$
$$= (\alpha^2 + \beta^2 + 1) z^2$$
$$\text{or, } z^2 = \frac{r^2}{1 + \alpha^2 + \beta^2}$$

But $l\,x$, $l\,y$, and $l\,z$ being the angles which either line makes with the axes of x, y and z respectively, we have from the second line

$$\cos. l\,x = \frac{x}{r} = \frac{\alpha z}{r} = \frac{\alpha}{\sqrt{1 + \alpha^2 + \beta^2}}$$
$$\cos. l\,y = \frac{y}{r} = \frac{\beta z}{r} = \frac{\beta}{\sqrt{1 + \alpha^2 + \beta^2}}$$
$$\cos. l\,z = \frac{z}{r} = \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}}$$

Also $(\cos. l\,x)^2 + (\cos. l\,y)^2 + (\cos. l\,z)^2 = 1$;

and this is the equation connecting the three angles which any straight line makes with the rectangular axes.

Since the system is rectangular, the angle which a line makes with any axis is the complement of the angle which it makes with the plane perpendicular to that axis : hence the angles which a line makes with the coordinate planes are given.

403. To find the cosine, sine, and tangent of the angle between two given straight lines.

Let the equations to the two straight lines be

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha' z + a' \\ y = \beta' z + b' \end{array}\right\}.$$

These two lines may meet, or they may not meet; but in either case their mutual inclination is the same as that of two straight lines parallel to them and passing through the origin; hence the problem is reduced to find the angle between the lines represented by the equations

$$\left.\begin{array}{l} x = \alpha z \\ y = \beta z \end{array}\right\} (1) \qquad \left.\begin{array}{l} x = \alpha' z \\ y = \beta' z \end{array}\right\} (2)$$

Let r = the distance of a point $x\ y\ z$ in (1) from the origin,

r_1 = ... $x_1 y_1 z_1$ in (2) . . . .

d = the distance between these points,

θ = the angle between the given lines,

then $d^2 = r^2 + r_1^2 - 2\,r\,r_1 \cos. \theta$

$$= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \quad (388)$$
$$= x^2 + y^2 + z^2 + x_1^2 + y_1^2 + z_1^2 - 2\,(x x_1 + y y_1 + z z_1)$$
$$= r^2 + r_1^2 - 2\,(x x_1 + y y_1 + z z_1)$$
$$\therefore r\,r_1 \cos. \theta = x x_1 + y y_1 + z z_1$$

Now $x x_1 + y y_1 + z z_1 = \alpha z\,\alpha' z_1 + \beta z\,\beta' z_1 + z z_1 = (\alpha\alpha' + \beta\beta' + 1)\,z z_1$

And $r\,r_1 = \sqrt{(x^2 + y^2 + z^2)}\,\sqrt{(x_1^2 + y_1^2 + z_1^2)}$

$$= z z' \sqrt{(\alpha^2 + \beta^2 + 1)}\sqrt{(\alpha'^2 + \beta'^2 + 1)}.$$

$$\therefore \cos. \theta = \frac{x x_1 + y y_1 + z z_1}{r\,r_1}$$

$$= \frac{\alpha\alpha' + \beta\beta' + 1}{\sqrt{(\alpha^2 + \beta^2 + 1)}\sqrt{(\alpha'^2 + \beta'^2 + 1)}}$$

Hence sin. $\theta = \dfrac{\sqrt{\{(\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2\}}}{\sqrt{(\alpha^2 + \beta^2 + 1)}\sqrt{(\alpha'^2 + \beta'^2 + 1)}}$.

And tan. $\theta = \dfrac{\sin.\ \theta}{\cos.\ \theta} = \dfrac{\sqrt{\{(\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2\}}}{\alpha\alpha' + \beta\beta' + 1.}$

The value of the cosine of the angle between two straight lines may also be expressed in terms of the angles which the two straight lines l and l_1 make with the co-ordinate axes.

For $x = r\cos.\ l\,x$, $y = r\cos.\ l\,y$, $z = r\cos.\ l\,z$,

and $x_1 = r_1\cos.\ l_1\,x$, $y_1 = r_1\cos.\ l_1\,y$, $z_1 = r_1\cos.\ l_1\,z$,

$$\therefore\ \cos.\ \theta = \frac{x\,x_1}{x\,x_1} + \frac{y\,y_1}{r\,r_1} + \frac{z\,z_1}{r\,r_1}$$

$$= \cos.\ l\,x \cos.\ l\,x + \cos.\ l\,y \cos.\ l\,y + \cos.\ l\,z \cos.\ l_1\,z.$$

404. If the lines are parallel, we must have sin. $\theta = 0$.

$$\therefore\ (\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2 = 0,$$

an equation which cannot be satisfied unless by supposing $\alpha = \alpha'$, $\beta = \beta'$, and $\alpha\beta' = \alpha'\beta$, the first two of these conditions are the same as those already shown to determine the parallelism of two lines (400), and the third condition is only a necessary consequence of the other two, and therefore implies nothing further.

405. If the lines are perpendicular to each other, we must have cos. $\theta = 0$.

$$\therefore\ \alpha\alpha' + \beta\beta' + 1 = 0.$$

or, $\cos.\ l\,x \cos.\ l_1\,x + \cos.\ l\,y \cos.\ l_1\,y + \cos.\ l\,z \cos.\ l_1\,z = 0$;

Now, one line in space is considered as perpendicular to a second straight line, whenever it is in a plane perpendicular to this second line; hence an infinite number of lines can be drawn perpendicular to a given line; and this appears from the above equation, for there are four constants involved in the equation to the perpendicular line, and only one equation between them.

406. If the lines also meet, we have then the additional equation,

$$(a' - a)\beta' - \beta = (b' - b)(\alpha' - \alpha) \quad (401),$$

However, even yet an infinite number of straight lines can be drawn, meeting the given line at right angles, for an infinite number of planes can be drawn perpendicular to the given line, and in each plane an infinite number of straight lines can be drawn passing through the given line.

407. To find the equation to a straight line passing through a given point $x_1 y_1 z_1$, and meeting a given line (1) at right angles.

Let the equations to the lines be,

$$\left.\begin{aligned} x &= \alpha z + a \\ y &= \beta z + b \end{aligned}\right\} (1) \qquad \left.\begin{aligned} x - x_1 &= \alpha'(z - z_1) \\ y - y_1 &= \beta'(z - z_1) \end{aligned}\right\} (2)$$

hence the two equations of condition are,

$$\alpha\alpha' + \beta\beta' + 1 = 0 \quad (3)$$

$$(a' - a)(\beta - \beta') - (b' - b)(\alpha - \alpha') = 0$$

or since $a' = x_1 - \alpha' z_1$, and $b' = y_1 - \beta' z_1$

$$(x_1 - \alpha' z_1 - a)(\beta - \beta') - (y_1 - \beta' z_1 - b)(\alpha - \alpha') = 0 \quad (4).$$

The elimination of α' and β' from (3) and (4) give the equations

$$\beta' = \frac{\{(x_1 - a)\,\alpha + z_1\}\,\beta - (y_1 - b)\,(1 + \alpha^2)}{(y_1 - b)\,\beta + (x_1 - a)\,\alpha - (\alpha^2 + \beta^2)\,z_1}$$

$$\alpha' = \frac{\{(y_1 - b)\,\beta + z_1\}\,\alpha - (x_1 - a)\,(1 + \beta^2)}{(y_1 - b)\,\beta + (x_1 - a)\,\alpha - (\alpha^2 + \beta^2)\,z_1}.$$

These values of α' and β' substituted in (2) give the final equation to the straight line, passing through a given point, and meeting a given straight line at right angles.

In particular cases other methods may be adopted, for example, to find the equations to a straight line passing through the axis of y at right angles to that axis:
here $x_1 = 0$, and $z_1 = 0$, therefore the equations to the line are

$$x = \alpha z$$
$$y - y_1 = \beta z$$

but because the line is perpendicular to the axis of y we have $\beta = 0$, hence the required equations are $x = a z$, $y = y_1$. By assuming the axes of co-ordinates to be conveniently situated, this and many other problems may be worked in a shorter manner. This will be shown hereafter.

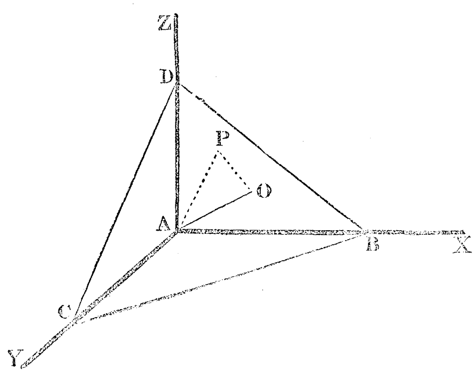
CHAPTER III.

THE PLANE.

408. A Plane may be supposed to be generated by the motion of a straight line about another straight line perpendicular to it.

Let A be the origin, A X, A Y, A Z the axes, B C D a portion of a plane, A O the perpendicular from the origin upon this plane, P any point in this plane; then, according to the above definition, we suppose the plane to be formed by the revolution of a line like O P round A O, the angle A O P being a right angle.

To find the equation to the plane.
Let x, y, z, be the co-ordinates of P, and x_1, y_1, z_1, those of O, and let the fixed distance AO $= d$.



Then the square on A P = the square on A O + the square on O P;

or, $x^2 + y^2 + z^2 = d^2 + (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$

$$= d^2 + x^2 + y^2 + z^2 + x_1^2 + y_1^2 + z^2 - 2\,x\,x_1 - 2\,y\,y_1 - 2\,z\,z_1.$$

$$\therefore\ 2\,(x\,x_1 + y\,y_1 + z\,z_1) = d^2 + d^2 = 2\,d^2$$

$$\text{or } x\,x_1 + y\,y_1 + z\,z_1 = d^2.$$

409. Let $\frac{x_1}{d^2} = m$, $\frac{y_1}{d^2} = n$, and $\frac{z_1}{d^2} = p$, then the above equation becomes

$$m\,x + n\,y + p\,z = 1.$$

And it is under this form that we shall generally consider the equation to the plane.

Let $\frac{d^2}{x_1} = a$, $\frac{d^2}{y_1} = b$, and $\frac{d^2}{z_1} = c$, then the equation to the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

And this is perhaps the most intelligible form in which the equation to the plane can be put, the constants a, b and c being equal to A B, A C and A D the respective distances of the origin from the intersection of the plane with the co-ordinate axes; this is found by putting y and z both $= 0$, hence $\frac{x}{a} = 1$, or A B $= a$, and similarly for the other lines.

410. Let the word "plane" be represented by the letter P, and let the angles which A O or d makes with the co-ordinate axes be represented by $d\,x$; $d\,y$; $d\,z$; and let the angles which the plane makes with the same axes be denoted by Px; Py; Pz; then, since A O B is a right angle, and A B O is the angle which the plane makes with A X, we have

$$d = a \cos.\, d\,x = a \sin.\, \mathrm{P}\,x$$

$$d = b \cos.\, d\,y = b \sin.\, \mathrm{P}\,y$$

$$d = c \cos.\, d\,z = c \sin.\, \mathrm{P}\,z;$$

therefore the last equation to the plane may be put in either of the forms

$$x \cos.\, d\,x + y \cos.\, d\,y + z \cos.\, d\,z = d$$

$$\text{or } x \sin.\, \mathrm{P}\,x + y \sin.\, \mathrm{P}\,y + z \sin.\, \mathrm{P}\,z = d.$$

411. Let P, $y\,z$ represent the angle which the plane makes with the co-ordinate plane $y\,z$, then since angle O A B. is equal to the angle of inclination of the plane to $y\,z$, we have cos. $d\,x$ = cos. P, $y\,z$, hence the equation of the plane becomes

$$x \cos.\, \mathrm{P}, y\,z + y \cos.\, \mathrm{P}, x\,z + z \cos.\, \mathrm{P}, x\,y = d.$$

412. Since by (386) $(\cos.\, d\,x)^2 + (\cos.\, d\,y)^2 + (\cos.\, d\,z)^2 = 1$

we have $(\cos.\, \mathrm{P}, y\,z)^2 + (\cos.\, \mathrm{P}\,x\,z)^2 + (\cos.\, \mathrm{P}\,x\,y)^2 = 1$ *.

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* If A be the area of a plane P, the projections of this area on the co-ordinate planes are represented by A cos. P, $x\,y$; A cos. P, $x\,z$; A cos. P, $y\,z$; hence $(\mathrm{A} \cos.\, \mathrm{P}, x\,y)^2 + (\mathrm{A} \cos.\, \mathrm{P}, x\,z)^2 + (\mathrm{A} \cos.\, \mathrm{P}, y\,z)^2 = \mathrm{A}^2 \left\{(\cos.\, \mathrm{P}, x\,y)^2 + (\cos.\, \mathrm{P}, x\,z)^2 + (\cos.\, \mathrm{P}, y\,z)^2\right\} = \mathrm{A}^2$ by (412). This theorem, referring to the numerical values of the projected areas, is of use in finding the area of a plane between the three co-ordinate planes. Thus, if the equation to a plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we have by the last figure the area A B C $= \frac{a\,b}{2}$; area A D C $= \frac{b\,c}{2}$, and area A B D $= \frac{a\,c}{2}$; hence the area B C D $= \sqrt{\tfrac{1}{4}(a^2 b^2 + a^2 c^2 + b^2 c^2)}$ by the above theorem. The volume of the pyramid A C D B $= \frac{c}{3} \cdot \frac{a\,b}{2} = \frac{a\,b\,c}{6}$.

413. To find the angles which a plane makes with the co-ordinate planes in terms of the co-efficients of the equation to the plane.

Let the equation to the plane be

$$m x + n y + p z = 1.$$

Now the equation to a plane expressed in terms of the angles which it makes with the co-ordinate planes is given by (411.)

$$x \cos. \text{P}, y z + y \cos. \text{P}, x z + z \cos. \text{P}, x y = d,$$

hence equating co-efficients, we have

$$m = \frac{\cos. \text{P}, y z}{d}, n = \frac{\cos. \text{P}, x z}{d}, p = \frac{\cos. \text{P}, x y}{d}$$

$$\therefore m^2 + n^2 + p^2 = \frac{1}{d^2} \text{ and } d = \frac{1}{\sqrt{m^2 + n^2 + p^2}};$$

$$\text{and } \cos. \text{P}, y z = m d = \frac{m}{\sqrt{m^2 + n^2 + p^2}}$$

$$\text{Cos. P}, x z = n d = \frac{n}{\sqrt{m^2 + n^2 + p^2}}$$

$$\text{Cos. P}, x y = p d = \frac{p}{\sqrt{m^2 + n^2 + p^2}}.$$

414. The equation to the plane will assume various forms according to the various positions of the plane.

Let the plane pass through the origin, then $d = 0$; therefore, putting $d = 0$ in the equation, art. (408), we have the equation to the plane passing through the origin; but as the equation to the plane has been obtained on the supposition of d being finite, it becomes necessary to give an independent proof for this particular case.

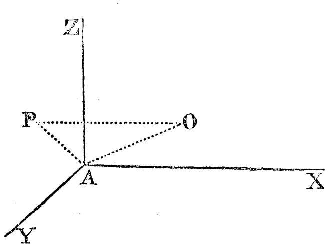
Let A O ($= d$) be the length of a perpendicular from the plane to a given point O; whose co-ordinates are x_1, y_1, z_1; x, y, z, as before, the co-ordinates of any point P in the plane, then

the square on O P = the square on A O + the square on A P;

or $(x - x_1)^2 + (y - y_1)^2 + (z - z_1) = d^2 + x^2 + y^2 + z^2$.

$$\therefore \; - 2 (x x_1 + y y_1 + z z_1) + d^2 = d^2,$$

$$\text{or } x x_1 + y y_1 + z z_1 = 0.$$



So that the equation to the plane in this case is the same as the original equation without the constant term.

415. Let the plane be parallel to any of the co-ordinate planes, as $x y$ for example, then $a = \infty$ and $b = \infty$; therefore the equation $\frac{x}{a} + \frac{y}{b} +$

$\frac{z}{c} = 1$ becomes $0\,x + 0\,y + \frac{z}{c} = 1$; $\therefore z = c$, $x = \frac{0}{0}$ and $y = \frac{0}{0}$; of these three equations the first signifies that every point in the plane is equidistant from the plane $x\,y$, and the other two signify that for this single value of z, every possible value of x and y will give points in the plane. The two latter equations are generally omitted; and we say that for a plane parallel to $x\,y$ the equation is $z = c$; similarly for the plane parallel to $x\,z$ it is $x = a$, and for a plane parallel to $y\,z$ the equation is $y = b$.

The equations to a co-ordinate plane, as $x\,y$ for example, are $z = 0$, $x = \frac{0}{0}$; $y = \frac{0}{0}$; or, more simply, $z = 0$.

416. The lines B C, B D, and D C, where the plane intersects the co-ordinate planes, are called the traces of the plane. The equations to these traces are found, from the equation to the plane, by giving to x, y, or z the particular values which they have when the plane intersects the co-ordinate planes.

Let the equation to the plane be $m\,x + n\,y + p\,z = 1$; then for the intersection B C we have the equations

$$z = 0,\ m\,x + n\,y = 1.$$

Similarly the equations to the traces B D and C D are respectively

$$y = 0,\ m\,x + p\,z = 1$$
$$x = 0,\ n\,y + p\,z = 1.$$

PROBLEMS ON THE PLANE.

417. To find the equation to a plane parallel to a given plane.

Let the given plane be $m\,x + n\,y + p\,z = 1$,
and the required plane be $m'\,x + n'\,y + p'\,z = 1$.
Then the planes being parallel, their traces on the co-ordinate planes must be parallel; now their traces on $x\,z$ are

$$m\,x + p\,z = 1,\ m'\,x + p'\,z = 1;$$

$$\therefore \frac{m}{p} = \frac{m'}{p'}, \text{ or } m' = \frac{m}{p}\,p'; \text{ similarly } n' = \frac{n}{p}\,p'.$$

Hence the required equation becomes

$$\frac{m}{p}\,p'\,x + \frac{n}{p}\,p'\,y + p'\,z = 1,$$

$$\text{or } m\,x + n\,y + p\,z = \frac{p}{p'}$$

In this case the resulting equation contains one indeterminate constant p', and therefore shows that an infinite number of planes can be drawn parallel to a given plane, which is also geometrically evident. Three conditions are apparently given, since the three traces of one plane are parallel to the three traces of the other plane; but if the traces on two of the co-ordinate planes be parallel, the traces on the third co-ordinate plane are necessarily parallel; for if $\frac{m}{p} = \frac{m'}{p'}$, and $\frac{n}{p} = \frac{n'}{p'}$, we have $\frac{m}{n} =$

$\frac{m'}{n'}$, or $m\,x + n\,y = 1$ parallel to $m'\,x + n'\,y = 1$. Thus, in reality, only two conditions are given to determine the three constants.

418. To find the equation to a plane parallel to a given plane, and passing through a given point x_1, y_1, z_1.

Let $m'\,x + n'\,y + p'\,z = 1$ be the required plane, then since the plane passes through $x_1\,y_1\,z_1$ we have

$$m'\,x_1 + n'\,y_1 + p'\,z_1 = 1$$

$$\therefore\ m'\,(x - x_1) + n'\,(y - y_1) + p'\,(z - z_1) = 0.$$

$$\text{Also } \frac{m'}{p'} = \frac{m}{p}, \text{ and } \frac{n'}{p'} = \frac{n}{p};$$

$$\therefore\ \frac{m}{p}\,p'\,(x - x_1) + \frac{n}{p}\,p'\,(y - y_1) + p'\,(z - z_1) = 0;$$

$$\text{or } m\,(x - x_1) + n\,(y - y_1) + p\,(z - z_1) = 0.$$

419. To find the intersection of a straight line and plane.

Let $m\,x + n\,y + p\,z = 1$ be the equation to the plane,

$$\left.\begin{array}{l} x = \alpha\,z + a \\ y = \beta\,z + b \end{array}\right\} \text{ the equations to the line;}$$

then, since the co-ordinates of the point of intersection are common, we have.

$$m\,(\alpha\,z + a) + n\,(\beta\,z + b) + p\,z = 1,$$

$$\therefore\ z = \frac{1 - m\,a - n\,b}{m\,\alpha + n\,\beta + p},$$

$$\text{and } x = \alpha\,z + a = \frac{\alpha - n\,b\,a + n\,\beta\,a + p\,a}{m\,\alpha + n\,\beta + p}$$

$$y = \beta\,z + b = \frac{\beta - m\,a\,\beta + m\,a\,b + p\,b}{m\,\alpha + n\,\beta + p}.$$

Thus the required point of intersection is found.

420. To find the conditions that the straight line and plane be parallel or coincide.

If they are parallel, the values of x, y, and z must be infinite

$$\therefore\ m\,\alpha + n\,\beta + p = 0.$$

If they coincide, the values of x, y, and z must be indeterminate, or each $= \frac{0}{0}$.

$$\therefore\ m\,\alpha + n\,\beta + p = 0, \text{ and } 1 - m\,a - n\,b = 0;$$

and these are the two conditions for coincidence, the numerators of x and y being both given $= 0$ by combining the last two equations.

Hence, to find the equation to a plane coinciding with a given straight line, we have the two conditions

$$m\,a + n\,b = 1,$$
$$m\,\alpha + n\,\beta + p = 0;$$

whence, by elimination, we have

$$m = \frac{\beta + p\,b}{a\,\beta - b\,\alpha} \text{ and } n = -\frac{\alpha + p\,a}{a\,\beta - b\,\alpha};$$

therefore the equation to the plane is

$$(\beta + p\,b)\,x - (\alpha + p\,a)\,y + p\,(a\,\beta - b\,\alpha)\,z = a\,\beta - b\,\alpha,$$

where p remains indeterminate.

421. To find the equation to a plane coinciding with two given lines.

$$m\,x + n\,y + p\,z = 1.$$

$$\left.\begin{array}{l} x = \alpha\,z + a \\ y = \beta\,z + b \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha'\,z + a' \\ y = \beta'\,z + b' \end{array}\right\}$$

the plane coinciding with the given lines, we have

$$m\,a + n\,b = 1 \quad (1) \qquad m\,\alpha + n\,\beta + p = 0 \quad (3)$$

$$m\,a' + n\,b' = 1 \quad (2) \qquad m\,\alpha' + n\,\beta' + p = 0 \quad (4)$$

From (1) and (2) we have m and n, and these values being substituted in (3) and (4), give two values of p, hence we have the equation of condition

$$(\beta' - \beta)\,(a - a') + (\alpha' - \alpha)\,(b - b') = 0.$$

This equation is verified either if the lines are parallel (in which case $\alpha' = \alpha$ and $\beta' = \beta$), or if they meet; hence in either of these cases a plane may be drawn coinciding with the two lines; the equation to this plane is found, from the values of m, n, and p, to be

$$(b' - b)\,x - (a' - a)\,y + \{\,(a' - a)\,\beta - (b' - b)\,\alpha\,\}\,z = a\,b' - a'\,b.$$

422. If it be required to find the equation to a plane which coincides with one given straight line, and is parallel to another given straight line, we have the three equations

$$\left.\begin{array}{r} m\,a + n\,b = 1 \\ m\,\alpha + n\,\beta + p = 0 \end{array}\right\}\text{ for coincidence with one line,}$$

$$m\,\alpha' + n\,\beta' + p = 0 \text{ for parallelism with the other;}$$

and from these three equations we may determine m, n, and p, and then substitute these values in the general equation to the plane.

423. To find the intersection of two given planes.

Let the equations to the two planes be

$$m\,x + n\,y + p\,z = 1$$

$$m'\,x + n'\,y + p'\,z = 1.$$

By the elimination of z we obtain an equation between x and y, which belongs to the projection of the intersection of the planes on $x\,y$,

hence $(m\,p' - m'\,p)\,x + (n\,p' - n'\,p)\,y = p' - p$

is the projection on $x\,y$ of the required intersection.

Similarly

$$(m\,n' - m'\,n)\,x + (p\,n' - p'\,n)\,z = n' - n$$

is the equation to the projection on $x\,z$.

But the equations to the projections of a line on two co-ordinate planes are called the equations to the line itself; hence the above two equations are the required equations to the intersection.

The third projection is given by the other two, or it may be found separately

$$(n\,m' - n'\,m)\,y + (p\,m' - m\,p')\,z = m' - m.$$

424. To find the intersection of three planes.

Let the intersection of the first and second, found as in the last article, be expressed by the equations

$$x = \alpha z + a$$
$$y = \beta z + b,$$

and let the intersection of the first and third planes be denoted by the equations

$$x = \alpha' z + a'$$
$$y = \beta' z + b'.$$

Then, finding the intersection of these two lines from their four equations, we have the values of x, y, and z, corresponding to the point of intersection of the two lines, and therefore to the point of intersection of the three planes.

In this manner we may find the relation among the co-efficients of any number of planes meeting in one point.

425. To find the relation among the coefficients of the equations to four planes so that they may meet in the same straight line.

Let the equations be

$$m x + n y + p z = 1$$
$$m_1 x + n_1 y + p_1 z = 1$$
$$m_2 x + n_2 y + p_2 z = 1$$
$$m_3 x + n_3 y + p_3 z = 1.$$

Then the first and second plane intersect in a line whose equations are

$$x = \alpha z + a$$
$$y = \beta z + b$$

The first and third intersect in the line

$$x = \alpha_1 z + a_1$$
$$y = \beta_1 z + b_1$$

And the first and fourth in the line

$$x = \alpha_2 z + a_2$$
$$y = \beta_2 z + b_2$$

Now, in order that these intersections all coincide, we must have

$$\alpha = \alpha_1 = \alpha_2 ; \ \beta = \beta_1 = \beta_2 ; \ a = a_1 = a_2 ; \text{ and } b = b_1 = b_2.$$

And the values of α, β, a and b are given in terms of m, n, p, &c., by article (423), hence the relation among the co-efficients is found.

The same relation exists among the co-efficients of any number o planes meeting in one point.

426. To find the relation among the co-efficients of a straight line and plane, so that they may be perpendicular to one another.

Let $(x_1 y_1 z_1)$ be the point in which the plane and line meet, then the equation to the plane is

$$m(x - x_1) + n(y - y_1) + p(z - z_1) = 0 \ (1)$$

And the equations to the line are

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} (2)$$

Also let the equations to a line perpendicular to (2) and passing through the point $(x_1 y_1 z_1)$ in (2) be

$$\left.\begin{array}{l} x - x_1 = \alpha' (z - z_1) \\ y - y_1 = \beta' (z - z_1) \end{array}\right\} \quad (3)$$

But since these two lines are perpendicular to one another, we have the cosine of the angle between them $= 0$,

$$\therefore \ \alpha\alpha' + \beta\beta' + 1 = 0 \quad (402)$$

Now, this equation combined with that to the last line (3), will give the relation among the co-ordinates of x, y, and z, so that the point to which they refer is always in a locus perpendicular to the first given line; hence substituting for α' and β', we have the equation to the plane which is the locus of all the lines perpendicular to (2), this equation is

$$\alpha \frac{x - x_1}{z - z_1} + \beta \frac{y - y_1}{z - z_1} + 1 = 0$$

$$\text{or } \alpha (x - x_1) + \beta (y - y_1) + z - z_1 = 0 \ (4)$$

and as this equation (4) must coincide with (2) we have, by equating the co-efficients,

$$\alpha = \frac{m}{p}, \text{ and } \beta = \frac{n}{p},$$

and these are the conditions required.

427. Hence, if the line be given, the equation to the plane perpendicular to it is

$$\alpha x + \beta y + z = \frac{1}{p}.$$

Or if the plane be given, the equations to the straight line perpendicular to it are

$$x = \frac{m}{p} z + a,$$

$$y = \frac{n}{p} z + b.$$

From the form of these equations to the plane and perpendicular straight line, it appears that the trace of the plane is perpendicular to the projection of the line upon the same co-ordinate plane.

428. If the plane pass through a given point $x_1\, y_1\, z_1$, and be perpendicular to a given straight line, ($x = \alpha z + a$, $y = \beta z + b$) its equation is

$$\alpha (x - x_1) + \beta (y - y_1) + z - z_1 = 0.$$

429. If the straight line pass through a given point, and be perpendicular to a given plane ($m x + n y + p z = 1$) its equations are

$$x - x_1 = \frac{m}{p} (z - z_1),$$

$$y - y_1 = \frac{n}{p} (z - z_1).$$

430. To find the length of a perpendicular from a given point on a given plane.

Let $x_1\, y_1\, z_1$ be the co-ordinates of the given point,

$m x + n y + p z = 1$ the equation to the given plane.

It was shown in Art. 413, that if d be the perpendicular distance of the origin from a plane, whose equation is

$$m x + n y + p z = 1$$

$$\text{we have } d = \frac{1}{\sqrt{m^2 + n^2 + p^2}}.$$

Now, the equation to the plane, parallel to the given plane, and passing through the given point, is

$$m (x - x_1) + n (y - y_1) + p (z - z_1) = 0 \ (418).$$

$$\text{or } \frac{m x + n y + p z}{m x_1 + n y_1 + p z_1} = 1.$$

Hence the distance d_1 of the origin from this plane is

$$d_1 = \frac{m x_1 + n y_1 + p z_1}{\sqrt{m^2 + n^2 + p^2}}.$$

But the distance of the given point from the given plane is evidently the distance between the two planes, that is, $= d_1 - d$

$$= \frac{m x_1 + n y_1 + p z_1 - 1}{\sqrt{m^2 + n^2 + p^2}}.$$

431. To find the distance of a point from a straight line.
Let the equations to the given line be $x = \alpha z + a$, $y = \beta z + b$, then the equation to the plane passing through the given point $x_1 y_1 z_1$, and perpendicular to the given line, is

$$\alpha (x - x_1) + \beta (y - y_1) + z - z_1 = 0.$$

Eliminating x, y, and z by means of the above equations to the straight line, we find

$$z = \frac{\alpha (x_1 - a) + \beta (y_1 - b) + z_1}{1 + \alpha^2 + \beta^2};$$

or, if this fraction $= \frac{M}{N}$, we have

$$z = \frac{M}{N}, \quad x = \alpha \frac{M}{N} + a, \quad y = \beta \frac{M}{N} + b.$$

These are the co-ordinates of the intersection of the given line, with the perpendicular plane passing through the given point; and the required perpendicular line (P) is the distance of the given point from this intersection.

Hence $P^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2$

$$= \left(x_1 - a - \alpha \frac{M}{N}\right)^2 + \left(y_1 - b - \beta \frac{M}{N}\right)^2 + \left(z_1 - \frac{M}{N}\right)^2$$

which, after expansion and reduction, becomes

$$= (x_1 - a)^2 + (y_1 - b)^2 + z_1^2 - \frac{M^2}{N}.$$

432. If the given point be the origin, we have $x_1 \, y_1 \, z_1$, each equal $= 0$

$$\therefore P^2 = a^2 + b^2 - \frac{(\alpha a + \beta b)^2}{1 + \alpha^2 + \beta^2}.$$

433. To find the angle θ between two given planes.
Let the equations to the planes be

$$m x + n y + p z = 1 \quad (1)$$
$$m_1 x + n_1 y + p_1 z = 1 \quad (2).$$

Then, if from the origin we draw perpendiculars on each of these planes, the angle between these perpendiculars is equal to the angle between the planes: let the equations to the two lines be

$$\left.\begin{array}{l} x = \alpha z \\ y = \beta z \end{array}\right\} (3) \qquad \left.\begin{array}{l} x = \alpha' z \\ y = \beta' z \end{array}\right\} (4).$$

In order that (3) may be perpendicular to (1), we must have

$$\alpha = \frac{m}{p}, \beta = \frac{n}{p} \text{ (426), and similarly } \alpha' = \frac{m_1}{p_1}, \beta' = \frac{n_1}{p_1}.$$

Then the angle between the two lines is found from the expression

$$\cos.\,\theta = \frac{\alpha\alpha' + \beta\beta' + 1}{\sqrt{(1 + \alpha^2 + \beta^2)}\sqrt{(1 + \alpha'^2 + \beta'^2)}} \text{ (403)}.$$

$$\therefore \cos.\,\theta = \frac{m m_1 + n n_1 + p p_1}{\sqrt{m^2 + n^2 + p^2}\;\sqrt{m_1^2 + n_1^2 + p_1^2}}.$$

434. This value of cos. θ may also be expressed in another form by means of Art. (413.)

$$\cos.\,\theta = \cos.\,P,x\cos.\,P',x + \cos.\,P,y\cos.\,P',y + \cos.\,P,z\cos.\,P',z.$$

or $\cos.\,\theta = \cos.\,P,yz\cos.\,P',yz + \cos.\,P,xz\cos.\,P',xz + \cos.\,P,xy\cos.\,P',xy.$

435. If the planes be perpendicular to each other, we have cos. $\theta = 0$.

$$\therefore m m_1 + n n_1 + p p_1 = 0.$$

Hence, if the equation to any plane be $mx + ny + pz = 1$, the equation to the plane perpendicular to it is

$$m_1 x + n_1 y - \frac{m m_1 + n n_1}{p} z = 1,$$

where two constants remain indeterminate.

436. If the planes be parallel, we have cos. $\theta = 1$; and putting therefore the expression for cos. θ equal to unity, we shall arrive at the results,

$$\frac{m}{n} = \frac{m_1}{n_1} \text{ and } \frac{m}{p} = \frac{m_1}{p_1};$$

the same as already obtained when two planes are parallel.

437. To find the angle between a straight line and a plane.

This angle is the angle which the line makes with its projection on the plane; and therefore, drawing a perpendicular from any point in the line to the plane, is the complement of the angle which this perpendicular makes with the given line.

Let the equations to the plane and the line be

$$mx + ny + pz = 1$$
$$x = \alpha z + a,\; y = \beta z + b,$$

then the equations to the perpendicular from any point $x_1\, y_1\, z_1$ in the line to the plane are $x = \frac{m}{p}(z - z_1), y = \frac{n}{p}(z - z_1)$. (429)

$$\therefore \cos.\,(\pi - \theta) = \sin.\,\theta = \frac{\alpha\frac{m}{p} + \beta\frac{n}{p} + 1}{\sqrt{1 + \alpha^2 + \beta^2}\;\sqrt{1 + \frac{m^2}{p^2} + \frac{n^2}{p^2}}}$$

$$= \frac{m\alpha + n\beta + p}{\sqrt{1 + \alpha^2 + \beta^2}\;\sqrt{m^2 + n^2 + p^2}}.$$

CHAPTER IV.

THE POINT, STRAIGHT LINE, AND PLANE REFERRED TO OBLIQUE AXES.

438. If the co-ordinate axes are not rectangular but inclined to each other at any given angles, they are then called oblique axes. The equations to the point, Art. (381.) remain exactly the same as before, but the quantities a, b, and c, are no longer the representatives of lines drawn perpendicular to the co-ordinate planes, but of lines respectively parallel to the oblique axes.

439. To find the distance of a point from the origin referred to oblique axes.

Let A X, A Y, A Z, be the oblique axes; and let x, y, z, be the co-ordinates of P, draw P N perpendicular on A Q produced,

then the sq. on AP = the sqs. on AQ and PQ + twice the rectangle AQ,QN.

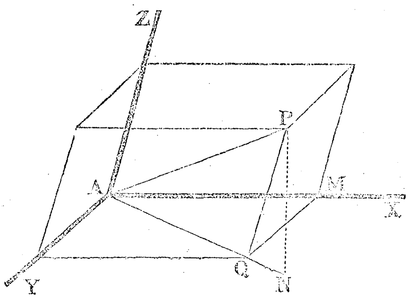
$$\text{Now, } Q\,N = P\,Q \cos. P\,Q\,N = z \cos. Z\,A\,Q$$

$$\text{and } A\,Q \cos. Z\,A\,Q = A\,M \cos. M\,A\,Z + M\,Q \cos. Y\,A\,Z \;(379)$$

$$= x \cos. X\,A\,Z + y \cos. Y\,A\,Z$$

$$\therefore \text{ the rectangle } A\,Q,\,Q\,N = z\,(x \cos. X\,Z + y \cos. Y\,Z)$$

$$\text{also the square on } A\,Q = x^2 + y^2 + 2\,x\,y \cos. Y\,X,$$



$$\therefore d^2 = x^2 + y^2 + z^2 + 2\,x\,y \cos. X\,Y + 2\,x\,z \cos. X\,Z + 2\,y\,z \cos. Y\,Z.$$

440. To find the distance between two points when the axes are oblique

Let $x\,y\,z$ be the co-ordinates of one point,

and $x_1\,y_1\,z_1$ . . the other point,

then the distance between these points is the diagonal of a parallelopiped, of which the sides are the differences of parallel co-ordinates (388); hence,

$$d^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + 2\,(x - x_1)\,(y - y_1) \cos. X\,Y$$
$$+ 2\,(x - x_1)\,(z - z_1) \cos. X\,Z + 2\,(y - y_1)\,(z - z_1) \cos. Y\,Z.$$

441. To find the equation to a straight line referred to oblique co-ordinates. The straight line must be considered to be the intersection of two planes formed by drawing straight lines through the several points of the given straight line parallel respectively to the planes of $x\,z$, $y\,z$; the traces of these planes on the co-ordinate planes are of the same *form* as for rect-

angular axes; that is, the equation to the traces, and therefore to the line itself are of the *form*

$$x = \alpha z + a$$
$$y = \beta z + b$$

but the values of α and β are not the tangents of any angles, but the ratio of the sines of the angles which each trace makes with the axes in its plane (51).

The quantities a and b remain the same as when the straight line is referred to rectangular co-ordinates, and since the equations are of the same form as before, those problems which do not affect the inclination of lines will remain the same as before.

442. To find the angle between two straight lines referred to oblique co-ordinates we shall follow the plan adopted in Art. 402.

Let the equations to the parallel lines through the origin be

$$\left.\begin{array}{l} x = \alpha z \\ y = \beta z \end{array}\right\} (1) \qquad \left.\begin{array}{l} x = \alpha' z \\ y = \beta' z \end{array}\right\} (2)$$

And let r be the distance of a point $x\,y\,z$ in (1) from the origin, and r_1 the distance of a point $x_1\,y_1\,z_1$ in (2) from the origin.

Then if d be the distance between these points, we have

$$d^2 = r^2 + r_1^2 - 2\,r\,r'\cos.\theta$$
$$= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + 2\,(x - x_1)\,(y - y_1)\cos.\,XY$$
$$+ 2(x - x_1)\,(z - z_1)\cos.\,XZ + 2\,(y - y_1)\,(z - z_1)\cos.\,YZ,$$
$$= r^2 + r_1^2 - 2\,(x\,x_1 + y\,y_1 + z\,z_1)$$
$$- 2\,\{(x_1\,y + x\,y_1)\cos.XY + (x_1\,z + x\,z_1)\cos.\,XZ + (y_1\,z + y\,z_1)\cos.\,YZ\}$$
$$\therefore r\,r_1\cos.\,\theta = x\,x_1 + y\,y_1 + z\,z_1$$
$$+ \{(x_1\,y + x\,y_1)\cos.\,XY + (x_1\,z + x\,z_1)\cos.\,XZ + (y_1\,z + y\,z_1)\cos.\,YZ\}$$

$$\therefore \cos.\,\theta = \frac{x\,x_1 + y\,y_1 + z\,z_1}{r\,r_1}$$

$$\frac{\alpha\,\alpha^1 + \beta\,\beta^1 + 1 + (\alpha^1\,\beta + \alpha\,\beta^1)\cos.\,X\,Y + (\alpha^1 + \alpha)\cos.\,XZ + (\beta^1 + \beta)\cos.YZ}{\sqrt{\{1 + \alpha^2 + \beta^2 + 2\alpha\,\beta\cos.XY + 2\,\alpha\cos.\,XZ + 2\beta\cos.YZ\}}\,\sqrt{\{1 + \alpha'^2 + \beta'^2\&c.\}}}$$

443. To find the equation to a plane referred to oblique axes.

We consider a plane as the locus of all the straight lines which can be drawn perpendicular to a given straight line, and passing through a given point in that given straight line.

Let the equations to the given line be

$$x = \alpha z + a$$
$$y = \beta z + b$$

Also the equations to the straight line passing through a point x_1, y_1, z_1, in the above line, are

$$x - x_1 = \alpha'\,(z - z_1)$$
$$y - y_1 = \beta'\,(z - z_1)$$

But these two last lines being perpendicular to each other, we have the angle θ between them $= 90°$, or $\cos.\,\theta = 0$; hence by the last article:

$$\alpha\,\alpha' + \beta\,\beta' + 1 + (\alpha'\,\beta + \alpha\,\beta')\cos.XY + (\alpha' + \alpha)\cos.XZ + (\beta' + \beta)\cos.YZ = 0$$

and eliminating α' and β'

$$\alpha \frac{x - x_1}{z - z_1} + \beta \frac{y - y_1}{z - z_1} + 1 + \left(\frac{x - x_1}{z - z_1} \beta + \frac{y - y_1}{z - z_1} \alpha\right) \cos. X Y$$

$$+ \left(\frac{x - x_1}{z - z_1} + \alpha\right) \cos. X Z + \left(\frac{y - y_1}{z - z_1} + \beta\right) \cos. Y Z = 0$$

or, $(\alpha + \beta \cos. X Y + \cos. X Z)(x - x_1) + (\beta + a \cos. X Y + \cos. Y Z)(y - y_1) + (1 + \alpha \cos. X Z + \beta \cos. Y Z)(z - z_1) = 0$

and this equation, which is the locus of all the straight lines meeting the given straight line at a given point and at right angles, is called the equation to the plane.

444. To find the conditions that a straight line be perpendicular to a given plane;

The method is the same as that in article 426.

The equation to a plane passing through a point $x_1 y_1 z_1$ in the given line is

$$m (x - x_1) + n (y - y_1) + p (z - z_1) = 0.$$

But the equations to the given line being

$$x = \alpha z + a, \; y = \beta z + b$$

the equation to the plane perpendicular to it is given at the end of the last article; hence, equating co-efficients we have

$$m = \alpha + \beta \cos. X Y + \cos. X Z,$$
$$n = \beta + a \cos. X Y + \cos. Y Z,$$
$$p = 1 + a \cos. X Z + \beta \cos. Y Z.$$

From these equations we have the values of m, n, p; or the values of α and β in terms of m, n, p.

445. To find the angle between a plane and straight line.

Let the given equations be

$$m x + n y + p z = 1 \quad (1)$$
$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} \quad (2)$$

And let the equations to a straight line perpendicular to the given plane be

$$\left.\begin{array}{l} x = \alpha' z + a^1 \\ y = \beta' z + b^1 \end{array}\right\} (3)$$

where α' and β' have the values of α and β in the last article.

Also the angle between the lines (2) and (3) is given in article (442.), and the angle between the plane and the line (1) being the complement of the angle between the two lines (2) and (3) may be obtained.

446. To find the angle between two planes.

The equations to the lines perpendicular to the given planes, and passing through the origin are given by Article (444.); and the angle between these lines, which is the angle between the given planes, is given by Article (442.)

CHAPTER V.

THE TRANSFORMATION OF CO-ORDINATES.

447. To transform an equation referred to an origin A to an equation referred to another origin A′, the axes in the latter case being parallel to those in the former.

The co-ordinates of the new origin being a, b, and c, it is evident that if a point be referred to this new origin and to the new axes, that each original ordinate is equivalent to the new ordinate together with the corresponding ordinate to the new origin; hence if x, y, z be the original co-ordinate of a point P, and X, Y, Z the new co-ordinates, we have

$$x = a + X,$$
$$y = b + Y,$$
$$z = c + Z;$$

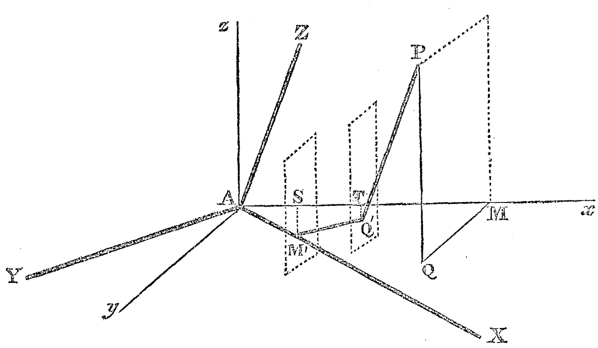
Substituting these values for x, y and z in the equation to the surface, we have the transformed equation between X, Y, and Z referred to the origin A′.

448. To transform the equation referred to rectangular axes to an equation referred to oblique axes having the same origin.

Let A x, A y, A z be the original axes,

A X, A Y, A Z the new axes,

$$\left.\begin{array}{l} AM = x \\ MQ = y \\ QP = z \end{array}\right\} \quad \left.\begin{array}{l} AM' = X \\ M'Q' = Y \\ Q'P = Z \end{array}\right\}$$



Through the points M′, Q′, P draw planes parallel to $y\,z$, or, which is the same thing, perpendicular to A x and meeting A x in S, T and M (these planes are represented by the dotted lines in the figure). Then A S, S T and T M are the respective projections of A M′, M′ Q′ and Q′ P on A x, also

$$A M = A S + S T + T M,$$

$$= A M' \cos. X A x + M' Q' \cos. Y A x + Q' P \cos. Z A x \quad (378)$$

$$\therefore \left.\begin{array}{l} x = X \cos. X x + Y \cos. Y x + Z \cos. Z x \\ y = X \cos. X y + Y \cos. Y y + Z \cos. Z y \\ z = X \cos. X z + Y \cos. Y z + Z \cos. Z z \end{array}\right\}$$

$$\text{Or,} \quad \left.\begin{array}{l} x = m X + m_1 Y + m_2 Z \\ y = n X + n_1 Y + n_2 Z \\ z = p X + p_1 Y + p_2 Z \end{array}\right\}$$

where m is put for cos. X x, &c.

We have also, by art. 397, the following equation between the angles which one straight line, as A X, makes with the axes of x, y, z.

$$(\cos. X x)^2 + (\cos. X y)^2 + (\cos. X z)^2 = 1,$$

Hence the following system,

$$\left.\begin{array}{l} m^2 + n^2 + p^2 = 1 \\ m_1^2 + n_1^2 + p_1^2 = 1 \\ m_2^2 + n_2^2 + p_2^2 = 1 \end{array}\right\} \quad 2.$$

449. If the new system be rectangular, we have also the equations in art. (405), which signify that the new axes are perpendicular to each other; hence the system

$$\left.\begin{array}{l} m m_1 + n n_1 + p p_1 = 0 \\ m m_2 + n n_2 + p p_2 = 0 \\ m_1 m_2 + n_1 n_2 + p_1 p_2 = 0 \end{array}\right\} \quad 3.$$

Hence we observe that of the nine cosines involved in the system (1) three are determined by the system (2), and other three by the system (3); and therefore that there are only three arbitrary angles remaining.

450. In the place of these three systems the following three may also be used:

$$\left.\begin{array}{l} X = m x + n y + p z \\ Y = m_1 x + n_1 y + p_1 z \\ Z = m_2 x + n_2 y + p_2 z \end{array}\right\} \quad 4.$$

$$\left.\begin{array}{l} m^2 + m_1^2 + m_2^2 = 1 \\ n^2 + n_1^2 + n_2^2 = 1 \\ p^2 + p_1^2 + p_2^2 = 1 \end{array}\right\} \quad 5. \qquad \left.\begin{array}{l} m n + m p + n p = 0 \\ m_1 n_1 + m_1 p_1 + n_1 p_1 = 0 \\ m_2 n_2 + m_2 p_2 + n_2 p_2 = 0 \end{array}\right\} \quad 6.$$

For, multiplying the values of x, y and z in (1) by m, n and p respectively; then adding the results together, and reducing by means of (2) and (3), we have $X = m x + n y + p z$; and repeating this operation with the other multipliers $m_1\, n_1\, p_1$ and $m_2\, n_2\, p_2$, we have the system (4). Also, since the distance of P from the origin is the same for both systems, we have $x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2$; putting here, for X, Y and Z, their values in (4), and then equating co-efficients on both sides, we have the two systems (5) and (6).

Whenever we see the systems (2) and (3), we may replace them by (5) and (6); this may be proved independently of any transformation of co-ordinates, by assuming the quantities $m\, n\, p$, &c. to be connected as in (1).

451. The transformation from oblique axes to others oblique, is effected by drawing a perpendicular from M in the last figure upon the

plane of $y\,z$, and by projecting x, X, Y, and Z on this perpendicular, we shall have

$$x \sin. x, y\,z = \mathrm{X} \sin. \mathrm{X}, y\,z + \mathrm{Y} \sin. \mathrm{Y}, y\,z + \mathrm{Z} \sin. \mathrm{Z}, y\,z;$$

and similarly for the other two, x and y,

$$y \sin. y, x\,z = \mathrm{X} \sin. \mathrm{X}, x\,z + \mathrm{Y} \sin. \mathrm{Y}, x\,z + \mathrm{Z} \sin. \mathrm{Z}, x\,z,$$

$$z \sin. z, x\,y = \mathrm{X} \sin. \mathrm{X}, x\,y + \mathrm{Y} \sin. \mathrm{Y}, x\,y + \mathrm{Z} \sin. \mathrm{Z}, x\,y.$$

452. Another useful method of transformation from rectangular axes to others also rectangular, is the following:

Let the equations to the axes of X, Y and Z be respectively

$$\left.\begin{array}{l} x = \alpha\,z \\ y = \beta\,z \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha_1\,z \\ y = \beta_1\,z \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha_2\,z \\ y = \beta_2\,z \end{array}\right\}$$

and let

$$m = \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}}, \quad m_1 = \frac{1}{\sqrt{1 + \alpha_1^2 + \beta_1^2}}, \quad m_2 = \frac{1}{\sqrt{1 + \alpha_2^2 + \beta_2^2}}$$

then by art. (402.) we have

$$\cos. \mathrm{X}\,x = m\,\alpha, \quad \cos. \mathrm{X}\,y = m\,\beta, \quad \cos. \mathrm{X}\,z = m; \ \&c.$$

Hence by substitution, the first formulas for transformation in art. (448.) become

$$x = m\,\alpha\,\mathrm{X} + m_1\,\alpha_1\,\mathrm{Y} + m_2\,\alpha_2\,\mathrm{Z}$$

$$y = m\,\beta\,\mathrm{X} + m_1\,\beta_1\,\mathrm{Y} + m_2\,\beta_2\,\mathrm{Z}$$

$$z = m\,\mathrm{X} + m_1\,\mathrm{Y} + m_2\,\mathrm{Z}.$$

And the nine angles in (1) are replaced by the six unknown terms $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2$.

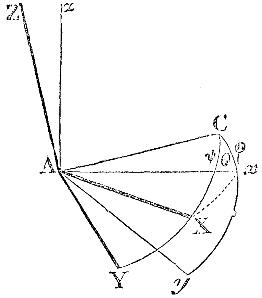
Instead of these systems, we may obtain a system involving only five arbitrary constants by supposing the solid trihedral angle formed by the original co-ordinate planes to turn about the origin into a new position: such a system has been ably discussed by M. Gergonne in the "Annales de Maths.," tome vii. p. 56.

453. It appears throughout these articles that only three arbitrary quantities are absolutely necessary; and therefore it might be supposed that formulas for transformation would be obtained involving only three angles: such formulas have been discovered by Euler, and as they are generally useful in various branches of analysis, we proceed to their investigation.

Let A C be the intersection of the original plane of $x\,y$ with the new plane of X Y, and suppose the plane C X Y A to lie above the plane C $x\,y$ A, which last we may assume to be the plane of the paper.

Let a sphere be described with centre A and radius unity, cutting all the axes in the points indicated by their respective letters.

Let $C\,x = \phi$, $C\,X = \psi$, and let the angle $X\,C\,x$ between the planes $x\,y$ and $X\,Y$ be called θ.



Then the object is to substitute in formula (1) art. (448.) the values of the cosines in terms of the new variables ϕ, ψ, and θ.

This is effected by means of the elementary theorem in spherical trigonometry for finding one side of a triangle in terms of the other two and the included angle. In the triangles $C\,X\,x$ and $C\,Y\,x$, we have

$$\cos. X\,x = \cos.\theta \sin\psi \sin.\phi + \cos.\psi \cos.\phi$$

$$\cos. Y\,x = \cos.\theta \sin.(90^\circ + \psi) \sin.\phi + \cos.(90^\circ + \psi) \cos.\phi$$

$$= \cos.\theta \cos.\psi \sin.\phi - \sin.\psi \cos.\phi.$$

Similarly $\cos. X\,y$ and $\cos. Y\,y$ may be found.

Also, supposing $Z\,x$ and $Z\,C$ to be joined by arcs of the sphere, we have from the triangle $Z\,C\,x$

$$\cos. Z\,x = \cos. Z\,C\,x \sin. Z\,C \sin. C\,x + \cos. Z\,C \cos. C\,x$$

$$= \cos.(90^\circ + \theta) \sin. 90^\circ \sin.\phi + \cos. 90^\circ \cos\phi$$

$$= -\sin.\theta \sin.\phi.$$

Similarly $\cos. Z\,y$, $\cos. X\,z$, and $\cos. Y\,z$ may be determined.

And $\cos. Z\,z = \cos.\theta$; hence the system (1) becomes

$$x = X(\cos.\theta \sin.\psi \sin.\phi + \cos.\psi \cos.\phi)$$
$$+ Y(\cos.\theta \cos.\psi \sin.\phi - \sin.\psi \cos.\phi)$$
$$- Z \sin.\theta \sin.\phi$$

$$y = X(\cos.\theta \sin.\psi \cos.\phi - \cos.\psi \sin.\phi)$$
$$+ Y(\cos.\theta \cos.\psi \cos.\phi + \sin.\psi \sin.\phi)$$
$$- Z \sin.\theta \cos.\phi$$

$$z = X \sin.\theta \sin.\psi + Y \sin.\theta \cos.\psi + Z \cos.\theta.$$

These are the formulas investigated, but in a different manner, by Laplace, "Méc. Cél." i. p. 58. They will be found in most works on this subject, but often with some slight alteration in the algebraic signs of the terms, arising from the various positions of A C.

THE INTERSECTION OF A SURFACE BY A PLANE.

454. The last system may be advantageously employed in finding the nature of the intersection of curve surfaces made by planes. If we propose to cut a surface, as a cone for example, by a plane, we should eliminate z from the equations to the surface and plane; but this gives us the equation to the projection of their intersection on $x\,y$, not the equation to the intersection itself; and as the projection will not always suffice to determine the nature of a curve, it is requisite to find the equation to that curve traced on the cutting plane.

This may be done by a transformation of co-ordinates.

Let the cutting plane be that of X Y, and the trace A C the axis of X, the surface will then be referred to new axes X, Y, Z, of which X and Y are in the cutting plane. By putting $Z = 0$ in the equation thus transformed, we shall have the intersection of the surface with the plane X Y, which is the intersection required.

Now, as the present object is only to obtain the curve of intersection, we may at first put $Z = 0$, and then transform the equation.

Let therefore $Z = 0$, and the angle C A X or $\psi = 0$, then the last formulas become

$$x = \quad X \cos.\phi + Y \sin.\phi \cos.\theta$$
$$y = - X \sin.\phi + Y \cos.\phi \cos.\theta$$
$$z = \quad Y \sin.\theta.$$

These formulas may be separately investigated, with great ease, without deduction from the general case.—See "Francœur," vol. ii. art. 369, or "Puissant, Géométrie," art. 134.

455. In applying these formulas to a particular case, a little consideration will greatly alleviate the labour of transformation: thus, in many cases, we may suppose the cutting plane to be perpendicular to $x\,z$, without at all diminishing the generality of the result, but only adding much to its simplicity; for in this case the trace A C either coincides with A y or y A produced, and therefore $\phi = 90°$; hence the ast formulas become

$$x = + Y \cos.\theta$$
$$y = - X$$
$$z = \quad Y \sin.\theta.$$

These formulas may be readily investigated by drawing a figure like the last, but letting A C, A X and y A produced coincide, $\phi = 90°$ and C Y $= 90°$, and then taking the original formulas (1) in art. 448.

456. If in the above cases the origin is also changed, we must intro duce the quantities a, b, c into the left side of the above equations.

CHAPTER VI.

THE SPHERE AND SURFACES OF REVOLUTION.

457. A CURVE surface as a sphere being given for discussion, we proceed as in plane geometry to find its equation from some known property of the surface; and generally we arrive at a relation between three unknown quantities x, y, and z, which relation is expressed by the symbol $f(x, y, z) = 0$, or $z = f(x, y)$. This equation is called the equation to the surface, and it corresponds to all points of the surface, and to it alone.

458. Conversely, an equation of the form $f(x, y, z) = 0$, where x, y, and z represent the co-ordinates of a point, refers to some surface. That it cannot belong to all the points in a solid may be thus shown.

Let there be two equations $f(x, y, z) = 0$, and $f'(x, y, z) = 0$; giving to x, y, and z the same values in both these equations, and then eliminating z, we have the equation to the intersection of the above loci projected on the plane of xy: this equation is of the form $\phi(x\,y) = 0$, and therefore it belongs to a *line*. Similarly the projections of the intersection on the other co-ordinate planes are *lines*; but if the projections of a locus on three different planes are lines, the locus itself must be a line, that is, it cannot be a surface. Hence the intersection of the two loci of $f(x, y, z) = 0$, and $f'(x, y, z) = 0$ being a line, each of these equations must belong to a surface.

459. Surfaces as well as lines are divided into orders, and for the same object, to avoid the confusion of ideas and to allow us to unite the important properties of generality and simplicity in our investigations as far as possible. Hence a plane which is the locus of a simple equation between three unknown quantities is called a surface of the first order; the locus of an equation of two dimensions between three unknown quantities is called a surface of the second order, and so on. The length, rather than the difficulty of the mathematical operations, renders this part of the subject tedious. Hence we shall omit many of the investigations which merely require manual labour, and rather dwell upon what we consider the important steps.

A much more serious difficulty arises from the state of the figures: we cannot give complete graphical illustrations of this part of geometry, and a mind unaccustomed to the conception of solid figures cannot always comprehend the meaning of the corresponding analytical results. We have endeavoured to obviate this difficulty as much as possible by descriptions of what the figures intend to represent, and to these descriptions we beg the particular attention of our readers, for we are convinced that this part of geometry is by no means difficult, if attention be paid to the form of the body; but without this care it is quite unintelligible.

We commence with the discussion of the Sphere.

THE SPHERE.

460. To find the equation to the surface of a sphere.

Let the surface be referred to rectangular axes, and let x, y, z be the co-ordinates of any point on the surface, and a, b, c the corresponding co-ordinates of the centre. Then since the surface is such that the distance of any point in it from the centre of the sphere is constant or equal to a line r, called the radius, we have by art. (388.)

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

461. This equation will assume various forms corresponding to the position of the centre.

Let the centre be in the plane of $x\,y$ $\therefore c = 0$,

$$\therefore (x-a)^2 + (y-b)^2 + z^2 = r^2.$$

Let the centre be on the axis of z $\therefore a = 0$, and $b = 0$,

$$\therefore x^2 + y^2 + (z-c)^2 = r^2.$$

462. Let the centre be the origin $\therefore a = b = c = 0$, and the equation is

$$x^2 + y^2 + z^2 = r^2.$$

And this is the equation to the surface of the sphere most generally used.

463. The general equation upon expansion becomes

$$x^2 + y^2 + z^2 - 2\,a\,x - 2\,b\,y - 2\,b\,z + a^2 + b^2 + c^2 - r^2 = 0.$$

And hence the sphere corresponding to any equation of this form may be described as for the circle, art. 67.

464. The sections of a surface made by the co-ordinate planes are called the principal sections of the surface, and the boundaries of the principal sections are called the traces of the surface on the co-ordinate planes.

The equation to a trace is determined by putting the ordinate perpendicular to the plane of the trace $= 0$ in the general equation. Thus, to find the curve in which the sphere cuts the plane of $x\,y$, put $z = 0$, and then we have the equation to the points where the plane and sphere meet, which in this case is

$$x - a)^2 + (y-b)^2 + c^2 = r^2.$$

Hence the section on $x\,y$ is a circle as long as x and y have real values And, similarly, the other traces are circles.

The theorem that the intersection of any plane with a sphere is a circle, is best proved geometrically, as in Geometry, *b.* v. 19.

465. To find the equation to the tangent plane to a sphere.

Let $x_1\,y_1\,z_1$ be the co-ordinates of the point on the surface through which the tangent plane passes, and let the equation to the spherical surface be

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2;$$

then the equation to the plane passing through the point $x_1\,y_1\,z_1$ is

$$m\,(x-x_1) + n\,(y-y_1) + p\,(z-z_1) = 0.$$

Also, the equations to the radius passing through the points $(a\ b\ c$ $(x_1\,y_1\,z_1)$ are

$$x - x_1 = \frac{x_1 - a}{z_1 - c}(z - z_1),\ y - y_1 = \frac{y_1 - b}{z_1 - c}(x - x_1).$$

And since every line in the tangent plane, and therefore the plane itself, is perpendicular to the radius at the point of tangence, we have from the equations to the plane and line

$$\frac{m}{p} = \frac{x_1 - a}{z_1 - c},\ \frac{n}{p} = \frac{y_1 - b}{z_1 - c}\ (426.)$$

Hence the equation to the tangent plane becomes

$$\frac{x_1 - a}{z_1 - c}(x - x_1) + \frac{y_1 - b}{z_1 - c}(y - y_1) + z - z_1 = 0,$$

or, $(x_1 - a)(x - x_1) + (y_1 - b)(y - y_1) + (z_1 - c)(z - z_1) = 0$

This equation may be modified by means of the condition

$$(x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2 = r^2,$$

or, $(x_1 - a)(x_1 - a) + (y_1 - b)(y_1 - b) + (z_1 - c)(z_1 - c) = r^2.$

Adding this equation, term by term, to the above one for the tangent plane, we have

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) + (z_1 - c)(z - c) = r^2.$$

466. If the origin is in the centre of the sphere, the equation to the tangent plane is

$$x x' + y y' + z z' = r^2,$$

which equation is at once obtained from that to the sphere $x^2 + y^2 + z^2 = r^2$, or, $x x + y y + z z = r^2$, by putting $x x'$, $y y'$, and $z z'$ for $x x$, $y y$, and $z z$ respectively.

The line in which the tangent plane cuts any co-ordinate plane is found by putting the ordinate perpendicular to that plane $= 0$; and the point in which the tangent plane cuts any axis is found by putting the two variables measured along the other axes each 0.

467. The equation to the spherical surface referred to oblique co-ordinates by (440.) is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 + 2(x - a)(y - b)\cos. XY + 2(x - a)(z - c)\cos. XZ + 2(y - b)(z - c)\cos. YZ = r^2.$$

ON COMMON SURFACES OF REVOLUTION.

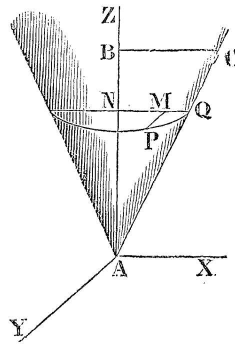
468. A right cone is formed by the revolution of the hypothenuse of a right-angled triangle about one of its sides.

Let A C be the side which revolves about A B as an axis, so that any section Q P perpendicular to the axis is a circle.

Let A X, A Y, A Z be the rectangular axes to which the cone is referred, having the origin at the vertex of the cone, and the axis of Z coincident with the axis of the cone.

Let $AN = z$, $NM = x$, $MP = y$ be the co-ordinates of any point on the surface.

Then the squares on N M and M P = the square on N P



and N P = N Q = A N tan. C A B,

therefore the equation to the surface is

$$x^2 + y^2 = e^2 z^2,$$

where e = tangent of the semiangle of the cone.

469. Let the line A C be a curve, as a parabola, for example, in which case the surface is called the common paraboloid.

Let the equation to the generating parabola A Q C be N Q = $\sqrt{p z}$.
Then the squares on N M, M P = the square on N P = the square on N Q,

$$\therefore x^2 + y^2 = p z.$$

470. Let A C be an ellipse, centre and origin at B.

Let B N = z, N M = x, and M P = y, C B = b and B A = a.

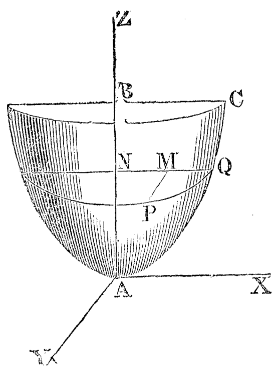
Then the squares on N M and M P = the square on N Q; and N Q being an ordinate to the ellipse A Q C, whose semiaxes are a and b, we have

$$\text{N Q} = \frac{b}{a} \sqrt{a^2 - z^2},$$

and therefore the equation to the surface is

$$x^2 + y^2 = \frac{b^2}{a^2} (a^2 - z^2);$$

$$\text{or, } x^2 + y^2 + \frac{b^2}{a^2} z^2 = b^2.$$



Let a and b change places in the equation, we have then for the surface of revolution round the axis minor the equation

$$x^2 + y^2 + \frac{a^2}{b^2} z^2 = a^2.$$

The former surface is called the prolate spheroid, the latter the oblate spheroid.

471. The equation to the hyperboloid round the transverse axis is

$$x^2 + y^2 - \frac{b^2}{a^2} z^2 = -b^2.$$

And putting a for b and b for a, we have the surface by revolution round the conjugate axis.

472. In general the equation to all these surfaces may be comprehended under the form $x^2 + y^2 = f(z)$ if A Z be the axis of revolution; or, $z^2 + y^2 = f(x)$ if A X be the axis of revolution.

To find the curve of intersection of a plane and a surface of revolution.

473. Let the section be made by a plane perpendicular to $x z$, and as the nature of the curve is the same in whatever part of the cutting plane we place the origin, we shall let the origin be in the plane $x z$.

Then the formulas for transformation are

$$x = a + y \cos.\theta$$
$$y = -x$$
$$z = c + y \sin.\theta.$$

Hence by substitution in the equation to a surface, we shall have the required curve of intersection.

474. Let the surface be a paraboloid

$$x^2 + y^2 = p z$$
$$\therefore (a + y \cos.\theta)^2 + x^2 = p\,(c + y \sin.\theta)$$

or, $y^2 (\cos.\theta)^2 + x^2 + (2\,a \cos.\theta - p \sin.\theta)\,y = 0$, since $a^2 = p\,c$;

hence the curve of intersection is a line of the second order.

It is an ellipse generally (76); a circle if $\theta = 0$; and a parabola similar to the generating one, if $\theta = 90°$.

475. Let the surface be the spheroid formed by the revolution of an ellipse round its axis major

$$x^2 + y^2 + \frac{b^2}{a^2} z^2 = b^2;$$

by substitution this equation becomes

$$y^2 \left\{ (\cos.\theta)^2 + \frac{b^2}{a^2} (\sin.\theta)^2 \right\} + x^2 + 2\,y \left\{ c \frac{b^2}{a^2} \sin.\theta - a_1 \cos.\theta \right\} = 0.$$

This is the equation to an ellipse generally, and to a circle when $\theta = 0$.

476. Let the surface be the hyperboloid, whose equation is

$$x^2 + y^2 - \frac{b^2}{a^2} z^2 = -b^2,$$

the sections will be found to depend on the value of $\tan.\theta$: if $\tan.\theta$ is less than $\frac{b}{a}$, the curve is an ellipse; if it is equal to $\frac{b}{a}$, the curve is a parabola; and if $\tan.\theta$ is greater than $\frac{b}{a}$, it is an hyperbola; and lastly, a circle if $\theta = 0$

CHAPTER VII.

SURFACES OF THE SECOND ORDER.

477. THE general equation to surfaces of the second order is

$$a x^2 + b y^2 + c z^2 + 2 d x y + 2 e x z + 2 f y z + 2 g x + 2 h y + 2 i z + k = 0$$

the number 2 being prefixed to some of the terms merely for convenience. In order to discuss this equation, that is, to examine the nature and position of the surfaces which it represents, we shall render it more simple by means of the transformation of co-ordinates.

Let the origin be transferred by putting

$$x = x' + m,\ y = y' + n,\ z = z' + p,$$

substituting these values in the general equation, and then putting the terms containing the first powers of the variables each $= 0$, we have the equation

$$a x'^2 + b y'^2 + c z'^2 + 2 d x' y' + 2 e x' z' + 2 f y' z' + k' = 0.$$

This equation remains the same if we change x', y', z', into $-x', -y', -z'$ respectively; thence we conclude that any straight line drawn through the origin, and intercepted by the surface, will be divided into two equal parts at the origin; this new origin therefore will be the centre of the surface, attributing to this expression the same signification as we did in treating of curves of the second order (81.)

478. The values of m, n, and p, are to be determined from the three equations

$$\begin{aligned} a m + d n + e p + g &= 0, \text{ co-efficient of } x', \\ b n + d m + f p + h &= 0, \quad \ldots \quad y', \\ c p + e m + f n + i &= 0, \quad \ldots \quad z'. \end{aligned}$$

Eliminate p from the first and second of these equations, and also from the first and third, then from the two resulting equations eliminate n, and we shall arrive at an equation of the first order involving m, whence we have the value of m, and therefore of n and p.

The denominator of the values of m, n and p is

$$a b c + 2 d e f - a f^2 - b e^2 - c d^2$$

hence, if this quantity $= 0$, the values of m and p are infinite, or the surface has no centre when there is this relation among the co-efficients of the original equation. This circumstance corresponds to the case of the parabola in lines of the second order (81.)

479. To destroy the co-efficients of the terms involving $x' y'$, $x' z'$, and $y' z'$, we must have recourse to another transformation of co-ordinates.

Taking the formulas in (452) we have

$$x' = m\,\alpha\,x'' + m_1\,\alpha_1\,y'' + m_2\,\alpha_2\,z''$$
$$y' = m\,\beta\,x'' + m_1\,\beta_1\,y'' + m_2\,\beta_2\,z''$$
$$z' = m\,x'' + m_1\,y'' + m_2\,z''$$

Substituting in the general equation, and then putting the co-efficients of $x''\,y''$, $x''\,z''$, and $y''\,z''$, each $= 0$, we have the three equations

$$(a\,\alpha + d\,\beta + e)\,\alpha_1 + (d\,\alpha + b\,\beta + f)\,\beta_1 + e\,\alpha + f\,\beta + c = 0 \ldots\ldots x''\,y''$$
$$(a\,\alpha + d\,\beta + e)\,\alpha_2 + (d\,\alpha + b\,\beta + f)\,\beta_2 + e\,\alpha + f\,\beta + c = 0 \ldots\ldots x''\,z''$$
$$(a\,\alpha_2 + d\,\beta_2 + e)\,\alpha_1 + (d\,\alpha_2 + b\,\beta_2 + f)\,\beta_1 + e\,\alpha_2 + f\,\beta_2 + c = 0 \ldots y''\,z''$$

Our object is now to ascertain if this transformation can always be effected, that is, to determine the possibility of the values of the six unknown quantities in the last three equations.

480. The equations to the new axis of y'' are $x = \alpha_1\,z$, $y = \beta_1\,z$ (452.); hence, by substitution, the first of the above three equations becomes

$$(a\,\alpha + d\,\beta + e)\,x + (d\,\alpha + b\,\beta + f)\,y + (e\,\alpha + f\,\beta + c)\,z = 0,$$

which is the equation to a plane passing through the origin.

Now the co-ordinates of every point in this plane satisfy the condition that the co-efficient of $x''\,y'' = 0$, that is, give the necessary relation between α_1 and β_1; hence, if the new axis of y'' be drawn in this plane the condition is still satisfied. Thus, the direction of the axis of x'' being quite arbitrary, that of y'' is determined to be in the particular plane given above; and the term $x''\,y''$ is gone.

Again, by a similar elimination of α_2 and β_2 from the co-efficient of $x''\,z''$, and from the equations of z'' ($x = \alpha_2\,z$, $y = \beta_2\,z$), we have, from the similarity of the equations, the same plane as before; hence, if the axis of z'' be also drawn in this plane, the term $x''\,z''$ will disappear.

Also, α_2 and β_2 being thus obtained, the relation between α_1 and β_1 may be found from the co-efficient of $y''\,z'' = 0$.

Thus, fixing upon any position of the axis of x'', that is, giving any values to α and β, we have determined a plane passing through the origin, in which plane any two straight lines whatever drawn from the origin may be the axes of y'' and z'', and one of them as z'' being so drawn, α_2 and β_2 are given, and then the relation between α_1 and β_1 is determined from the co-efficient of $x''\,y'' = 0$.

But since the relation between these quantities α_1 and β_1, and not the quantities themselves, is given by the last equation, it appears that there are an infinite number of systems to which, if the axes be transferred, the products of the variables may be destroyed.

481. Let the new axes be rectangular.

In this case the axis of x'' must be perpendicular to the plane of $y''\,x''$, or the line whose equations are $x = \alpha\,z$, $y = \beta\,z$ is perpendicular to the plane

$$(a\,\alpha + d\,\beta + e)\,x + (d\,\alpha + b\,\beta + f)\,y + (e\,\alpha + f\,\beta + c)\,z = 0$$

$$\therefore\ a\,\alpha + d\,\beta + e = (e\,\alpha + f\,\beta + c)\,\alpha \ (426)$$
$$d\,\alpha + b\,\beta + f = (e\,\alpha + f\,\beta + c)\,\beta$$

Substituting in the first of these equations the value of α obtained from the second, we have the following equation for β:

$$\{(a-b)fe+(f^2-e^2)d\}\beta^3$$
$$+\{(a-b)(c-b)e+(2d^2-f^2-e^2)e+(2c-a-b)fd\}\beta^2$$
$$+\{(c-a)(c-b)d+(2c^2-f^2-d^2)d+(2b-a-c)fe\}\beta$$
$$+\{(a-c)fd+(f^2-d^2)e\}=0.$$

This equation of the third degree has at least one real value for β, and hence a real value of α; thus the position of the axis of x'' is found, and also the position of the perpendicular plane in which y'' and z'' are situated.

Again, we might find a plane $x''\,z''$ perpendicular to y'', and such that the terms in $x''\,y''$, $y''\,z''$ should disappear, and the necessary conditions will, as appears from the similarity of the equations, lead to the same equation of the third degree in β_1, and the same is true for the axis of z''.

Hence the three roots of the above equation of the third degree are the three real values of β, β_1 and β_2.

These three quantities give the three corresponding values of α, α_1 and α_2, and since there are only one value of each quantity, it appears that there is only one system of rectangular axes to which the curve surface can be referred so as not to contain the products of the variables. For further information on this subject, see "Annales Math." ii. p. 144.

482. By the last transformation, the equation when the locus has a centre is reduced to the form

$$a_1\,x''^2+b_1\,y''^2+c_1\,z''^2+k_1=0$$
$$\text{or, } \mathrm{L}\,x^2+\mathrm{M}\,y^2+\mathrm{N}\,z^2=1$$

by substitution and the suppression of accents, which are no longer necessary.

The order of transformation might have been inverted, by first destroying the products of the variables exactly in every respect as in the last article, and then the resulting equation must be deprived of three terms by a simple change of the origin; the result, after both transformations, is

$$\mathrm{L}\,x^2+\mathrm{M}\,y^2+\mathrm{N}\,z^2+\mathrm{P}\,x=0.$$

483. The central equation involves three distinct cases, which depend on the signs of the quantities L, M, and N.

(1) They may be all positive.
(2) Two may be positive, and the third negative.
(3) One may be positive, and the other two negative.

They cannot be all negative.

Substituting for L, M and N, the constants $\frac{1}{a^2}\ \frac{1}{b^2}\ \frac{1}{c^2}$ respectively, where a is $> b$ and $b > c$, the three cases are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

The readiest way of obtaining the form of these surfaces is by sections either in planes parallel to the co-ordinate planes, or on the co-ordinate planes. We remark again, that in the latter case they are called the principal sections or traces.

THE ELLIPSOID.

484. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

For the trace on $x\,y$, $z = 0$, $\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

.... $x\,z$, $y = 0$, $\therefore \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$

.... $y\,z$, $x = 0$, $\therefore \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Therefore the principal sections are ellipses.

Let $z = m$ $\therefore$ the section parallel to $x\,y$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{m^2}{c^2}$ *

$y = n$ .... $x\,z$ is $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{n^2}{c^2}$

$x = p$ .... $y\,z$ is $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{p^2}{c^2}$.

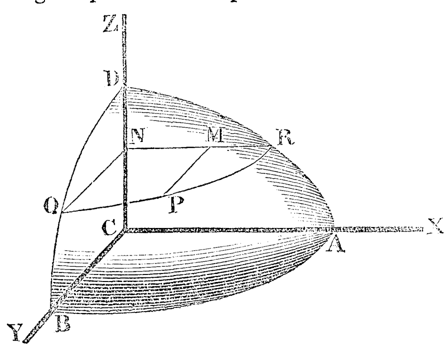
The first of these equations is an ellipse from m, or $z = 0$ to $z = c$; when $z = c$ the curve becomes a point, and when z is greater than c the ellipse is imaginary, therefore the surface is limited in the direction of z. Similarly it may be proved, that the other sections are ellipses, and the surface is limited in the directions of x and y. From the circumstance of this surface being thus limited in every direction, and also from the above sections being all ellipses, this surface is called the ellipsoid.

The diameters $2\,a$, $2\,b$, $2\,c$ of the principal sections are called the diameters of the ellipsoid, and their extremities are the vertices of the surface.

If $b = a$, the equation becomes $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$, which is the equation to a spheroid by revolution round the axis of z.

If any other two co-efficients are equal, we have spheroids round the other axes; and if $a = b = c$, the surface becomes a sphere.

485. To render the conception of this surface clear we subjoin a figure representing the eighth part of an ellipsoid.



* This equation belongs to the projection on $x\,y$, but since the plane of $x\,y$ is parallel to that of $z = m$, the projection is exactly the same in form as the curve of section itself.

A B is part of the ellipse on $x\,y$
A D . . . . $x\,z$
B D . . . . $y\,z$,

and the section Q P R parallel to $x\,y$ is also an ellipse.

The surface may be conceived to be generated by a variable ellipse C A B moving upwards parallel to itself with its centre in C Z. Let N Q R be one position of this variable ellipse; and let

C N $= z$,	C A $= a$,	N R $= x_1$;
N M $= x$,	C B $= b$,	N Q $= y_1$
M P $= y$,	C D $= c$,	

Then from the ellipse Q P R we have

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1$$

Also from the ellipses D R A and D Q B we have

$$\frac{x_1^2}{a^2} + \frac{z^2}{c^2} = 1, \text{ and } \frac{y_1^2}{b^2} + \frac{z^2}{c^2} = 1$$

Therefore $\frac{x_1^2}{a^2} = \frac{y_1^2}{b^2}$; and multiplying the first equation by $\frac{x_1^2}{a^2}$ or its equal $\frac{y_1^2}{b^2}$, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x_1^2}{a^2} = 1 - \frac{z^2}{c^2}$.

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

THE HYPERBOLOID.

486. Case 2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

The principal sections are

$$\text{on } x\,y,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

$$\text{on } x\,z,\ \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \quad (2)$$

$$\text{on } y\,z,\ \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (3).$$

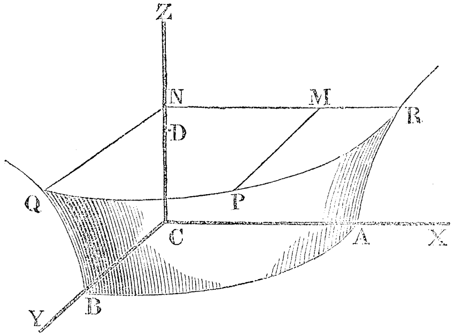
(1) is the equation to an ellipse whose axes are $2\,a$ and $2\,b$; (2) and (3) are hyperbolas with the same imaginary conjugate axis $2c\sqrt{-1}$; if x is less than a, or y less than b, z is imaginary.

Giving to z, y, and x the values m, n, and p, respectively, we have the section parallel to $x\,y$ an ellipse, to $y\,z$ and $x\,z$ hyperbolas.

487. The accompanying figure represents a portion of the eighth part of this surface. A B is the ellipse on $x\,y$, A R the hyperbola on $x\,z$, and B Q is the other hyperbola on $y\,z$. This surface may also be conceived

to be generated by a variable ellipse C A B moving parallel to itself with it centre in C Z. Let N Q R be one position of this variable ellipse ; and le

$CN = z$,	$CA = a$,	$NR = x_1$;
$NM = x$,	$CB = b$,	$NQ = y_1$;
$MP = y$,	$CD = c$,	



Then from the ellipse P Q R, we have

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1$$

Also from the hyperbolas A R and B Q we have

$$\frac{x_1^2}{a^2} - \frac{z^2}{c^2} = 1, \text{ and } \frac{y_1^2}{b^2} - \frac{z^2}{c^2} = 1,$$

therefore $\frac{x_1^2}{a^2} = \frac{y_1^2}{b^2}$; and multiplying the first equation by $\frac{x_1^2}{a^2}$ or its equal $\frac{y_1^2}{b^2}$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x_1^2}{a^2} = 1 + \frac{z^2}{c^2}$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

This surface is called the hyperboloid of one sheet because it forms one continuous surface or sheet.

If $a = b$ the surface becomes the common hyperboloid of revolution round the *conjugate* axis.

488. Through the origin draw a line, whose equations are $x = \alpha z$, $y = \beta z$, and substituting in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, we have

$$\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - \frac{1}{c^2}\right) z^2 = 1$$

$$\therefore z = \pm \frac{a\,b\,c}{\sqrt{b^2 c^2 \alpha^2 + a^2 c^2 \beta^2 - a\ b^2}};$$

hence this line meets the surface as long as the denominator of the fraction is real and finite; let $b^2 c^2 \alpha^2 + a^2 c^2 \beta^2 = a^2 b^2$, then the line only meets the surface at an infinite distance, or is an asymptote to the surface. The last equation gives the relation between α and β, when the corresponding line is an asymptote; and if for α and β we substitute their general values $\frac{x}{z}$ and $\frac{y}{z}$, we obtain an equation between x, y, z, whose locus will consist of all the asymptotes to the surface, because the co-ordinates of any point in it have the required relation above.

The equation to this surface is

$$b^2 c^2 x^2 + a^2 c^2 y^2 = a^2 b^2 z^2$$

$$\text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}:$$

We shall hereafter show (art. 514.) that this is the equation to a cone whose vertex is the origin, and whose base, or section parallel to the axis, is an ellipse.

489. Case 3.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The principal sections are

$$\text{on } x y, \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1)$$

$$\text{on } x z, \ \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad (2)$$

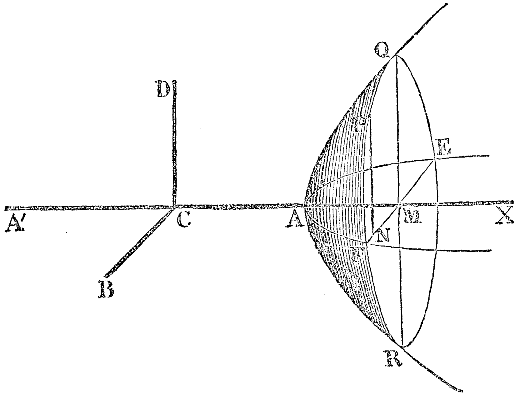
$$\text{on } y z, \ \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \qquad (3).$$

(1) is an hyperbola whose axes are $2a$ and $2b\sqrt{-1}$; (2) is an hyperbola whose axes are $2a$ and $2c\sqrt{-1}$; (3) is imaginary, therefore the plane of yz never meets the surface.

Of the sections parallel to the co-ordinate planes, those parallel to xy and xz are hyperbolas, and that parallel to yz is an ellipse, whose equation is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{p^2}{a^2} - 1;$$

hence this ellipse is imaginary, if p or x is less than $\pm a$; therefore, if two planes are drawn parallel to yz, and at distances $\pm a$ from the centre, no part of the surface can be between these planes.



In the figure E A F represents the hyperbolic section on $x y$, and Q A R that on $x z$; E Q F R is an elliptic section parallel to $y z$. There is an equal and opposite sheet with its vertex at A′; hence the surface is called the hyperboloid of two sheets.

490. The equation to the surface is deduced from the figure; let A M $= x$, M N $= y$, N P $= z$; Q M $= z_1$, M F $= y_1$;

Then from the elliptic section Q P F R we have

$$\frac{z^2}{z_1^2} + \frac{y^2}{y_1^2} = 1.$$

Also from the hyperbolas E A F and Q A R we have

$$\frac{y_1^2}{b^2} - \frac{x^2}{a^2} = -1, \text{ and } \frac{z_1^2}{c^2} - \frac{x^2}{a^2} = -1$$

therefore $\frac{y_1^2}{b^2} = \frac{z_1^2}{c^2}$; and multiplying the first equation by $\frac{z_1^2}{c^2}$ or its equal $\frac{y_1^2}{b^2}$, we have

$$\therefore \frac{z^2}{c^2} + \frac{y^2}{b^2} = \frac{z_1^2}{c^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

491. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, is the equation to the conical asymptote; hence both in case (2) and (3) we have the conical asymptote by omitting the constant term in the equations.

ON SURFACES WHICH HAVE NO CENTRE.

492. In this case the general equation can be deprived of the products of the variables, as in (479); it will then be of the form

$$a x^2 + b y^2 + c z^2 + 2 g x + 2 h y + 2 i z + k = 0.$$

In order to deprive this equation of three more terms, let

$$x = m + x', \ y = n + y', \ z = p + z',$$

$$\therefore ax'^2 + by'^2 + c z'^2 + 2 (a m + g) x' + 2 (b n + h) y' + 2 (c p + i) z' + k' = 0 :$$

Let the co-efficients of x', y' and $z' = 0$;

$$\therefore m = -\frac{g}{a}, n = -\frac{h}{b}, p = -\frac{i}{c} :$$

But since this class has no centre, the values of some, or all the quantities m, n, p, must be infinite; therefore, either one, two, or three of the co-efficients a, b, c, must $= 0$. Thus the original transformation which deprived the equation of the terms $x y$, $x z$, and $y z$, has of itself destroyed one or two of the co-efficients of x^2, y^2, or z^2; this corresponds to the case in art. 92. Now, all three co-efficients cannot $= 0$, for then we fall upon the equation to a plane: hence we have only two cases left, when a vanishes, or when a and b both vanish.

493. Let $a = 0$, then, as we have three quantities, m, n and p to determine, we may let $k' = 0$ as well as the co-efficients of y' and z'; hence the equation is reduced to the form

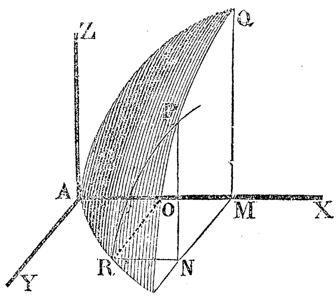
$$b\,y'^2 + c\,z'^2 + 2\,g\,x' = 0.$$

$$\text{or}\left(-\frac{b}{2\,g}\right)y'^2 + \left(-\frac{c}{2\,g}\right)z'^2 = x'.$$

This equation has two varieties depending upon the signs of the quantities $-\dfrac{b}{2\,g}$ and $-\dfrac{c}{2\,g}$.

494. Case 1. Let the signs of y'^2 and z'^2 be both alike and positive, (if they were negative we should change the sign of x' to reduce the equation to the same form) substituting $\dfrac{1}{l}$ for $-\dfrac{b}{2\,g}$, and $\dfrac{1}{l'}$ for $-\dfrac{c}{2\,g}$, and suppressing the accents on x, y and z as no longer necessary, the equation is of the form

$$\frac{y^2}{l} + \frac{z^2}{l'} = x.$$



For the principal sections we have

on $x\,y$, $y^2 = l\,x$. (1)
on $x\,z$, $z^2 = lx'$. (2)
on $y\,z$, $l'y^2 + l\,z^2 = 0$ (3)

(1) and (2) are parabolas extending on the side of x positive; (3) is a point, which is the origin itself.

For the sections parallel to

$$x\,y,\ \text{put } z = p,\ \therefore\ \frac{y^2}{l} = x - \frac{p^2}{l'} \quad (1)$$

$$x\,z,\ \text{put } y = n,\ \therefore\ \frac{z^2}{l'} = x - \frac{n^2}{l} \quad (2)$$

$$y\,z,\ \text{put } x = m,\ \therefore\ \frac{y^2}{l} + \frac{z^2}{l'} = m \quad (3)$$

(1) and (2) are parabolas, equal to those of the principal sections respectively, (the equation differing by a constant term, implies that the origin is differently situated with regard to the curve): (3) is an ellipse.

495. In the figure A Q and A R are parts of the parabolas on $x\,z$ and $x\,y$, and the surface is described by the motion of the parabola A Q, parallel to itself, its vertex moving along the parabola A R. Let P R N be one position of the generating parabola, and let A M $= x$, M N $= y$, N P $= z$, and draw R O parallel to A Y or M N; then from the parabola R P we have

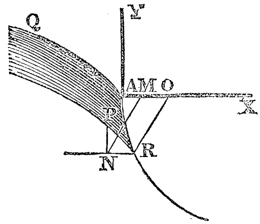
$$z^2 = l'\,\text{R N} = l'\,(\text{A M} - \text{A O}) = l'\left(x - \frac{y^2}{l}\right)$$

$$\therefore \frac{z^2}{l'} + \frac{y^2}{l} = x.$$

This surface is called the elliptic paraboloid, and is composed of one entire sheet, like the paraboloid of revolution.

496. Case 2. Let the signs of y'^2 and z'^2 be different.

$$\therefore \frac{y^2}{l} - \frac{z^2}{l'} = x.$$



For the principal sections we have

on $x y$, $y^2 = l x$ (1)
on $x z$, $z^2 = - l'x$ (2)
on $y z$, $l'y^2 - l z^2 = 0$ (3)

(1) and (2) are parabolas, the first corresponding to x positive, and the second to x negative; (3) belongs to two straight lines through the origin.

The sections in planes parallel to $x y$ and $x z$ are parabolas, and those parallel to $y z$ are hyperbolas.

497. A Q is the parabola on $x z$, and A R is that on $x y$; and the surface is described by the motion of the parabola A Q parallel to itself, its vertex moving along the parabola A R. Let R P N be one position of the generating parabola, and let A M $= x$, M N $= y$, and N P $= z$, and draw R O parallel to M N; then from the parabola R Q we have

$$z^2 = l' \text{ R N} = l' (\text{A O} - \text{A M}) = l' \left(\frac{y^2}{l} - x, \right)$$

$$\therefore \frac{y^2}{l} - \frac{z^2}{l'} = x,$$

This surface is called the hyperbolic paraboloid.

498. The equations to the elliptic and hyperbolic paraboloids may be deduced from those of the ellipsoid and hyperboloid of one sheet, as the equation to the parabola was deduced from that to the ellipse (228) by supposing the centre to be infinitely distant.

Let the origin be transferred to a vertex of the surface, by putting $x - a$ for x, then the equation to the ellipsoid and hyperboloid is

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1.$$

Let m and m' be the distances of the vertex from the foci or the sections on $x y$ and $x z$;

$$\therefore b^2 = a^2 - (a - m)^2 = 2 a m - m^2$$

$$\text{and } c^2 = 2 am' - m'^2;$$

therefore, by substitution, the equation

$$\frac{x^2}{a^2} - \frac{2 x}{a} + 1 + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

$$\text{becomes } \frac{x^2}{a^2} - \frac{2 x}{a} + \frac{y^2}{2 a m - m^2} \pm \frac{z^2}{2 a m' - m'^2} = 0$$

$$\text{or } \frac{x^2}{a} - 2 x + \frac{y^2}{2 m - \frac{m^2}{a}} \pm \frac{z^2}{2 m' - \frac{m'^2}{a}} = 0,$$

$$\text{or } \frac{y^2}{2 m} \pm \frac{z^2}{2 m'}, - 2 x = 0, \text{ when } a \text{ is infinite.}$$

And hence results obtained for the ellipsoid and hyperboloid will be rue for the paraboloids, after making the above substitutions.

499. We stated in article 492, that both a and b might vanish; in this case the equation will be

$$c z^2 + 2 g x + 2 h y + 2 i z + k = 0.$$

And by the transformation in art. 492, we cannot destroy the co-efficients of x and y, but we may destroy that of z, and also the constant term k; hence the transformed equation is reduced to the form

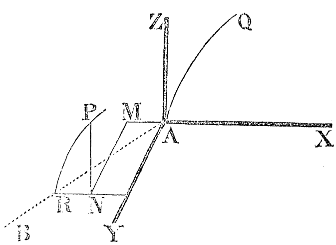
$$c z^2 + 2 g x + 2 h y = 0;$$

$$\text{or } z^2 = l x + l' y \text{ if } -\frac{2g}{c} = l, \text{ and } -\frac{2h}{c} = l'.$$

500. There are two cases depending on the signs of l and l', which may be both positive, or one positive and the other negative.

Case 1. l and l' both positive.

The section on $x y$ is $l x + l' y = o$ (1)
on $x z$ is $z^2 = l x$ (2)
on $y z$ is $z^2 = l' y$ (3)



(1) is a straight line A B; (2) is a parabola A Q; (3) is also a parabola, not in the figure; the sections on the planes parallel to the above are similar in each case. The surface is formed by the motion of the parabola A Q parallel to itself, its vertex describing the straight line A R; let R P N be one position of the generating parabola; let A M $= x$, M N $= y$, N P $= z$,

$$\text{then } z^2 = l.\, \text{R N} = l\left(\frac{l'}{l} y + x\right) = l' y + l x.$$

Since this surface is a cylinder with a parabolic base, it is not usually classed among the surfaces of the second order.

Case 2. If the signs of l and l' be different, the surface will be the same, but situated in a different manner.

CHAPTER VIII.

CYLINDRICAL AND CONICAL SURFACES.

501. Our notion of surfaces will be very much enlarged, if we take into consideration the general character of classes of surfaces, defining them by their peculiar method of generation, and then expressing that definition in a general algebraical form. For example, we have been accustomed, in common geometry, to consider a cylinder as a surface generated by a straight line, which is carried round the circumference of a given circle, and always parallel to a given straight line. (Geom. b. v,

def. 1.) But it is evident that if the base be not a circle, but any other curve, as a parabola, for instance, we shall have a surface partaking of the essential cylindrical character, and which, with others of the same kind, come under a more extended definition; and similarly for conical and many other surfaces.

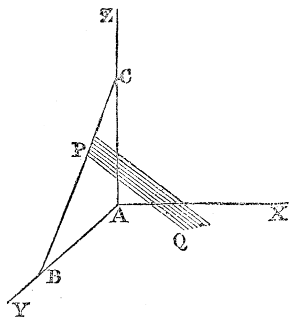
Having seized upon this general character, method of generation, or law by which the lines are compelled to move, the next step is to express this fact in algebraical language; that is, to obtain an equation between co-ordinates x, y, and z, of any point on the surface, which equation shall belong to the class of surfaces in the first instance, and then can be adapted to any particular surface in that class.

THE PLANE.

502. In order to prepare the reader for this subject, we shall take a simple case: to find the surface generated by the motion of a straight line, parallel to itself, and constrained to pass through a given straight line.

Let A X, A Y, A Z be rectangular axes, and let the equations to the given straight line B C (supposed for the sake of simplicity to be in the plane of y z) be

$$\left.\begin{aligned} n\,Y + p\,Z &= 1 \\ X &= 0 \end{aligned}\right\} \quad 1$$



Also, let the equations to the generating line P Q, in any one of its positions, be

$$\left.\begin{aligned} x &= \alpha\, z + a \\ y &= \beta\, z + b \end{aligned}\right\} \quad 2.$$

Now, α and β are the tangents of the angles which the projections of P Q make with the axes A X and A Y respectively; and in the motion of P Q, parallel to itself, the projections also remain parallel to themselves respectively; and hence α and β are always constant, and therefore are known or given quantities. But a and b being the co-ordinates of the point where the line P Q meets the plane of x y, they change with every change of position of P Q; and therefore, being variable, must not appear in the final equation to the surface. Now, these variable quantities, a and b, can be expressed in terms of the other variable quantities x, y, z; and hence we can thus estimate them from the two given systems above.

At the point P, where P Q meets B C, we have, by comparison of (1) and (2),

$$X = x = 0$$
$$Z = z = -\frac{a}{\alpha}$$
$$Y = y = -\frac{a\beta}{\alpha} + b$$

But the system (1) is true for any values of X, Y, Z; therefore, by substitution in (1), we have

$$n\left(-\frac{a\beta}{\alpha} + b\right) + p\left(-\frac{a}{\alpha}\right) = 1$$

and this is the equation connecting a and b together, or expressing the relation which the variable quantities a and b have to each other, or the relation which any quantities equal to a and b have to each other; that is, substituting for a and b the quantities $x - \alpha z$, and $y - \beta z$ from (2), we shall have the relation between the quantities x, y, and z, which is called the equation to a surface.

$$\therefore -n\frac{\beta}{\alpha}\left(x - \alpha z\right) + n\,(y - \beta z) - \frac{p}{\alpha}\left(x - \alpha z\right) = 1;$$

$$\text{or, } -\frac{n\beta + p}{\alpha}\,x + n y + p z = 1,$$

which is the equation to a plane; and this is the most general method of determining the equation to a plane; for it can be thus found for any system of co-ordinate axes, and it is determined from the most obvious character of the plane.

We now proceed to the discussion of surfaces formed by the motion of a straight line constrained to move after some given law or condition.

ON CYLINDRICAL SURFACES.

503. Definition. A cylindrical surface is generated by a straight line, which moves parallel to itself in space, and describes, with its extremity, a given curve.

The straight line which moves is called the Generatrix; and the given curve is called the Directrix.

To find the equation to the surface,

Let the equation to the generatrix, in any one of its positions, be

$$x = \alpha z + a$$
$$y = \beta z + b$$

Now, the generatrix, in its movement, always moving parallel to itself; the quantities α and β remain the same for every position of the generatrix; but the quantities a and b, which are the co-ordinates of the point where the generatrix meets the plane of $x\,y$, are constant for the same position of the generatrix, but vary when the generatrix passes from one position to another. Thus, when any point on the surface changes its position without quitting the generatrix, a and b are both constant; and when the point moves from one position of the generatrix to another, a and b are both variable; hence these two quantities, being constant together, and variable together, must be dependent on each other in some way or another; which general dependence is expressed by saying that one of them is a function of the other

$$\therefore b = \phi(a);$$

or, putting for b and a their values as above, we have

$$y - \beta z = \phi(x - \alpha z),$$

which is the general equation to cylindrical surfaces.

504. The form of the function ϕ will depend upon the nature of the directrix in any particular case.

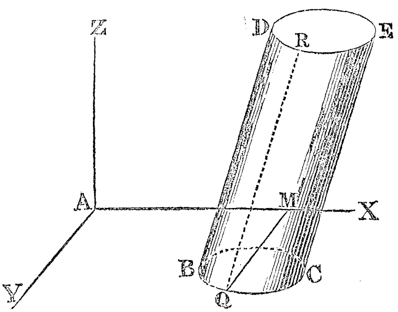
Let the equations to the directrix be

$$\left.\begin{array}{l} F(X, Y, Z,) = 0 \\ f(X, Y, Z,) = 0 \end{array}\right\}$$

Then as the generatrix must in all its positions meet the directrix, the equations to this curve and to the generatrix must exist simultaneously for the points of intersection; thus having four equations we may eliminate x, y, z, and arrive at an equation between a, b, and constant quantities, which will determine the form of the function ϕ.

Substituting in this equation for a and b their values $x - \alpha z$, $y - \beta z$, we have the actual equation to the particular cylinder required.

505. Ex. 1. Let the directrix be the circle B Q C, in the plane of $x\,y$, and let x_1 and y_1 be the co-ordinates of its centre; then the equations to the directrix are



$$\left.\begin{array}{l} (X - x_1)^2 + (Y - y_1)^2 = r^2 \\ Z = 0 \end{array}\right\} \quad (1)$$

Let B D, Q R, C E, be various positions of the generatrix whose general equation is

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} \quad (2)$$

to express that the generatrix meets the circle as at Q, the equations (1) and (2) must exist together

$$\therefore Z = z = 0$$
$$X = x = a$$
$$Y = y = b$$

substituting these values in (1), we have

$$(a - x_1)^2 + (b - y_1)^2 = r^2 \qquad (3)$$

hence the form of the function ϕ is determined.

Substituting in (3) the values of a and b from (2), we have

$$(x - \alpha z - x_1)^2 + (y - \beta z - y_1)^2 = r^2$$

This is the equation to an oblique cylinder, with circular base, situated in the plane of $x\,y$.

506. Let the centre of the circle be at the origin,

$$\therefore x_1 = 0 \text{ and } y_1 = 0$$
$$\therefore (x - \alpha z)^2 + (y - \beta z)^2 = r^2$$

And if the origin be at the extremity of a diameter parallel to the axis of x,

$$(x - \alpha z)^2 + (y - \beta z)^2 = 2r(x - \alpha z)$$

507. Let the axis of the cylinder be parallel to the axis of z; then α and β

each $= 0$, since they are the tangents of the angles which the projection of the generatrix on $x\,z$ and $y\,z$ make with A Z;

$$\therefore (x - x_1)^2 + (y - y_1)^2 = r^2;$$

and if the axis coincide with A Z, $x^2 + y^2 = r^2$, $z = 0$;
in these cases the cylinder is called a right cylinder, and its equation is the same as that of the directrix.

If the directrix be a circle on $x\,z$, the equation to the right cylinder will be

$$x^2 + z^2 = r^2.$$

508. Let the directrix be a parabola on $x\,y$, vertex at the origin, and axis coincident with the axis of x.

Then the equations to the directrix and generatrix are

$$\left.\begin{array}{l} Y^2 = p\,X \\ Z = 0 \end{array}\right\} 1 \qquad \left.\begin{array}{l} x = \alpha\,z + a \\ y = \beta\,z + b \end{array}\right\} 2;$$

therefore at the points of junction we have

$$\begin{array}{l} Z = z = 0 \\ X = x = a \\ Y = y = b; \end{array}$$

then by substituting in (1) we have

$$b^2 = p\,a$$

$$\therefore (y - \beta\,z)^2 = p\,(x - \alpha\,z)$$

which is the equation to an oblique parabolic cylinder, whose base is on $x\,y$.

509. Let the directrix be a parabola on $x\,z$, axis A X, and vertex at A; and let the generatrix be parallel to the plane $x\,y$.

The equations are

$$\left.\begin{array}{l} Z^2 = p\,X \\ Y = 0 \end{array}\right\} 1 \qquad \left.\begin{array}{l} y + \alpha\,x = a \\ z = b \end{array}\right\} 2$$

Then the equation to the surface is

$$z^2 = \frac{p}{\alpha}\,y + p\,x. \qquad \text{See article (499).}$$

ON CONICAL SURFACES.

510. Definition. A conical surface is generated by the movement of a straight line, which passes constantly through a given point, and also describes a given curve.

The given point is called the centre of the surface, the straight line which moves is called the generatrix, and the given curve is called the directrix.

Let a, b, c, be the co-ordinates of the centre; then the equations to the generatrix are

$$\begin{array}{l} x - a = \alpha\,(z - c) \\ y - b = \beta\,(z - c). \end{array}$$

Now when a point on the surface changes its position without quitting the generatrix, the quantities α, β are constant, but when the point passes from one generatrix to another, they are both variable; hence being constant together, and variable together, they are functions of one another;

$\therefore \beta = \phi\,(\alpha)$, or substituting their equals,

$$\frac{y - b}{z - c} = \phi\left(\frac{x - a}{z - c}\right)$$ which is the general equation to conical surfaces.

511. The form of the function ϕ will depend upon the nature of the directrix in any particular case.

By combining the equations to the generatrix and directrix we may, as for cylindrical surfaces, eliminate x, y, z, in a particular case, and thus arrive at an equation between α and β, which will determine the form of the function ϕ.

Substituting in this equation for α and β their values $\frac{x-a}{z-c}$ and $\frac{y-b}{z-c}$ we obtain the actual equation to the particular conical surface.

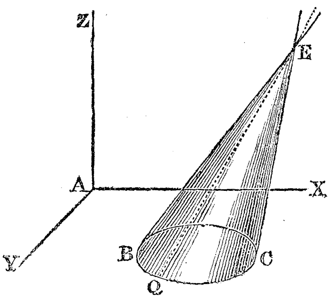
512. Ex. Let the directrix be a circle B Q C in the plane of $x\,y$.

The equations to this directrix are

$$\left.\begin{array}{c}(X - x_1)^2 + (Y - y_1)^2 = r^2 \\ Z = 0\end{array}\right\} \quad (1)$$

And the equations to the generatrix B E, or Q E passing through the point E (a, b, c), are

$$\left.\begin{array}{l}x - a = \alpha\,(z - c) \\ y - b = \beta\,(z - c)\end{array}\right\} \quad (2)$$



To express that the generatrix meets the circle, the equations (1) and (2) must coexist.

$$\therefore Z = z = 0$$
$$X = x = a - \alpha\,c$$
$$Y = y = b - \beta\,c$$

hence by substitution in (1) we have

$$(a - \alpha\,c - x_1)^2 + (b - \beta\,c - y_1)^2 = r^2 \quad (3)$$

Putting for α and β their values from (2) and reducing

$$\left(\frac{a\,z - c\,x}{z - c} - x_1\right)^2 + \left(\frac{b\,z - c\,y}{z - c} - y_1\right)^2 = r^2$$

This is the equation to an oblique cone with a circular base situated in the plane of $x\,y$.

Let the centre of the circle be at the origin $\therefore x_1 = o$ and $y_1 = o$;

$$\therefore (a\,z - c\,x)^2 + (b\,z - c\,y)^2 = r^2\,(z - c)^2$$

513. Let the axis of the cone be parallel to the axis of z $\therefore a = x_1$ and $b = y_1$, and the general equation becomes

$$\left(\frac{x - a}{z - c}\right)^2 + \left(\frac{y - b}{z - c}\right)^2 = \frac{r^2}{c^2}$$

In this case the cone is called a right cone.

Also, if in this case the origin be at the centre of the circle, we have $a = o$ and $b = o$,

$$\therefore x^2 + y^2 = \frac{r^2}{c^2}\,(z - c)^2$$

514. Directrix an ellipse on $x\,y$, whose centre is the origin, and the centre of the cone in the axis of z; then the equation to the cone is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{z - c}{c}\right)^2$$

or putting z, for $z-c$ that is, measuring from the centre of the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}.$$

In this simple case, the equation to the surface is easily found by the method in article (468). Taking the figure in that article, and supposing every section, like P Q, to be an ellipse, whose axes x_1 and y_1 are always proportional to the axes a and b of an ellipse whose centre is in A Z, and at a distance c from A, we have the equation to P Q

$$\frac{x^2}{x_1^{\,2}}+\frac{y^2}{y_1^{\,2}}=1,$$

$$\text{but } y_1=\frac{b}{a}\,x_1,\text{ and } x_1=\frac{a}{c}z$$

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}.$$

515. Let the directrix be a parabola parallel to $x\,y$, and vertex in the axis of z. The equations to the directrix and generatrix are

$$\left.\begin{matrix}Y^2=p\,X\\ Z=d\end{matrix}\right\}1 \qquad \left.\begin{matrix}x-a=\alpha\,(z-c)\\ y-b=\beta\,(z-c)\end{matrix}\right\}$$

at the points of junction we have

$$\begin{aligned}Z&=z=d\\ X&=x=a+\alpha\,(d-c)\\ Y&=y=b+\beta\,(d-c)\end{aligned}$$

hence the final equation is

$$\left\{b+\frac{y-b}{z-c}(d-c)\right\}^2=p\left\{a+\frac{x-a}{z-c}(d-c)\right\}$$

516. Let the vertex or centre of the cone be at the origin $\therefore\ a=b=c=o$, and the equation to a cone whose directrix is $\{\,y^2=p\,x,\ z=d\,\}$ and whose vertex is at the origin, is

$$d\,y^2=p\,x\,z.$$

517. The following method of finding the equation to a right cone whose vertex is at the origin, is sometimes useful.

Let the length of the axis of the cone be k, and suppose this axis to pass through the origin, and be perpendicular to a given plane or base whose equation therefore will be of the form

$$\alpha\,x+\beta\,y+\gamma\,z=k$$

where α, β, γ are the co-sines of the angles which k makes with the axis of x, y, and z (410).

Also suppose x, y, and z to be the co-ordinates of a point on the circumference of this base, and let θ be the angle which the generatrix of the cone makes with its axis, then by the property of the right-angle triangle we have the equation

$$k=\sqrt{(x^2+y^2+z^2)}\cos.\theta$$

Hence by equating the values of k we have the equation,

$$(\alpha\,x+\beta\,y+\gamma\,z)^2=(x^2+y^2+z^2)\,(\cos.\theta)^2$$

And this is the equation to any point in the surface, since α, β, γ remain the same for a plane parallel to the base and passing through any point $(x\ y\ z)$ of the surface.

If the axis of the cone coincides with the axis of z, we have $\alpha = \beta = o$ and $\gamma = 1$; $\quad z^2 = (x^2 + y^2 + z^2)\ (\cos.\ \theta)^2$

518. To find the curve of intersection of a plane and an oblique cone, we may suppose the cutting plane to pass through the origin of co-ordinates without detracting from the generality of the result. Substituting for x, y, z, in the equation, their values in 455, we readily find that the sections are lines of the second order and their varieties.

ON CONOIDAL SURFACES.

519. Definition. A conoidal surface is generated by the movement of a straight line constantly parallel to a plane, one extremity of the line moving along a given straight line, the other describing a given curve.

We shall commence with a simple case. Let the axis of z be one directrix, and let the generatrix be parallel to the plane of $x\ y$: then the equations to the generatrix in any one position are

$$\left.\begin{array}{l} y = \alpha\, x \\ z = b \end{array}\right\}$$

Now it is evident that when a point moves on the surface without quitting the generatrix, α and b are both constant, but when it passes from one position of the generatrix to another α and b are both variable; hence these quantities, being constant together and variable together, are functions of one another.

$\therefore\ b = \phi\ (\alpha)$ or substituting their values.

$$z = \phi\left(\frac{y}{x}\right)$$

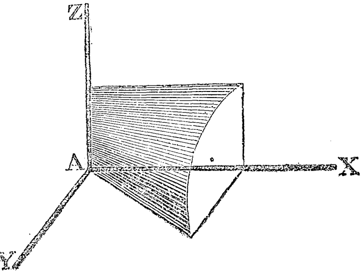
which is the general equation to all conoidal surfaces.

520. The form of the function ϕ will depend upon the nature of the second directrix.

By combining the equations to the generatrix and this directrix, we may, as before, eliminate x, y, z, and arrive at an equation between b and α, we must then substitute the values of b and α, their general values z and $\frac{y}{x}$, and we shall obtain the equation to the particular conoidal surface.

521. Let the second directrix be a circle parallel to $y\ z$, and the centre in the axis of x, therefore the equations to this directrix are

$$\left.\begin{array}{l} Z^2 + Y^2 = r^2 \\ X = a \end{array}\right\} \ (1)$$



Then where this directrix meets the generatrix we have

$$Z = z = b$$
$$X = x = a$$
$$Y = y = \alpha\, a$$
$$\therefore\ b^2 + \alpha^2 a^2 = r^2$$

Hence the required equation is

$$z^2 + a^2 \frac{y^2}{x^2} = r^2.$$

This surface partaking of the form and generation of both the cone and the wedge, was called the cono-cuneus by Wallis, who investigated many of its properties.

If the axis of x be one directrix, and the other be a circle parallel to $x\,z$, and the generatrix be parallel to $y\,z$, the equation is

$$x^2 + \frac{a^2 z^2}{y^2} = r^2.$$

522. Let the axis of z be one directrix, any straight line the other, and let the generatrix move parallel to $x\,y$.

Then the equations to the second directrix are

$$\mathrm{X} = \mu\,\mathrm{Z} + m$$
$$\mathrm{Y} = \nu\,\mathrm{Z} + n$$

Also the equations to the generatrix being $y = \alpha\,x$, $z = b$, we have at the points of junction

$$\mathrm{Z} = z = b$$
$$\mathrm{Y} = y = \nu\,b + n$$
$$\mathrm{X} = x = \frac{\nu\,b + n}{\alpha}$$

$$\therefore \frac{\nu\,b + n}{\alpha} = \mu\,b + m$$

$$\therefore (\nu\,z + n)\frac{x}{y} = \mu\,z + m$$

or $$\nu\,z\,x - \mu\,z\,y + n\,x - m\,y = 0.$$

523. Let the axis of z be one directrix, and let the second directrix be the thread of a screw whose axis is coincident with the axis of z.

The thread of a screw, or the curve called the helix, is formed by a thread wrapped round the surface of a right cylinder, so as always to make the same angle with the axis; or if the base of a right-angled triangle coincide with the base of the cylinder, and the triangle be wrapped round the cylinder, the hypothenuse will form the helix A P.

To find the equations to the helix,

Let the centre of the cylindrical base be the origin of rectangular axes. C M $= x$, M Q $= y$, P Q $= z$ and the radius of the cylinder $= a$.

Then P Q bears a constant ratio to A Q; namely, that of the altitude to the base of the describing triangle

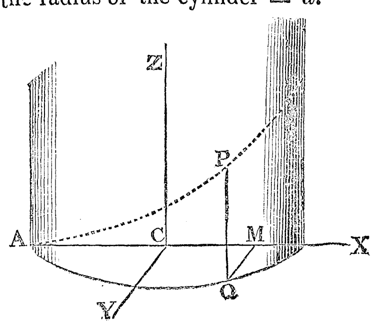
$\therefore$ P Q $= e$ A Q

and A Q is a circular arc whose sine is y and radius a:

$$\therefore\ z = e\,a\,\sin.^{-1}\frac{y}{a},$$

$$\text{or } z = e\,a\cos.^{-1}\frac{x}{a};$$

$$\text{also } x^2 + y^2 = a^2$$



And these are the equations to the projections of the helix.

To return to the problem, which is to find the surface described by a line subject to the conditions that it be parallel to the base of the cylinder, that it passes through the axis, and that it follows the course of the helix.

The equations to the directrix (if c be the interval between two threads) are

$$z = e\,a \sin.^{-1} \frac{y}{a} + c$$

$$x^2 + y^2 = a^2$$

And the equations to the generatrix being $y = \alpha\, x$, $z = b$; we have

$$z = b;\ x = \frac{y}{\alpha} = \sqrt{a^2 - y^2} \therefore y = \frac{\alpha\, a}{\sqrt{1 + \alpha^2}}$$

$$\therefore b = e\, a \sin.^{-1} \frac{\alpha}{\sqrt{1 + \alpha^2}} + c$$

hence the equation to the surface is

$$z = e\, a \sin.^{-1} \frac{y}{\sqrt{y^2 + x^2}} + c$$

This surface is the under side of many spiral staircases.

524. A straight line passes through two straight lines whose equations are $x = a$, $y = b$; and $x = a_1$, $z = b_1$; and also through a given curve $z = f(y)$ in the plane of $z\,y$; to find the equation to the surface traced out by the straight line.

The three directrices are

$$\left.\begin{array}{l} X = a \\ Y = b \end{array}\right\} 1 \qquad \left.\begin{array}{l} X = a_1 \\ Z = b_1 \end{array}\right\} 2 \qquad \left.\begin{array}{l} Z = f(Y) \\ X = o \end{array}\right\} (3)$$

And let the equations to the generatrix be

$$x = \alpha\, z + m$$

$$y = \beta\, z + n$$

and consequently $y = \frac{\beta}{\alpha} x + p$, if $p = n - \frac{\beta}{\alpha} m$;

Then since this line meets the three given lines, we have the following equations

$$b = \frac{\beta}{\alpha} a + p,\quad a_1 = \alpha\, b_1 + m,\quad - \frac{m}{\alpha} = f\left(- \frac{m}{\alpha} \beta + n\right)$$

We must now eliminate α, b, m, n from these equations, and that to the generatrix.

By subtraction we have

$$y - b = \frac{\beta}{\alpha}(x - a);\ x - a_1 = \alpha\,(z - b_1) \quad \therefore \alpha = \frac{x - a_1}{z - b_1},$$

$$\therefore \frac{m}{\alpha} = \frac{x}{\alpha} - z = \frac{a_1\, z - b_1\, x}{x - a_1}, \text{ and}$$

$$\frac{n\,\alpha - m\,\beta}{\alpha} = \frac{\alpha\,(y - \beta\, z) - \beta\,(x - \alpha\, z)}{\alpha} = \frac{\alpha\, y - \beta\, x}{\alpha} = y - \frac{\beta\, x}{\alpha} = \frac{b\, x - a\, y}{x - a}.$$

Hence the final equation is

$$\frac{b_1\, x - a_1\, z}{x - a_1} = f\left(\frac{b\, x - a\, y}{x - a}\right)$$

525. The following problem is easily solved in the same manner. To find the equation to a surface formed by a straight line moving parallel to the plane of $x z$, and having its extremities in two given curves $z = f(y)$ on $z y$, and $x = \phi(y)$ on $x y$.

The equation is $$\frac{z}{f(y)} + \frac{x}{\phi(y)} = 1.$$

526. In questions of this kind some care is requisite in selecting the position of the axes and co-ordinate planes, so that the equations, both those given and those to be found, may present themselves in the simplest form. For example, —to find the surface formed by the motion of a straight line constantly passing through three other given straight lines;

Take three lines parallel to the given lines for the axes of co-ordinates; then the equations to the three directrices are

$$\left.\begin{array}{l} X = a_1 \\ Y = b_1 \end{array}\right\} \qquad \left.\begin{array}{l} X = a_2 \\ Z = c_2 \end{array}\right\} \qquad \left.\begin{array}{l} Y = b_3 \\ Z = c_3 \end{array}\right\}$$

and the equations to the generating line in any position are

$$x = \alpha z + a, \; y = \beta z + b,$$

and consequently $y = \frac{\beta}{\alpha} x + c$, where $c = b - \frac{\beta}{\alpha} a$;

Then since this line meets each of three given lines, we have the following equations:

$$b_1 = \frac{\beta}{\alpha} a_1 + c \; ; \; a_2 = \alpha c_2 + a \; ; \; b_3 = \beta c_3 + b.$$

We must now eliminate a, b, α, β from these three equations and that to the generatrix; by subtraction we have

$$y - b_1 = \frac{\beta}{\alpha}(x - a_1) \; ; \; x - a_2 = \alpha(z - c_2) \; ; \; y - b_3 = \beta(z - c_3)$$

hence, eliminating α and β, we have the required equation

$$(x - a_1)(y - b_3)(z - c_2) = (x - a_2)(y - b)(z - c_3)$$

which is of the second order, since the term $x y z$ disappears. See Hymers's Anal. Geom. p. 23, Cambridge, 1830.

CHAPTER IX.

ON CURVES OF DOUBLE CURVATURE.

527. Definition. A curve of double curvature is one whose generating point is perpetually changing not only the direction of its motion, as in plane curves, but also the plane in which it moves.

If a circle be described on a flat sheet of paper, it is a plane curve; let the sheet of paper be rolled into a cylindrical form, then the circle has two curvatures, that which it originally had, and that which it has acquired by the flexion of the paper, hence in this situation it is called a curve of double curvature.

528. Curves of double curvature arise from the intersection of two surfaces; for example, place one foot of a pair of compasses on a cylindrical surface, let the other in revolving constantly touch the surface, it will describe a curve of double curvature, which, though not a circle, has yet all

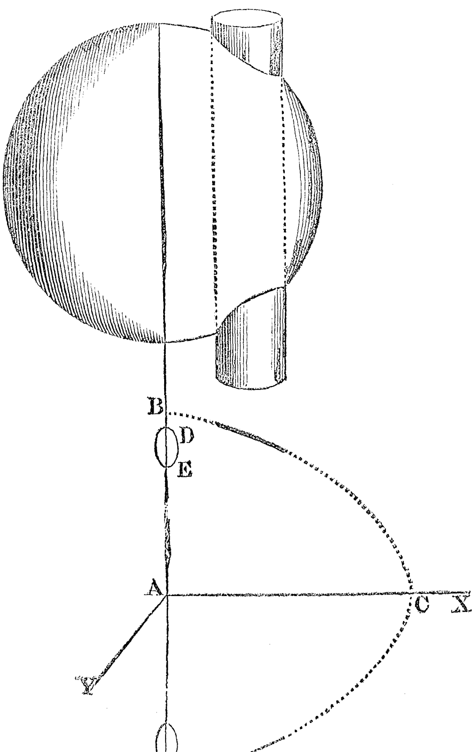
its points at equal distances from the fixed foot of the compasses. The curve is then part of a spherical surface, whose radius is equal to the distance between the feet of the compasses, and consequently is the intersection of this sphere with the cylinder.

529. The equations to the two surfaces taken together are the equations to their intersection, and consequently are the equations to the curve of double curvature.

By the separate elimination of the variables in the two equations, we obtain the respective projections of the curve upon the co-ordinate planes. Two of these are sufficient to define the curve of double curvature; for we may pass two cylinders through two projections of the curve, at right angles to each other, and to the co-ordinate planes, the intersection of these cylinders is the required curve. This is analogous to the consideration of a straight line, being the intersection of two planes.

We proceed to examine curves of double curvature arising from the intersections of surfaces.

530. Let the curve arise from the intersection of a sphere and right cylinder; the origin of co-ordinates being at the centre of the sphere, the axis of the cylinder in the plane $x\,z$ and parallel to the axis of z.



Let the distance between the centres of the sphere and cylinder $= c$, then the equation to the sphere is $x^2 + y^2 + z^2 = a^2$, and the equation to the cylinder is $(x - c)^2 + y^2 = b^2$, (507.)

eliminating y, $z^2 = a^2 + c^2 - b^2 - 2\,c\,x$ (1),

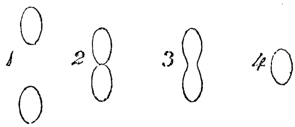
eliminating x, $z^2 = a^2 - b^2 - c^2 \mp 2\,c\,\sqrt{b^2 - y^2}$ (2).

From (1) the projection of the curve on $x\,z$ is a portion of a parabola B C whose vertex is C, where A C $= \dfrac{a^2 + c^2 - b^2}{2\,c}$. and A B $= \sqrt{a^2 + c^2 - b^2}$.

From (2) the projection on $y\,z$ consists of two ovals, whose positions are determined by the two extreme values of z,

$$\text{A D} = \pm\sqrt{a^2 - (b - c)^2}$$

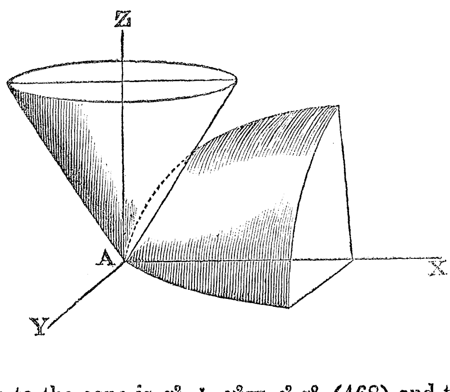
$$\text{A E} = \pm\sqrt{a^2 - (b + c)^2}.$$



As c increases, that is, as the cylinder moves further from A, A E decreases, and the ovals approach nearer to each other, as in fig. (1); when $c = a - b$, that is, when the sphere but just encloses the cylinder A E $= 0$, and the ovals meet, fig. (2). As c increases, we obtain fig. (3), which gradually approaches fig. (4); and lastly, when $c = a$ vanishes entirely.

Different values, as c, $\frac{a}{2}$, &c., may be given to b, and we may then trace the projections: they offer no difficulty, but we recommend their investigation, as the complete examination of one example greatly facilitates the comprehension of all others.

531. Ex. 2. A right cone and a paraboloid of revolution have their vertices coincident, the axis of the cone being perpendicular to the axis of the paraboloid.



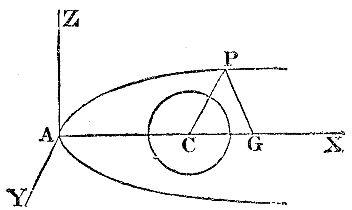
The equation to the cone is $x^2 + y^2 = e^2\,z^2$, (468) and that to the paraboloid, $y^2 + z^2 = p\,x$ (469); hence the projection on $x\,z$ is $x^2 + p\,x = (1 + e^2)\,z^2$, which is an hyperbola, whose axes are p and $\dfrac{p}{\sqrt{e^2 + 1}}$ (157).

Again, $x^2 + y^2 = e^2 (p x - y^2) \therefore (1 + e^2) y^2 = e^2 p x - x^2$; hence the projection on $x y$ is an ellipse, whose vertex is A and axes $e^2 p$ and $\frac{e^2 p}{\sqrt{1 + e^2}}$ (103).

The equation to the projection on $y z$ is $(y^2 + z^2)^2 + p^2 y^2 = e^2 p^2 z^2$; this is the equation to a Lemniscata, and becomes the Lemniscata of Bernouilli, when $e = 1$, that is, when the cone is right-angled (314).

532. To find the curve of intersection of two surfaces, we have eliminated the variables separately, and thus obtained the equations to the projections on the co-ordinate planes; conversely, by combining these last equations either by addition or multiplication, &c., so as to have an equation between the three variables, we may obtain the surface on which the curve of double curvature may be described. This surface does not at all define the curve of double curvature; since an infinite number of curves may be traced on this individual surface, to all of which the general equation to the surface belongs.

The results of the above combination are often interesting. For example: Let the curve be the intersection of a parabolic cylinder on $x y$, with a circular cylinder on $x z$, the origin being the vertex of the parabola, and the centre of the circle being in the axis of the parabola, which is also the axis of x.



Let $y^2 = 2 p x$ be the equation to the parabola A P on $x y$,
$(x - a)^2 + z^2 = r^2$.... circle on $x z$,
Combining these equations by addition,
$(x - a)^2 - 2 p x + y^2 + z^2 = r^2$,
or $(x - a - p)^2 + y^2 + z^2 = r^2 + p^2 + 2 a p$.

Which is the equation to a sphere whose centre is at a distance A G $= a + p$, measured from A along A X. Now, p is the subnormal C G to the point P of the parabola, P C being the ordinate at C (242); hence all the points of the curve of double curvature are on the surface of a sphere whose centre is at the extremity of the subnormal of a point in the parabola, the ordinate of which point passes through the centre of the given circle.

533. The intersections of surfaces are not always curves of double curvature, but often they are plane curves. We proceed, then, to show how plane curves may be detected, and their equations determined.

Whenever we obtain a straight line for a projection, the curve cannot be one of double curvature.

Ex. Let the curve be the intersection of two parabolic cylinders, whose equations are

$$x^2 = a z$$
$$b y = x^2.$$

Eliminating x, we have $b y = a z$, hence the projection on $y z$ is a straight line; and as no projection of a curve of double curvature can be a straight line, it follows that the curve of intersection is a plane curve.

534. Again, If we can so combine the equations to the projections as to produce the general equation to a plane, the curve, which is necessarily traced on that plane, is itself a plane curve. For example: let the curve

arise from the intersection of two parabolic cylinders, whose equations are

$$x^2 = a z$$
$$b y = x^2 + c x.$$

In the second equation, substituting $a z$ for x^2, we obtain

$$b y = a z + c x;$$

which equation belonging to a plane, the curve is a plane curve.

535. There is another and more general method of detecting plane curves.

From the two equations to the surfaces eliminate one of the variables, as z, for example, we obtain an equation $F(x, y) = 0$.

Now, if the curve be plane, it may arise from the intersection of either of the surfaces with a plane whose equation is $z = m x + n y + p$; eliminate z between this equation to the plane and that to one of the surfaces, the result is $f(x, y) = 0$, which must be identical with $F(x, y) = 0$; therefore, comparing $F(x, y) = 0$, with $f(x, y) = 0$, we may obtain various equations to determine m, n, and p; which values of m, n, and p must satisfy all the equations in which these quantities appear; if not, the curve is one of double curvature.

For example; take the intersection of a sphere and cylinder, art. 530.

The equation to the Sphere is $x^2 + y^2 + z^2 = a^2$ (1)
.... Cylinder $(x - c)^2 + y^2 = b^2$ (2)
.... Plane $z = m x + n y + p$ (3)

Eliminating z between (1) and (3), we have $f(x, y) = 0$

$$(m^2+1) x^2 + (n^2+1) y^2 + 2 m n x y + 2 m p x + 2 n p y + p^2 - a^2 = 0 \quad (4)$$

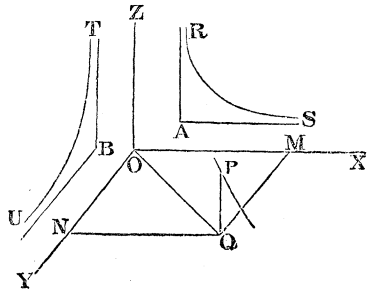
Comparing (2) and (4), we have $m = 0$, $n = 0$ from the co-efficients of x^2 and y^2; but the condition of $m = 0$ destroys the coefficient of x in (4); and thereby shows that (4) cannot be made identical with (2). The curve is therefore a curve of double curvature.

But let the equation to the cylinder be $x^2 + y^2 = b^2$, then $m = 0$ and $n = 0$ render (4) and (2) identical; therefore the curve is a plane curve, situated in a plane, whose equation is $z = \sqrt{a^2 - b^2}$; this is clear, also, from geometrical considerations.

536. To find the curve represented by the equations

$$\frac{a}{x} + \frac{c}{z} = 1, \quad \frac{b}{y} + \frac{c}{z} = 1.$$

These equations, taken separately, belong to two right hyperbolic cylinders; one with the base in $x z$, and the other in $y z$. (209, Ex. 3.)



R S is the hyperbola on $x z$, its centre being at A; T U is the hyperbola on $y z$, its centre being at B.

Also, $\frac{a}{x} = \frac{b}{y}$, or $y = \frac{b}{a} x$.

Hence the projection of the intersection of the above cylinders on $x y$ is a straight line O Q, and therefore the curve is a plane curve, situated in the plane Z O Q, perpendicular to $x y$.

537. As we cannot have a very clear notion of the curve itself, merely from the idea of the two hyperbolic cylinders, we shall find the equation to the curve in the plane Z O Q; that is, in its own plane.

Let P be any point in the curve; O M $= x$, M Q $= y$, P Q $= z$. Then, in order to find the relation between O Q ($= u$) and Q P ($= z$), we shall express O M and O N in terms of O Q, and substitute in the given equations.

The equation to O Q is $y = \frac{b}{a} x = x \tan. \theta$ (if $\frac{b}{a} = \tan. \theta$),

$$\therefore \text{O M} = \text{O Q} \cos. \theta, \text{ and O N} = \text{O Q} \sin. \theta.$$

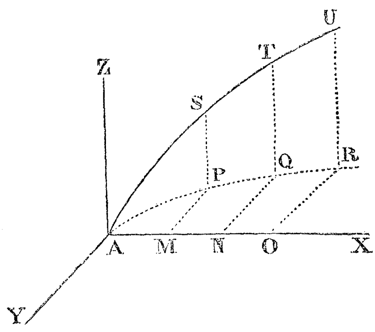
Hence the equation $\frac{a}{x} + \frac{c}{z} = 1$ becomes $\frac{a}{u \cos. \theta} + \frac{c}{z} = 1$,

and the equation $\frac{b}{y} + \frac{c}{z} = 1$ becomes $\frac{b}{u \sin. \theta} + \frac{c}{z} = 1$.

Since $b = a \tan. \theta$, or $b \cos. \theta = a \sin. \theta$, these two equations are the same, and either of them belongs to the required curve; hence the curve is an hyperbola, whose equation referred to its centre is

$$u\, z = \frac{a\, c}{\cos. \theta} = \frac{b\, c}{\sin. \theta}. \quad (209)$$

538. To describe a curve of double curvature by points



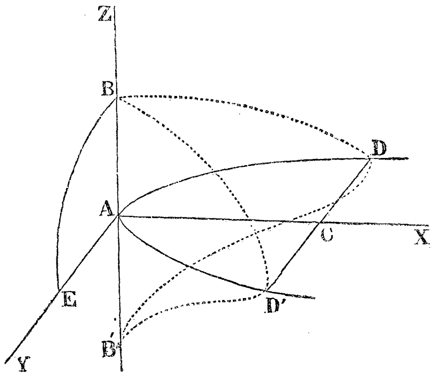
Let $f(x, y) = 0$, and $\phi(x\ z) = 0$, be two of its projections.

Upon $x\,y$ trace the curve A P Q R, whose equation is $f(x, y) = 0$.

For any value of x, as A M, we obtain a corresponding value M P of y; from $\phi(x, z) = 0$, we can also obtain a corresponding value of z. From P draw P S perpendicular to $x\ y$, and equal to this value of z; then S is a point in the curve. By repeating this process we may obtain any number of points S T U, &c., in the curve.

It is evident, that if any value given to x or y renders z imaginary, no part of the curve can be constructed corresponding to such values of x or y. Also, that if z be negative, P S must be drawn below the plane $x\ y$.

539. Ex. 1. Let the curve arise from the intersection of a parabolic cylinder on $x\ y$, and a circular cylinder on $y\ z$, the axes perpendicular to each other; and the vertex of the parabola together with the centre of the circle at the origin of co-ordinates.



Let $y^2 = a\,x$ be the equation to the parabola D A D′,

$y^2 + z^2 = a^2$ .... circle E B,

$\therefore\ z^2 + a\,x = a^2$ is a parabola on $x\,z$.

Let A B $= a$, A C $= a$, and let the ordinate C D $= a$.

To trace the curve, we have the three equations on the co-ordinate planes,

$$z = \pm\sqrt{a\,(a - x)}$$
$$z = \pm\sqrt{a^2 - y^2}$$
$$y = \pm\sqrt{a\,x}.$$

If $x = 0$, $y = 0$, and $z = a$, $\therefore$ the curve passes through B ; as x increases, y increases, and z diminishes ;

When $x = a$, $y = a$, and $z = 0$, therefore the curve decreases in altitude from B down to meet the parabola in D. This gives the dotted branch BD.

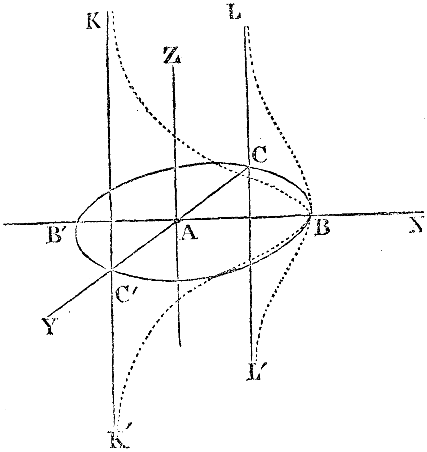
If x is greater than a, z is imaginary ; therefore the curve does not extend beyond D.

But since $z = \pm\sqrt{a\,(a-x)}$ there is another ordinate corresponding to every value of x between o and a ; hence there is another branch, equal and opposite to B D, but below the plane $x\,y$. This is represented by D B′.

Again, since when y is negative, the values of z do not change, there is another arc, B D′ B′, represented by the double dotted line, which is exactly similar to B D B′.

Therefore, the curve is composed of four parts, B D, D B′, B D′, and D′ B′, equal to one another, and described upon the surface of the parabolic cylinder, whose base is D A D′. These branches form altogether a figure something like that of an ellipse, of which the plane is bent to coincide with the cylinder.

540. Ex. 2. Let the circle, whose equation is $x^2 + y^2 = a^2$, be the projection of the curve of double curvature on $x\,y$; and the curve, of which the equation is $a^2\,y^2 = a^2\,z^2 - y^2\,z^2$, be the projection on $y\,z$, to trace the curve.



Let B C B′ C′ be the circle on $x\ y$ whose equation is $x^2 + y^2 = a^2$; then the equation on $y\ z$ being $a^2 y^2 = a^2 z^2 - y^2 z^2$, the equation on $x\ z$ is $x^2 z^2 = a^4 - a^2 x^2$.

$$\therefore z = \pm \frac{a}{x} \sqrt{a^2 - x^2},$$

$$\text{or } z = \pm \frac{a\,y}{\sqrt{a^2 - y^2}},$$

$$\text{and } y = \pm \sqrt{a^2 - x^2}.$$

If $x = 0$, $y = a$, $z =$ infinity, therefore the vertical line C L through C is an asymptote to the curve. As x increases, y decreases, and z decreases, therefore the curve approaches the plane of $x\ y$. If $x = a$, $y = 0$, $z = 0$, therefore the curve passes through B. If x is greater than a, y and z are each impossible, therefore no part of the curve is beyond B: for any value of y there are two of z, therefore for the values of y in the quadrant A C B, there are two equal and opposite branches, L B, B L′.

Similarly there are two other equal branches, K B, B K′, for the quadrant B A C′; and as the same values of y and z recur for x negative, there are four other branches equal and opposite to those already drawn, which correspond to the semicircle C B′ C′, and which proceed from B′.

These two examples are taken from Clairaut's Treatise on Curves of Double Curvature; a work containing numerous examples and many excellent remarks on this subject.

Printed by W. Clowes, Duke Street, Lambeth.

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ERRATA.

Page 7, line 1, read Let $x = \sqrt{3} = \sqrt{2 + 1}$. In the last figure let A B, B C, and C D each be equal to the linear unit, then A D $= \sqrt{3}$.

40.....2, read $\therefore y' = \pm \frac{r}{\sqrt{1 + a^2}}$.

40.....3, for $\frac{r^2}{y}$ read $\frac{r^2}{y'}$.

40....17, for 24 read 25.

48....12, for y^3 read y^2.

110....24, for e read c.

111....17, for tan. θ . tan. $= \frac{c}{a}$, read tan. θ . tan. $\theta' = \frac{c}{a}$.

112....20, for $\frac{y^2}{b^3}$ read $\frac{y^2}{b^2}$.

114....18, for conjugate read semi-conjugate.

123....30, for $x' + m$ S = P, read $x' + m =$ S P.

153, in the table, column 7, insert c.

190, line 5 from bottom, for $3\,a$, read $3\,a^2$.

209, line 10, read cos. $l\,x$ cos. $l_1\,x$ + cos. $l\,y$ cos. $l_1\,y$.

217, line 13, for (2) read (1).

221, line 27, read cos. $\theta = \frac{x\,x_1 + y\,y_1 + z\,z_1 + \&c.}{r\,r_1} =$

224, line 10, for 397 read 402.

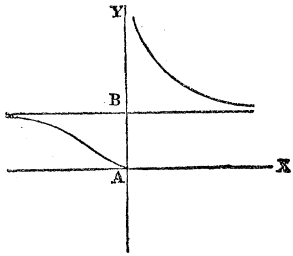
247, line 3, $\therefore (x - x_1)^2 + (y - y_1)^2 = r^2$;

and if the axis coincide with A Z, $x^2 + y^2 = r^2$, $z = 0$;

249, line 1, for $r - c$ read z, and for c read $z - c$.

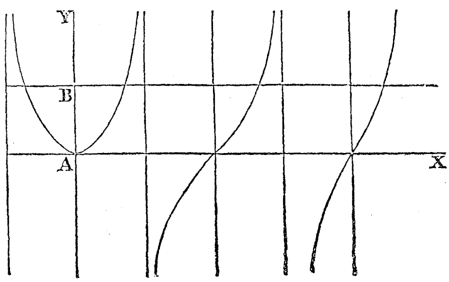
ERRATA IN THE FIGURES.

Art. 352. $y = a^{\frac{1}{x}}$



Art. 353 $y = x^x$. See the figure in the same page just above the Art. 353 ; the letter B should be at the point where the upper curve meets A Y.

Art. 355 $y = x \tan. x$



Art. 363. The involute of the circle